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Primitive Groups with a Determination of the Primitive Groups of Degree 20.

By Elizabeth R. Bennett.

Introduction.

Substitution groups were first considered in connection with the solution of algebraic equations. The earliest work that devotes considerable attention to these groups (Ruffini, *Teoria generale delle equazioni*, Bologna, 1799) divides the substitution groups, or permutations, as Ruffini calls them, into two main classes, viz., the cyclic and non-cyclic permutations. The non-cyclic permutations are again divided into permutations of the first, second, and third kind, which correspond to what are now known as intransitive, imprimitive, and primitive groups.

The primitive group notion was emphasized later by Galois, who defined primitive groups in connection with primitive equations* and who was the first to show the close relation which exists between the theory of substitution groups and the solution of algebraic equations. From his time mathematicians have recognized the fact that the determination of all the substitution groups of a given degree is a fundamental problem of algebra. Since the determination of intransitive groups is based on a knowledge of the transitive groups of lower degree and the determination of imprimitive groups requires a knowledge of primitive groups of lower degree, it is evident that the determination of the primitive groups of different degrees is important for the solution of this fundamental problem. Numerous theorems have been published which aid in this determination, and the enumeration of all the primitive groups of a given degree has been carried through degree 19.†

* Galois, *Œuvres publiés par E. Picard*, Paris, 1897, pp. 11, 53

† The possible primitive groups of degree 19 are mentioned by Jordan, Paris *Comptes Rendus*, Vol. LXXIX (1874), p. 1150. Extensive references as regards the primitive groups of lower degree are given in the *Encyclopédie des Sciences Mathématique*, Vol. I, p. 564.

The first part of the present paper contains a number of theorems concerning the determination of simply transitive primitive groups. In Part II the primitive groups of degree 20 are determined, while in Part III certain primitive groups whose maximal subgroup contains a transitive constituent of prime order are considered.

PART I.

Simple Transitive Primitive Groups. General Theorems.

Let G represent a primitive group of degree n and let G_1 represent its maximal subgroup of degree $n-1$. When G is simply transitive, G_1 is intransitive and must contain certain transitive constituents. The determination of all the simply transitive primitive groups of a given degree n, therefore, requires the determination of the intransitive groups of degree $n-1$ which can be maximal subgroups of a primitive group. It is important to know what groups may occur in G_1 as transitive constituents and what relations, if any, must exist between the degrees or between the orders of such transitive constituents. Various theorems which aid either directly or indirectly in the determination of these transitive constituents have been proved by Jordan,* Miller,† and Rietz.‡

In the present paper extensive use has been made of the following theorem and corollary with respect to the transitive constituents of G_1:

THEOREM. If a transitive constituent of G_1 is of a prime order, the order of G_1 is the same prime number, and G is of class $n-1$.

COROLLARY. If G_1 contains a constituent of degree 2, its order is 2, and the degree of G is a prime number.§

The corollary, in particular, has suggested the consideration of what additional transitive constituents G_1 must contain when it contains either a transitive constituent of degree 3 or a transitive constituent of degree 4. The results of these considerations are stated in Theorems VII to X.

Certain conditions regarding the transitive constituents which must occur in G_1 are obtained by considering a subgroup K of G_1 which will now be defined. Suppose one of the transitive constituents M of G_1 is a non-regular

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* Jordan, *Traité des Substitutions*, p. 284.

† Miller, *Proc. London Math. Soc.*, Vol. XXVIII, pp. 535, 536.

‡ Rietz, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVI, pp. 5, 6.

§ Miller, *Proc. London Math. Soc.*, Vol. XXVIII, p. 536.

group of degree m and consider the isomorphism that exists between this group and the group formed from all the other transitive constituents of G_1. The invariant subgroup of G_1 corresponding to identity in the transitive constituent of degree m will be denoted by H and the subgroup composed of all the substitutions leaving a letter of the transitive constituent of degree m fixed will be denoted by K.

In Theorems I to VI inclusive it is assumed that H is of degree $n - m - 1$ and that G_1 contains a transitive constituent M of degree m, but no other transitive constituent whose degree divides m.

THEOREM I. *If the subgroup which is both in K and also in a transitive constituent of degree m_1 of G_1 is transitive, K must contain another transitive constituent of degree m_1 which is not a transitive constituent of G_1 and which is similar to the one previously known to be in K.*

Since K leaves at least one letter of G_1 fixed, it must occur in some conjugate of G_1, as G_1'. The group G_1' may then be considered with respect to the isomorphism which exists between its transitive constituent M' of degree m and the group formed from all the other transitive constituents of G_1'. Since K is composed of all the substitutions leaving a letter of the transitive constituent of degree m fixed, the order of K is one m-th the order of G_1 and, therefore, its order must be one m-th the order of G_1'. Moreover, since the subgroup K leaves a letter of the transitive constituent M of G_1 fixed, when it occurs in G_1' it must also leave a letter of some transitive constituent of G_1' fixed. However, none of the transitive constituents of G_1' except M' have degrees which divide m. Therefore, K could not be contained in a subgroup composed of all the substitutions leaving a letter of a transitive constituent of G_1' fixed, M' excepted, for the order of K would not divide the order of such a subgroup. The subgroup K also could not leave fixed all the letters of a transitive constituent of G_1', for in $G_1 K$ contained letters from all the transitive constituents of G_1. The subgroup K in G_1' must then be composed of all the substitutions leaving a letter of the transitive constituent M' of degree m fixed and must occupy in G_1' a place similar to the one which it occupied in G_1. The transitive constituent m_1 in G_1' cannot have exactly the same letters as in G_1, for if it had this property, G_1 and G_1' would generate an intransitive group. This is impossible since G_1 is a maximal subgroup of G. It is evident that when the subgroup of the transitive constituent of degree m_1 in K is transitive, the letters cannot be partly different from what they were in G_1. Therefore, all

the letters of the transitive constituent of degree m_1 in G_1' must be entirely different from what they were in G_1, and K must contain another transitive constituent of degree m_1 which is similar to the one already known to be in it. Since G_1 and G_1' generate G, this additional transitive constituent of degree m_1 in K cannot be a transitive constituent of G_1.

THEOREM II. *If the subgroup which is both in K and also in a transitive constituent of degree m_1 of G_1 is intransitive, then K must contain a transitive constituent similar to one of the transitive constituents occurring in this intransitive subgroup.*

Since K leaves at least one letter of G_1 fixed, it must occur in some conjugate of G_1, as G_1'. When G_1' is considered with respect to the isomorphism which exists between its transitive constituent M' of degree m and the group formed from all the other transitive constituents of G_1', it may be shown as in Theorem I that K occupies in G_1' a place similar to the one which it occupied in G_1. The subgroup which is both in K and also in a transitive constituent of degree m_1 of G_1' is intransitive and will contain in K certain transitive constituents of the same degree. At least one of these must be entirely different from any such constituent occurring in the intransitive subgroup of the constituent of degree m_1 of G_1 which is in K. If such were not the case, G_1 and G_1' would again generate an intransitive group. Therefore, K must contain a transitive constituent similar to one occurring in the subgroup of the group of degree m_1 which is in K and entirely different from it.

If in place of the transitive constituent of degree m_1 in G_1, the transitive constituent M of degree m is considered, a theorem with respect to the subgroup of M composed of all the substitutions leaving a letter of this transitive constituent fixed similar to Theorems I and II may be proved. The theorem is as follows:

THEOREM III. *If l represents the degree of one of the transitive constituents of the subgroup of M composed of all the substitutions leaving a letter of this transitive constituent fixed, or the degree of this subgroup itself when it is transitive, then K must contain a transitive constituent of degree l not in M.*

As in the previous theorems the subgroup K must occur in some conjugate of G_1, as G_1', and when G_1' is considered with respect to the isomorphism which exists between its transitive constituent M' of degree m and the group formed from all the other transitive constituents of G_1', K is seen to occupy in G_1' a place similar to the one which it occupied in G_1. The subgroup of M' which is

composed of all the substitutions leaving a letter of M' fixed cannot be entirely the same as that occurring in the transitive constituent M of degree m of G_1. If such were the case, G_1 and G_1' would generate an intransitive group, which is impossible, since G_1 is maximal. If the subgroup of M' leaving a letter fixed is transitive, none of its letters can be the same as those of a corresponding subgroup composed of all the substitutions leaving a letter of the transitive constituent M of G_1 fixed, since each set of l letters is transitively connected. Therefore, K must contain a transitive constituent of degree l which is entirely different from the one appearing in the subgroup of M which is in K. Should the subgroup of M' leaving a letter fixed be intransitive, it will contain certain transitive constituents. At least one of these must be entirely different from any such constituent occurring in the subgroup of M which is in K; otherwise G_1 and G_1' would again generate an intransitive group. Therefore K must again contain a transitive constituent of degree l different entirely from any such constituent occurring in the subgroup of M in G, where l is assumed to represent the degree of any one of the transitive constituents of the subgroup composed of all the substitutions of M which leave a letter fixed.

THEOREM IV. *If G_1 contains a primitive group M_1 of degree m_1 as a transitive constituent, the subgroup of M_1 occurring in K is transitive. Should this transitive subgroup in K also be primitive, G_1 must contain an imprimitive group whose systems of imprimitivity are of degree m_1.*

The subgroup K must occur in some conjugate of G_1, as G_1', and must occupy in G_1' a place similar to the one which it occupied in G_1. Since H is of degree $n-m-1$ and a primitive group cannot contain an intransitive, invariant subgroup,* the subgroup of the transitive constituent of degree m_1 in K must be transitive. The subgroup K must then contain another transitive constituent of degree m_1 similar to the one already in K and which is not a transitive constituent of G_1. If this additional transitive constituent is primitive and is also a part of some transitive constituent of G_1 of degree greater than m_1, it must be a system of imprimitivity of this constituent of larger degree. This follows from the fact that H is of degree $n-m-1$ and the considered primitive constituent of K of degree m_1 which is a part of a transitive constituent of G_1 has a transitive subgroup in H.

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*Compare Burnside, *Theory of Groups*, p. 187.

COROLLARY I. *If a primitive group of degree m_1 occurs as a factor in G_1, then G_1 must contain an imprimitive group whose systems of imprimitivity are of degree m_1.*

The transitive constituent of degree m_1 occurs as a primitive group in K and, therefore, has a transitive subgroup of degree m_1 in H.

COROLLARY II. *All the transitive constituents of G_1 cannot be primitive groups.*

THEOREM V. *If the transitive constituent M of degree m of G_1 is a primitive group, and if no transitive constituent of degree l occurs in G_1, where l represents the degree of one of the transitive constituents of the subgroup of M leaving a letter fixed, or the degree of this subgroup itself when it is transitive, then when the transitive constituent of degree l of K which is not in M has a transitive subgroup in H, G_1 must contain an imprimitive group of degree ml.*

According to Theorem III the subgroup K must contain a transitive constituent of degree l which is entirely different from any such constituent occurring in the part of K that is in M. Since the order of K is one m-th the order of G_1, this transitive constituent of degree l must be transformed into itself by one m-th the substitutions of G_1. Then since the subgroup composed of all the substitutions of M leaving a letter fixed is a maximal subgroup of a primitive group, if the transitive constituent of degree l in K were transformed into itself by more than one m-th the substitutions of G_1, it would be transformed into itself by the entire G_1. This is impossible, since G_1 is assumed to contain no transitive constituent of degree l. The transitive constituent of degree l in K has a transitive subgroup in H and is not a transitive constituent of G_1; therefore, it must be a system of imprimitivity of some transitive constituent of G_1. Then since the constituent of degree l is transformed into itself by only one m-th the substitutions of G_1, G_1 must contain an imprimitive group of degree ml.

COROLLARY I. *If the transitive constituent of degree l in K which is not in M is a primitive group, G_1 always contains an imprimitive group of degree ml.*

Since H is of degree $n - m - 1$ and an invariant subgroup of a primitive group is transitive, the subgroup of the constituent of degree l in H is always transitive.

COROLLARY II. *If the transitive constituent M of degree m of G_1 is a K-times transitive group, $K \geqq 3$, G_1 always contains an imprimitive group of degree $m(m-1)$.*

The subgroup composed of all the substitutions leaving a letter of M fixed

is at least doubly transitive and, therefore, a primitive group. It is also of degree $m-1$. Therefore, G_1 contains an imprimitive group of degree $m(m-1)$.

THEOREM VI. *If G_1 contains an imprimitive group M of degree m as a transitive constituent, and if no transitive constituent of degree l occurs in G_1, where l represents the degree of any one of the transitive constituents of the subgroup of M composed of all the substitutions leaving a letter of M fixed, then when the transitive constituent of degree l which is not in M has a transitive subgroup in H, G_1 must contain a transitive constituent of degree $m'l$, where m' represents some divisor of m different from* 1.

According to Theorem III the subgroup K of G_1 must contain a transitive constituent of degree l entirely different from any such constituent occurring in the part of K in M. Since the order of K is one m-th the order of G_1, this transitive constituent of degree l must be transformed into itself by at least one m-th the substitutions of G_1. It cannot be transformed into itself by G_1 or G_1 would contain a transitive constituent of degree l. However, the transitive constituent of degree l may be transformed into itself by certain subgroups of G_1 containing K. The orders of such subgroups must then be multiples of the order of K or one m'-th the order of G_1, where m' is some divisor of m different from 1. Then if the transitive constituent of K of degree l is transformed into itself by one m'-th the substitutions of G_1 and the part of the constituent of degree l in H is transitive, it must be a system of imprimitivity of some transitive constituent of G_1, and G_1 must contain a transitive constituent of degree $m'l$.

THEOREM VII. *If G_1 contains a transitive constituent of degree* 3, *it contains another transitive constituent of degree* 3 *or* 6.*

When the transitive constituent of degree 3 is of order 3, the theorem is self-evident, since all the other transitive constituents must be of the same degree and order.† If the constituent of degree 3 is of order 6 and G_1 is of this same order, all the other transitive constituents of G_1 must be of degrees 3 or 6, for only transitive constituents of these degrees can be put in simple isomorphism with the symmetric group of degree 3. It may then be assumed that the order of G_1 is greater than 6, and since the theorem would require no proof if G_1 contained a second transitive constituent of degree 3, it may be assumed that G_1 contains only one transitive constituent of degree 3. Under

* This theorem was proved by Professor G. A. Miller in a lecture course during the year 1909–10.

† Miller, *Proc. London Math. Soc.*, Vol. XXVIII, p. 536.

this assumption H is of degree $n-4$ and Theorem V then states that G_1 must contain a transitive constituent of degree 6.

THEOREM VIII. *If G_1 contains a group of degree 4 whose order is a power of 2 as a transitive constituent, it must contain another transitive constituent of degree 4 or of degree 8.*

When the transitive constituent of degree 4 is of order 4, G_1 is of order 4 and G is of class $n-1$.* The group of order 8 and degree 4 can be represented only on 4 or 8 letters; therefore, when the transitive constituent of degree 4 is of order 8 and G_1 is of the same order, G_1 can contain only transitive constituents of degrees 4 or 8. Since any transitive constituent in simple isomorphism with the transitive constituent of degree 4 and order 8 must be of degree 4 or 8, it may be assumed that the order of G_1 is greater than 8. The subgroup of the transitive constituent of degree 4 composed of all the substitutions leaving a letter of this constituent of degree 4 fixed is of degree 2. Therefore, K must contain a transitive constituent of degree 2 entirely different from the one occurring in the subgroup composed of all the substitutions leaving a letter of the transitive constituent of degree 4 fixed. Since G_1 cannot contain a transitive constituent of degree 2 and the constituent of degree 2 in K may be transformed into itself by one-fourth and also by one-half of the substitutions of G_1, G_1 must contain a transitive constituent of degree 4 or of degree 8.

THEOREM IX. *If G_1 contains the alternating group of degree 4 as a transitive constituent and the order of G_1 exceeds 12, G_1 must contain a transitive constituent of degree 12 whose order is at least 36.*

When the order of G_1 exceeds 12, there may be an intransitive constituent R of degree r and order 12 formed from the simple isomorphism of certain groups with the considered alternating group of degree 4. The subgroup H will then be of degree $n-r-1$. If such an isomorphism does not occur in G_1, H is of degree $n-5$. In either case K must contain a transitive constituent of degree 3 entirely different from any such constituent occurring in the part of K in the alternating group of degree 4 or in R when it occurs. This constituent of degree 3 is the cyclic group of degree 3. If a constituent of degree 3 occurs in G_1, it must be the symmetric group of degree 3 and occur wholly in H, for G_1 could not contain a constituent of degree 3 and order 3 and

* Miller, *Bull. Amer. Math. Soc.*, 6 (1899–1900), p. 104.

no isomorphism exists between the symmetric group of degree 3 and the alternating group of degree 4. Therefore, the cyclic constituent of degree 3 cannot be a part of any symmetric constituent of degree 3 that may occur in G_1. Since the additional cyclic group of degree 3 in K must be in H, is transformed into itself by only one-fourth the substitutions of G_1 and could not occur as a transitive constituent of G_1, G_1 must contain a transitive constituent of degree 12. The order of the subgroup of this transitive constituent of degree 12 occurring in H cannot be less than 3. Therefore, G_1 must contain a transitive constituent of degree 12 whose order is at least 36.

THEOREM X. *If G_1 contains the symmetric group of degree 4 as a transitive constituent and no transitive constituent of degree less than 4, then if the order of G_1 is 24, it must contain another transitive constituent of degree 4, 6, 8, 12 or 24. If the order of G_1 is greater than 24, and H is of degree $n-5$, G_1 must contain a transitive constituent of degree 12.*

The symmetric group of degree 24 can be represented only on 4, 6, 8, 12, or 24 letters.* Therefore, if the order of G_1 is 24 and no transitive constituent of degree less than 4 occurs, the other transitive constituents, when such occur, can be only of degrees 4, 6, 8, 12, or 24. If the order of G_1 is greater than 24 and H is of degree $n-5$, from Theorem IV it is evident that a transitive constituent of degree 12 must occur.

PART II.

§ 1. *Simply Transitive Primitive Groups of Degree 20.*

We first observe that the order of a transitive constituent of G_1 and, therefore, the order of G_1 could not be divisible by any one of the primes 5, 7, 11, 13, and 17. If such were the case, G_1 would contain substitutions of these orders† and of degrees $5q$, $7q$, $11q$, $13q$, and $17q$, where q represents the number of cycles in the substitutions of the given orders. When $q=1$, the degree of the primitive group generated could not exceed $p+2$,‡ a degree which is less than 20 for each of the primes considered. When $q=2$, a possibility that occurs for the primes 5 and 7, or when $q=3$, as is possible for the prime 5, the limit for the degree is again less than 20§ for each of the primitive groups that

* Dyck, *Math. Ann.*, XXII (1883), pp. 84–86, pp. 105–106.

† Cauchy, *Exercises d' analyse*, 3 (1844), p. 250.

‡ C. Jordan, *Bulletin de la Société Mathématique de France*, Vol. I (1873), pp. 40–71 and pp. 175–221.

§ Manning, *Trans. Amer. Math. Soc.*, Vol. X (1909), p. 252.

might be generated. Then since the order of a transitive constituent of G_1 is not divisible by these primes, G_1 could contain no transitive constituent whose degree is divisible by any one of them.* In other words, we see at once that transitive constituents of degrees 17, 15, 13, 11, 10, 7, and 5 cannot occur in G_1. It is also known that G_1 cannot contain a transitive constituent of degree 2. These results together with Theorem VII show immediately that the largest possible degree for any transitive constituent of G_1 is 9.

We shall first consider all the possible groups when the maximum degree of a transitive constituent of G_1 is 9; then those when this maximum degree is 8, etc.

A transitive constituent of degree 9 cannot occur in G_1. The possible systems of intransitivity which require special consideration are 9, 6, 4 and 9, 4, 3, 3.

When the systems of intransitivity of G_1 are 9, 6, 4 and G_1 is considered with respect to the isomorphism which exists between its transitive constituent of degree 4 and the group formed from the transitive constituents of degrees 9 and 6, it is seen that the order of G_1 must exceed the order of the transitive constituent of degree 4. The subgroup H corresponding to identity in the transitive constituent of degree 4 will be of degree $n - 5$, for the class of G_1 must be at least 12.† Then according to Theorems IX and X, G_1 must contain a transitive constituent of degree 12. This is impossible with the systems of intransitivity 9, 6, 4; therefore, such systems cannot occur in G_1.

When G_1 contains a transitive constituent of prime degree p, the order of G_1 is not divisible by p^2.‡ It is evident, therefore, that the systems of intransitivity 9, 4, 3, 3 cannot occur in G_1.

If G_1 contains a transitive constituent of degree 8, but no transitive constituent of higher degree, its possible systems of intransitivity are 8, 8, 3 and 8, 4, 4, 3. Since only one constituent of degree 3 occurs in either case, Theorem VII shows at once that a transitive constituent of degree 8 cannot occur in G_1.

When G_1 contains a transitive constituent of degree 6, but none of larger degree, the systems of intransitivity of G_1 may be 6, 6, 4, 3 or 6, 4, 3, 3, 3.

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* Compare Burnside, *Theory of Groups*, p. 140.

† Jordan, C. R., Vol. LXXIII (1871), p. 853; Manning, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVIII (1906), p. 226.

‡ Rietz, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVI (1904), p. 6.

It is evident that the transitive constituent of degree 4 must be either the symmetric or the alternating group of this degree and that the constituent of degree 3 must be the symmetric group of degree 3. Since the class of G_1 must be at least 12, in both cases G_1 may be considered as formed from a simple isomorphism between the intransitive group of degree 9 and the intransitive group of degree 10.

If the systems of intransitivity of G_1 are 6, 4, 3, 3, 3, the intransitive group of degree 9 contains only one Sylow subgroup of order 3^a while the intransitive group of degree 10 contains more than one such subgroup. The simple isomorphism, therefore, cannot be established.

If the systems of intransitivity of G_1 are 6, 4, 6, 3, the constituent of degree 6 in the intransitive group of degree 9 must be of order 48, 24, or 12, for the intransitive group of degree 10 has more than one Sylow subgroup of order 3^a. The order of the intransitive group of degree 9 cannot be divisible by 9, for the order of G_1 cannot be divisible by 3^2. The order of G_1 is, therefore, equal to the order of the transitive constituent of degree 6 in the intransitive group of degree 9 or is 48, 24, or 12. It is at once evident that the order of G_1 cannot be 12 since the constituent of degree 3 is the symmetric group of degree 3. The order of G_1 also cannot be 24* and on account of class a G_1 of order 48 cannot be constructed. Therefore, G_1 can contain no transitive constituent of degree 6.

A transitive constituent of degree 4 cannot occur in G_1. The possible systems of intransitivity are 4, 3, 3, 3, 3, 3 and 4, 4, 4, 4, 3. If the systems of intransitivity are 4, 3, 3, 3, 3, 3, G_1 may be considered as formed from a simple isomorphism between an intransitive group of degree 10 and an intransitive group of degree 9. The intransitive group of degree 9 will contain only one Sylow subgroup of order 3^a while the intransitive group of degree 10 contains more than one such subgroup. The considered isomorphism is then impossible. Theorem VI shows at once that the systems of intransitivity 4, 4, 4, 4, 3 cannot occur in G_1.

Since a constituent of degree 2 cannot occur in G_1 and 19 is not divisible by 3, a transitive constituent of degree 3 cannot occur in G_1.

This completes the consideration of the simply transitive primitive groups of degree 20 and shows that no such groups exist.

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*Miller, *Proc. London Math. Soc.*, Vol. XXVIII, p. 534.

§ 2. *Multiply Transitive Groups of Degree* 20.

A multiply transitive group of degree 20 must have a transitive group of degree 19 as the maximal subgroup composed of all the substitutions leaving a letter fixed. According to the determination made by Jordan, besides the alternating and symmetric groups of degree 19, there are only six transitive groups of this degree, the metacyclic group and its subgroups.* The orders of the latter groups are 19, 2 . 19, 3 . 19, 6 . 19, 9 . 19, and 18 . 19.

Since a transitive group of degree n and order $n(n-1)$ whose operations other than identity displace all, or all but one, of the symbols, can exist only when n is the power of a prime,† no primitive group of degree 20 can have the group of degree and order 19 as a maximal subgroup.

If a primitive group of degree 20 and order 20 . 19 . 2 existed, it would contain both positive and negative substitutions. The subgroup of its positive substitutions would be a group of degree 20 and order 20 . 19. Since such a group has been shown not to exist, the required group of order 20 . 19 . 2 cannot exist.

A primitive group of degree 20 and order 20 . 19 . 3 cannot exist. Such a group must contain

380 substitutions of degree 18, order 3.
360 " " " 19, " 19.
399 " " " 20.

The substitutions of order 5 contained in the group must be of degree 20 having four systems of intransitivity. A subgroup of order 5 cannot be transformed into itself by substitutions of degree 19 nor by substitutions of degree 18 and order 3. Therefore, the number of subgroups of order 5 must be a multiple of 57. From Sylow's theorem the number of subgroups of order 5 must be of the form $5K + 1$; there must then be at least 171 subgroups of order 5. Since it is impossible to form 171 subgroups of order 5 from 399 substitutions of degree 20, a primitive group of order 20 . 19 . 3 cannot exist.

A primitive group of degree 20 and order 20 . 19 . 3 . 2 cannot exist. Such a group would be both positive and negative and contain a positive subgroup of order 20 . 19 . 3. Since the positive subgroup does not exist, the group considered cannot exist.

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* Jordan, C. R., Vol. LXXIX, p. 150.

† Jordan, *Liouville's Journal*, 2me sér., Vol. XVII, p. 355.

Mathieu* has proved that there is at least one triply-transitive group of degree $p+1$ and order $p(p^2-1)$ and one doubly-transitive group of degree $p+1$ and order $\frac{p}{2}(p^2-1)$, p being any prime number. The latter group is composed of the positive substitutions of the former. Frobenius has further proved that there is only one group of each order except for the case $p=7$ when there are two groups of order $\frac{p}{2}(p^2-1)$.† For degree 20 there is then only one primitive group of order $20 \cdot 19 \cdot 9$ and only one primitive group of order $20 \cdot 19 \cdot 18$.

This completes the determination of the primitive groups of degree 20. Besides the symmetric and alternating groups of this degree, there are only two primitive groups and these are the well-known Mathieu groups of orders $p(p^2-1)$ and $\frac{p}{2}(p^2-1)$.

Part III.

Certain Primitive Groups whose Maximal Subgroup Contains a Transitive Constituent of Prime Order.

In the following discussion G will be considered as denoting any primitive group of degree n whose maximal subgroup contains a transitive constituent of order p, p an odd prime. We first observe that G_1 must be of order p and G of class $n-1$.‡ Then according to Frobenius, the degree of G is of form q^a, q a prime, and G contains a characteristic abelian subgroup H of degree q^a and type $(1, 1, \ldots, 1)$.§

Theorem 1. *If the number of transitive constituents of prime order p of G_1 is 2^α, $\alpha > 0$, the degree of G is of form $2^\alpha p + 1 = q^a$, q an odd prime. If a is even, a primitive group G can exist only when $a \equiv 0$, mod. 4, or when $a = 2$ and q belongs to exponent 2, mod. p, and there is only one group G for each set of values of p and α for which the equation $2^\alpha p + 1 = q^2$ is satisfied. If a is odd, the group G can exist only when a is 1 or a prime. When $a = 1$, G is a subgroup of the metacyclic group. In case $a \neq 1$ and odd, α can be only 1 or some power of 2, as 2^γ. When $a \neq 1$ and $\alpha = 1$, there is one and only one primitive group G of degree 3^a for*

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* Mathieu, *Journal de Mathématiques*, Vol. V (1860), p. 37.

† Frobenius, *Berliner Sitzungsberichte* (1902), pp. 455–459.

‡ Miller, *Proc. London Math. Soc.*, Vol. XXVIII, p. 536.

§ Frobenius, *Berliner Sitzungsberichte* (1902), pp. 455–59.

each prime p for which $2p+1=3^a$. When $a \neq 1$ and $a=2^\gamma$, one and only one primitive group G exists for each set of values of p and γ for which $2^{2\gamma}p+1=q^a$.

Since G_1 contains 2^α transitive constituents of order p, the degree of G_1 is $2^\alpha p$ and the degree of G is $2^\alpha p+1$, an odd number. But the degree of G must be the power of a prime; therefore,

$$2^\alpha p+1=q^a, \tag{I}$$

q an odd prime. It will now be shown that Equation (I) can exist, if a is even, only when $a=2$ or $a\equiv 0$, mod. 4. We have, when a is even,

$$2^\alpha p=q^a-1=(q-1)(q+1)(q^{a-2}+q^{a-4}+\ldots.+1)=(q-1)(q+1)K.$$

If $a>2$ and $\not\equiv 0$, mod. 4, K must be odd and, therefore, $K=p$ and $2^\alpha=q^2-1$. The latter equation is satisfied only when $\alpha=3$ and $q=3$. Equation (I) then becomes $8p+1=3^a$, an equation which does not exist under the given conditions. Therefore, Equation (I) cannot exist when a is even, greater than 2 and $\not\equiv 0$, mod. 4.

If $2^\alpha p+1=q^2$, q must belong to exponent 2, mod. p, or exponent 1, mod. p. Suppose q belongs to exponent 2, mod. p. The order of the group of isomorphisms of the abelian group of order q^2 is divisible by p while q^2-1 is not divisible by p^x, $x>1$. Then for each set of values of p and α only one primitive group G can exist for $a=2$.* The primitive group of order 75 and degree 25 is an example of this type of group. Here $\alpha=3$, $p=3$, and $q=5$.

When q belongs to exponent 1, mod. p, the number of subgroups of order q is not divisible by p. The subgroup G_1 is then not maximal and G is not primitive.

Now consider that a is odd and first assume that $a=mn$, m and n odd primes. Then

$$2^\alpha p=q^{mn}-1=(q-1)(q^{m-1}+q^{m-2}+\ldots.+1)[(q^m)^{n-1}+(q^m)^{n-2}+\ldots.+1],$$

which is at once seen to be impossible unless either m or n is 1 or both m and n are 1. Thus a must be 1 or a prime.

When $a=1$, we have $2^\alpha p+1=q$ and G is a subgroup of the metacyclic group. There is only one such group for each set of values of p and α for which the equation $2^\alpha p+1=q$ is satisfied.

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*Rietz, AMERICAN JOURNAL, Vol. XXVI (1904), p. 12.

When a is an odd prime, $2^\alpha = q - 1$ or $q = 2^\alpha + 1$. Assume that α is odd. Then we have

$$q = 2^\alpha + 1 = (2 + 1)(2^{\alpha-1} - 2^{\alpha-2} + \ldots + 1).$$

Therefore $\alpha = 1$ when odd and $q = 3$. Assume that α is even and that $\alpha = Kl$, where K consists of powers of 2 and l consists of odd factors only. Then

$$q = 2^\alpha + 1 = 2^{kl} + 1 = (2^k + 1)[(2^k)^{l-1} - \ldots + 1].$$

Therefore, $l = 1$ and $\alpha = 2^\gamma$ when α is even.

When a is an odd prime and $\alpha = 1$, we have $2p + 1 = 3^a$ and G is of class $2p$. Since all the groups of class $2p$ have been determined,* it is known at once that this condition gives a primitive group G of degree 3^a for each prime p for which $2p + 1 = 3^a$.

When a is an odd prime and $\alpha = 2^\gamma$, we have

$$2^{2^\gamma}p = q^a - 1 = (q - 1)(q^{a-1} + q^{a-2} + \ldots + 1)$$

or

$$q - 1 = 2^{2^\gamma} \text{ and } p = q^{a-1} + q^{a-2} + \ldots + 1.$$

Therefore, if the equation $2^{2^\gamma}p + 1 = q^a$ is satisfied, a is the exponent to which q belongs, mod. p. The order of the group of isomorphisms of the abelian group of order q^a is divisible by p and $q^a - 1$ is not divisible by p^x, $x > 1$. Then there is one and only one group G for each set of values of p and α for which $2^{2^\gamma} + 1 = q^a$.† The primitive group of degree 125 and order $125 \cdot 31$ is an example of the above type of group.

THEOREM II. *If the number of transitive constituents of prime order p is an odd prime r, then a primitive group G may only exist when its degree is of form $rp + 1 = 2^a$, where a is 4, an odd prime, or the square of an odd prime. There is only one primitive group G when $a = 4$. If $a = \beta^2$, β an odd prime, for each set of values of r and p for which the equation $rp + 1 = 2^{\beta^2}$ exists only one primitive group G occurs. If a is an odd prime β, for each set of values of r and p for which the equations $rp + 1 = 2^\beta$ exists, $r \neq p$, two primitive groups always occur.*

Since G_1 contains r transitive constituents of prime order p, the degree of G_1 will be represented by rp and the degree of G by the expression $rp + 1$. But $rp + 1$ is even and the degree of G must be the power of a prime; therefore,

$$rp + 1 = 2^a. \qquad \text{(I)}$$

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* Manning, *Trans. Amer. Math. Soc.*, Vol. IV, pp. 351–57.

† Rietz, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVI (1904), p. 12.

It is first assumed that a is even. If $a \equiv 2$, mod. 4, we have

$$rp = 2^a - 1 = 2^{2\beta} - 1 = (2^{\beta-1} + \ldots + 1) \cdot 3 \cdot (2^{\beta-1} - 2^{\beta-2} + \ldots + 1);$$

that is, rp would have at least three factors unless $\beta = 1$ when it has only one. Both conditions are impossible.

If $a \equiv 0$, mod. 4, we have

$$rp = 2^a - 1 = 2^{4\beta} - 1 = (2^\beta - 1)(2^\beta + 1)(2^{2\beta} + 1),$$

which has three factors unless $\beta = 1$. When $\beta = 1$ and $p = 5$, a primitive group of degree 16 and order 80 exists. The group is not primitive, if $p = 3$.

Now assume that a is odd. The exponent a may be the product of like or unlike odd primes or a prime.

If $a = \beta\gamma$, β an odd prime and γ one or more old primes, $\beta < \gamma$,

$$rq = 2^{\beta\gamma} - 1 = (2^\beta - 1)[(2^\beta)^{\gamma-1} + (2^\beta)^{\gamma-2} + \ldots + 2^\beta + 1] = L \cdot M, \quad \text{(II)}$$
$$rq = (2^\gamma)^\beta - 1 = (2^\gamma - 1)[(2^\gamma)^{\beta-1} + (2^\gamma)^{\beta-2} + \ldots + 2^\gamma + 1] = S \cdot T. \quad \text{(III)}$$

If L is not a prime, (II) contains too many prime factors. If L is a prime, it must divide either S or T and rq is again equal to more than two prime factors, which is impossible. Therefore, $a \neq \beta\gamma$. If $\beta = \gamma$ or $\gamma = 1$, that is, $a = \beta^2$ or $a = \beta$, it is evident that (I) may exist. Then if a is odd, it must be the square of a prime or a prime.

Assume that $\beta = \gamma$, that is, $a = \beta^2$. Then

$$rp = 2^{\beta^2} - 1 = (2^\beta - 1)[(2^\beta)^{\beta-1} + (2^\beta)^{\beta-2} + \ldots + 1].$$

If $r < p$ and the equation $rp + 1 = 2^{\beta^2}$ exists,

$$r = 2^\beta - 1 \quad \text{and} \quad p = (2^\beta)^{\beta-1} + \ldots + 2^\beta + 1.$$

In this case 2 belongs to exponent β, mod. r, and exponent β^2, mod. p.

When β^2 is the exponent to which 2 belongs, mod. p, the group G whose G_1 contains r systems of order p is primitive. Only one such group can exist for each prime p. The order of the holomorph L of the abelian group of order 2^{β^2} is $(2^{\beta^2} - 1)(2^{\beta^2} - 2) \ldots (2^{\beta^2} - 2^{\beta^2-1}) \cdot 2^{\beta^2}$. This is divisible by p, but not by p^2. Then all the subgroups of order p in L are conjugate and, therefore, all subgroups of order $2^{\beta^2}p$ are also conjugate. There is then for each set of values of r and p, $r < p$, only the one primitive group whose G_1 contains systems of order p. Such a primitive group occurs when $r = 7$, $p = 73$, and $\beta^2 = 9$.

If $p < r$, $p = 2^\beta - 1$ and $r = (2^\beta)^{\beta-1} + \ldots + 2^\beta + 1$. Then 2 belongs to exponent β, mod. p. In this case there is no primitive group G containing r systems of order p in its G_1. The number of subgroups of order 2^β occurring in the abelian group H of order 2^{β^2} is

$$\frac{(2^{\beta^2}-1)(2^{\beta^2-1}-1)\ldots(2^{\beta^2-(\beta-1)}-1)}{(2-1)(2^2-1)\ldots(2^\beta-1)} = N.$$

But N is not divisible by p. Then some subgroups of H of order 2^β are invariant under p or G_1 is not maximal and a primitive group G does not exist. Therefore, for each set of values of r and p for which the equation $rp + 1 = 2^{\beta^2}$ exists, only one primitive group G occurs.

Now assume that $\gamma = 1$, that is, $a = \beta$. Then $rp + 1 = 2^\beta$, β an odd prime, and β is the exponent to which 2 belongs, mod. r, and also mod. p. The order of the holomorph of the abelian group H of order 2^β is divisible by p and r, but by no higher powers of these primes. Then it may be shown as in the previous cases that there exists a primitive group G containing r systems of order p. However, if the equation $rp + 1 = 2^\beta$ exists, this existence is not destroyed if r and p exchange values. Suppose this change is effected. Then since 2 belongs to exponent β, mod. r, it will also belong to exponent β, mod. p, where p has the value previously held by r. For this latter condition by processes similar to those already used, it may be shown that another primitive group G of degree 2^β exists. Therefore, for each set of values of r and p, $r \neq p$, for which the equation $rp + 1 = 2^\beta$ exists, two primitive groups always occur. Two such primitive groups exist, for instance, for the relation $23 \cdot 89 = 2^{11} - 1$. Both are of degree 2^{11}, but the G_1 of one contains 23 transitive constituents of order 89, while the G_1 of the other contains 89 transitive constituents of order 23.

THEOREM III. *If the number of transitive constituents of prime order p occurring in G_1 is $2r$, r an odd prime, then a primitive group G can only exist when the degree is of form $2rp + 1 = q^a$, where q is a prime of form $4n + 3$ and a is 1, an odd prime, or the square of an odd prime. If $a = 1$, the group G is a subgroup of the metacyclic group. If $a = \beta^2$, β an odd prime, the degree of G is of form 3^{β^2} and for each set of values of r and p for which the equation $2rp + 1 = 3^{\beta^2}$ exists, only one primitive group G occurs. If $a = \beta$ and $q \neq 3$ for each set of values of r and p for which the equation $2rp + 1 = q^\beta$ exists, only one primitive group G is possible, while if $a = \beta$ and $q = 3$, $r \neq p$, two groups are possible.*

Since the number of transitive constituents of prime order p in G_1 is $2r$, the degree of G may be represented by $2rp+1$. But the degree of G must also be the power of a prime; therefore, we have

$$2rp+1=q^a, \tag{I}$$

q an odd prime.

The left-hand member of this expression is always congruent to 3, mod. 4, whatever values p and r may have. Therefore, q must be a prime of form $4n+3$ and a an odd number.

Consider $2rp=q^{\beta\gamma}-1$, where $a=\beta\gamma$, β being an odd prime and γ one or more odd primes, $\beta<\gamma$,

$$\begin{aligned} 2rp=q^{\beta\gamma}-1&=(q-1)(q^{\beta-1}+\ldots.+1)[(q^\beta)^{\gamma-1}+(q^\beta)^{\gamma-2}+\ldots.+1]\\ &=(q-1)L\,.\,R, \end{aligned} \tag{II}$$

$$\begin{aligned} 2rp=q^{\beta\gamma}-1&=(q-1)(q^{\gamma-1}+\ldots.+1)[(q^\gamma)^{\beta-1}+(q^\gamma)^{\beta-2}+\ldots.+1]\\ &=(q-1)M\,.\,N. \end{aligned} \tag{III}$$

It is evident that Equations (II) and (III) can exist only if $q=3$. Then if L in (II) is not a prime, $q^{\beta\gamma}-1$ has too many prime factors. If L is a prime, it must divide either M or N since it is different from both and a number can be resolved into its prime factors in only one way. Again $q^{\beta\gamma}-1$ has more than two odd prime factors. Therefore, (I) cannot exist when $a=\beta\gamma$.

An inspection of Equations (II) and (III) shows that the existence of the equation $2rp=q^a-1$ may be considered possible if $\beta=\gamma=1$, $\beta=\gamma$, or $\gamma=1$, that is, $a=1$, $a=\beta^2$, or $a=\beta$.

When $a=1$, $2rp+1=q$ and G is a subgroup of the metacyclic group.

When $\gamma=\beta$, we have

$$2rp=q^{\beta^2}-1=(q-1)(q^{\beta-1}+q^{\beta-2}+\ldots.+1)[(q^\beta)^{\beta-1}+\ldots.+1].$$

In order that $q^{\beta^2}-1$ may not contain too many prime factors, q must be 3. Assume $r<p$. Then

$$r=3^{\beta-1}+3^{\beta-2}+\ldots.+1$$

and

$$p=(3^\beta)^{\beta-1}+(3^\beta)^{\beta-2}+\ldots.+3^\beta+1.$$

These values for r and p show that 3 belongs to exponent β^2, mod. p, and exponent β, mod. r. Since β^2 is the exponent to which 3 belongs, mod. p, a primitive group G exists. Also since the order of the group of isomorphisms of the abelian group H of order 3^{β^2} is divisible by p and $3^{\beta^2}-1$ is not divisible

by p^x, $x > 1$, only one such group G exists for each set of values of r and p for which the equation $2rp + 1 = 3^{\beta^2}$ is satisfied.

Assume $r > p$. Then $p = 3^{\beta-1} + 3^{\beta-2} + \ldots + 1$ and 3 belongs to exponent β, mod. p. In this case no primitive group G exists. For the number of subgroups of order 3^β in the abelian group H of order 3 is not divisible by p. Then G_1 is not maximal and there can be no primitive group G for $r > p$.

If $p = 757$, $r = 13$, and $\beta = 3$, a group G illustrating the case when $r < p$ occurs.

When $\gamma = 1$ and $q \neq 3$, we have $a = \beta$ and

$$2rp = q^\beta - 1 = (q-1)(q^{\beta-1} + q^{\beta-2} + \ldots + q + 1).$$

If $r < p$, $2r = q - 1$ and $p = q^{\beta-1} + q^{\beta-2} + \ldots + 1$. Then q belongs to exponent 1, mod. r, and exponent β, mod. p. As in the previous cases, it may be shown that one and only one primitive group G occurs for each set of values of r and p for which the equation $2rp + 1 = q^\beta$ exists. As an illustration, this equation is satisfied when $\beta = 5$, $q = 7$, $r = 3$, and $p = 2801$ and a primitive group G exists whose G_1 contains 6 transitive constituents of order 2801.

If $r > p$, there is no primitive group G existing. Then $p = \frac{q-1}{2}$ and the number of subgroups of order q in the abelian group H of order q^β is not divisible by p and G_1 is not maximal.

When $q = 3$ and $\gamma = 1$, $a = \beta$ and

$$2rp = 3^\beta - 1 = 2(3^{\beta-1} + 3^{\beta-1} + \ldots + 3 + 1)$$

or

$$rp = 3^{\beta-1} + 3^{\beta-2} + \ldots + 3 + 1 = S \,.\, T.$$

Assume $S \neq T$. If $p = T$, and again when $p = S$, 3 belongs to exponent β, mod. p. Also for both values of p the order of the group of isomorphisms of the abelian group H of order 3^β is divisible by p and $3^\beta - 1$ is not divisible by p^2. Then for each of these values of p, one and only one primitive group G exists. Therefore, if $a = \beta$ and $q = 3$ for each set of values of r and p, $r \neq p$, for which the equation $2rp + 1 = 3^\beta$ is satisfied, two primitive groups exist. The identity $2 \,.\, 23 \,.\, 3851 = 3^{11} - 1$ gives one set of values of r and p for which two primitive groups exist.

THEOREM IV. *If the number of transitive constituents of prime order p of G_1 is rq, r and q odd primes, the degree of G is of form $rqp + 1 = 2^\alpha$. Then if 2 belongs to an even exponent with respect to any one of the primes r, q, and p,*

a primitive group G can exist only when its degree is 2^a or 2^{2b}, b an odd prime. If $a = 8$, only one primitive group G exists. If $a = 2b$, b an odd prime, for each set of values of r, q, and p for which the equation $rqp + 1 = 2^{2b}$ is satisfied, only one primitive group G is possible.

Since G_1 contains rq transitive constituents of order p, the degree of G_1 is rqp and the degree of G is $rqp + 1$, an even number. But the degree of G is also the power of a prime; therefore,

$$rqp + 1 = 2^a.$$

If 2 belongs to an even exponent with respect to any one of the primes r, q, and p, a must be even. Consider $a \equiv 2$, mod. 4, and assume $a = 2bc$, b an odd prime and c the other odd factors, if such exist. Then

$$rqp = 2^{2bc} - 1 = (2^b - 1)[(2^b)^{c-1} + (2^b)^{c-2} + \ldots + 1]$$
$$(2^b + 1)[(2^b)^{c-1} - \ldots + 1].$$

This equation is evidently impossible unless $c = 1$. It then becomes

$$rqp = (2^b - 1) \cdot 3 \cdot (2^{b-1} - 2^{b-2} + \ldots + 1).$$

Therefore, when $a \equiv 2$, mod. 4, $a = 2b$, b an odd prime. Let

$$p = (2^{b-1} - 2^{b-2} + \ldots + 1).$$

Then 2 belongs to exponent $2b$, mod. p, and by considering the holomorph of the abelian group of order 2^{2b}, it may be shown by methods similar to those used in Theorem II that for this value of p only one primitive group G exists. If p were equal to either one of the other factors of $2^{2b} - 1$, no primitive group G could exist. Therefore, for each set of values of r, q, and p for which the equation $rqp + 1 = 2^a$ is satisfied, only one primitive group G exists. Such a primitive group occurs when $rq = 9$ and $p = 7$, also when $rq = 93$ and $p = 11$.

Now consider that $a \equiv 0$, mod. 4, and assume $a = 4c$. Then we must have

$$rqp = 2^{4c} - 1 = (2^c - 1)(2^c + 1)(2^{2c} + 1).$$

If $c = 1$, the above equation is impossible; and if $c > 1$ and odd, it is also impossible. Therefore, c must be even, or $a = 8d$ and

$$rqp = 2^{8d} - 1 = (2^d - 1)(2^d + 1)(2^{2d} + 1)(2^{4d} + 1).$$

In order that the above equation may exist, d must be 1. Therefore, if $a \equiv 0$, mod. 4, $a = 8$. When $a = 8$, $rq = 15$ and $p = 17$ and only one primitive group G exists.

UNIVERSITY OF ILLINOIS, *June*, 1910.

On the Arithmetical Theory of Pencils of Binary Quadratic Forms.

By D. N. Lehmer.

1. Given two binary quadratic forms:

$$A = ax^2 + 2bxy + cy^2,$$
$$B = a'x^2 + 2b'xy + c'y^2,$$

where the coefficients are integers, positive or negative or zero, the totality of forms obtained by giving to α and β all positive or negative or zero integer values in the expression $\alpha A + \beta B$ will be called a *pencil of forms* and will be denoted by (A, B). The forms A and B will be called the *base forms* of the pencil.

2. If $\overline{A} = \alpha A + \beta B$ and $\overline{B} = \gamma A + \delta B$ are two forms of the pencil, then the pencil $(\overline{A}, \overline{B})$ is said to be *contained* in the pencil (A, B). It is easily seen that any form that appears in $(\overline{A}, \overline{B})$ appears also in (A, B). The converse is not necessarily true. If two pencils are such that each is contained in the other, they are said to be *equivalent.*

3. Theorem. *If two pencils are equivalent and their bases are related by the equations* $\overline{A} = \alpha A + \beta B$, $\overline{B} = \gamma A + \delta B$, *then* $\alpha\delta - \beta\gamma = \pm 1$.

For solving the two equations for A and B, we have

$$A = \frac{\delta}{\varepsilon}\overline{A} - \frac{\beta}{\varepsilon}\overline{B},$$
$$B = -\frac{\gamma}{\varepsilon}\overline{A} + \frac{\alpha}{\varepsilon}\overline{B},$$

where $\varepsilon = \alpha\delta - \beta\gamma$. It is necessary therefore that $\frac{\delta}{\varepsilon}$, $\frac{\beta}{\varepsilon}$, $\frac{\gamma}{\varepsilon}$ and $\frac{\alpha}{\varepsilon}$ be integers, and therefore that $\frac{\alpha\delta}{\varepsilon^2} - \frac{\beta\gamma}{\varepsilon^2}$ should be. But this expression is equal to $\frac{1}{\varepsilon}$, whence $\varepsilon = \pm 1$. The equivalence will be called proper or improper according as $\varepsilon = +1$ or $\varepsilon = -1$.

4. In the study of pencils two kinds of transformations will be used. The first, as above, connects the base forms of the two pencils. The pencil (A, B) will be said to be transformed into the pencil $(\overline{A}, \overline{B})$ by the *α-substitution* $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$. If, in the second place, the transformation $x = \lambda x' + \mu y'$, $y = \nu x' + \rho y'$ be applied to the *variables* in the two base forms A and B, two other forms are produced which may be taken as base forms of a new pencil $(\overline{A}', \overline{B}')$. The pencil will then be said to be transformed into the pencil $(\overline{A}', \overline{B}')$ by the *λ-substitution* $\begin{pmatrix}\lambda & \mu\\ \nu & \rho\end{pmatrix}$. It will be found that in this case the two pencils are not necessarily equivalent, even when $\lambda\rho - \mu\nu = \pm 1$. If $\lambda\rho - \mu\nu = \pm 1$, the pencils will be called *similar*, and the similarity will be called proper or improper according as $\lambda\rho - \mu\nu = +1$ or -1.

5. We shall denote by D and D' the determinants of A and B respectively, and the invariant $2bb' - ac' - a'c$ by F. The binary quadratic form $Dx^2 + Fxy + D'y^2$, which is of fundamental importance in the theory, we shall denote by H, and call the harmonic form of the pencil. The form $(ab' - a'b)x^2 + (ac' - a'c)xy + (bc' - b'c)y^2$, which except for a constant factor is the Jacobian of the two forms, will be denoted by J and called the Jacobian form of the pencil. The resultant of A and B, which is also the determinant of H and J, will be denoted by R. It has the value

$$R = F^2 - 4DD' = (ac' - a'c)^2 - 4(ab' - a'b)(bc' - b'c).$$

6. THEOREM. *Two equivalent pencils have the same resultant.*

If $\overline{A} = \alpha A + \beta B$, $\overline{B} = \gamma A + \delta B$, where $\alpha\delta - \beta\gamma = \pm 1$, we have

$$\overline{A} = (\alpha a + \beta a')x^2 + 2(\alpha b + \beta b')xy + (\alpha c + \beta c')y^2,$$
$$\overline{B} = (\gamma a + \delta a')x^2 + 2(\gamma b + \delta b')xy + (\gamma c + \delta c')y^2.$$

If now we compute the determinants $\overline{D}$ and $\overline{D}'$, of $\overline{A}$ and $\overline{B}$, we find:

$$\overline{D} = \alpha^2 D + \alpha\beta F + \beta^2 D',$$
$$\overline{D}' = \gamma^2 D + \gamma\delta F + \delta^2 D'.$$

Also

$$\overline{F} = 2\alpha\gamma D + F(\alpha\delta + \beta\gamma) + 2\beta\delta D'.$$

From these equations $\overline{R} = F^2 - 4DD' = (\alpha\delta - \beta\gamma)^2 R = R$.

7. From the form of the expressions for $\overline{D}$, $\overline{D}'$ and $\overline{F}$ in the preceding paragraph, we infer immediately the following theorem:

THEOREM. *If the pencil* $(\bar{A}, \bar{B})$ *is contained in the pencil* (A, B) *and the bases are related by the equations* $\bar{A} = \alpha A + \beta B$, $\bar{B} = \gamma A + \delta B$, *then the form* H *of the pencil* (A, B) *goes into the form* $\bar{H}$ *of the pencil* $(\bar{A}, \bar{B})$ *by the substitution* $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$.

8. As a particular case of the last theorem when $\alpha\delta - \beta\gamma = \pm 1$, we have the theorem:

THEOREM. *Two equivalent pencils have equivalent forms* H, *and the equivalence of the forms is proper or improper according as the equivalence of the pencils is proper or improper.*

9. THEOREM. *If* J *and* $\bar{J}$ *are the Jacobian forms of two equivalent pencils, then* $J = \pm \bar{J}$, *the upper sign or the lower occurring according as the equivalence of the pencils is proper or improper.*

Let the pencils be as given in paragraph 6. The form $\bar{J}$ belonging to the pencil $(\bar{A}, \bar{B})$ is found to be

$$\begin{aligned} &[(\alpha a + \beta a')(\gamma b + \delta b') - (\alpha b + \beta b')(\gamma a + \delta a')]\, x^2 \\ + &[(\alpha a + \beta a')(\gamma c + \delta c') - (\alpha c + \beta c')(\gamma a + \delta a')]\, xy \\ + &[(\alpha b + \beta b')(\gamma c + \delta c') - (\gamma b + \delta b')(\alpha c + \beta c')]\, y^2. \end{aligned}$$

This easily reduces to

$$(\alpha\delta - \beta\gamma)\,[(ab' - a'b)x^2 + (ac' - a'c)xy + (bc' - b'c)y^2],$$

whence $\bar{J} = (\alpha\delta - \beta\gamma) J$.

10. THEOREM. *Two similar pencils have the same forms* H.

Let the two pencils be (A, B) and $(\bar{A}', \bar{B}')$, where $A = (a\,b\,c)$, $B = (a'\,b'\,c')$, $\bar{A}' = (\bar{a}\,\bar{b}\,\bar{c})$, $\bar{B}' = (\bar{a}'\,\bar{b}'\,\bar{c}')$ Then if the λ-substitution which connects the two pencils be $\begin{pmatrix} \lambda & \mu \\ \nu & \rho \end{pmatrix}$, we have the following equations:

$$\begin{aligned} \bar{a} &= a\lambda^2 + 2b\lambda\nu + c\nu^2, \\ \bar{b} &= a\lambda\mu + b(\lambda\rho + \mu\nu) + c\nu\rho, \\ \bar{c} &= a\mu^2 + 2b\mu\rho + c\rho^2, \\ \bar{a}' &= a'\lambda^2 + 2b'\lambda\nu + c'\nu^2, \\ \bar{b}' &= a'\lambda\mu + b'(\lambda\rho + \mu\nu) + c'\nu\rho, \\ \bar{c}' &= a'\mu^2 + 2b'\mu\rho + c'\rho^2. \end{aligned}$$

It is easily verified that

$$\bar{b}^2 - \bar{a}\bar{c} = (b^2 - ac)(\lambda\rho - \mu\nu)^2,$$
$$\bar{b}'^2 - \bar{a}'\bar{c}' = (b'^2 - a'c')(\lambda\rho - \mu\nu)^2,$$

and

$$2\bar{b}\bar{b}' - \bar{a}\bar{c}' - \bar{a}'\bar{c} = (2bb' - ac' - a'c)(\lambda\rho - \mu\nu)^2.$$

The H forms are thus identical, since $(\lambda\rho - \mu\nu)^2 = 1$.

11. THEOREM. *Two similar pencils have equivalent forms J; and the equivalence is proper or improper according as the similarity of the pencils is proper or improper.*

It is easily verified, using the notation of the preceding theorem, that

$$\bar{a}\bar{b}' - \bar{a}'\bar{b} = (ab' - a'b)\lambda^2 + (ac' - a'c)\lambda\nu + (bc' - b'c)\nu^2,$$
$$\bar{a}\bar{c}' - \bar{a}'\bar{c} = 2(ab' - a'b)\lambda\mu + (ac' - a'c)(\lambda\rho + \mu\nu) + 2(bc' - b'c)\nu\rho,$$
$$\bar{b}\bar{c}' - \bar{b}'\bar{c} = (ab' - a'b)\mu^2 - (ac' - a'c)\mu\rho + (bc' - b'c)\rho^2.$$

These are precisely the coefficients that would be obtained by transforming J by the substitution $\begin{pmatrix}\lambda & \mu\\ \nu & \rho\end{pmatrix}$.

12. THEOREM. *If two pencils are equivalent and also similar, then their λ-substitution is an automorph of J, and their α-substitution is an automorph of H.*

13. THEOREM. *Given a pencil, an equivalent pencil may always be found in which one of the base forms shall lack any desired coefficient.*

Let the pencil be (A, B), where $A = (abc)$ and $B = (a'b'c')$. If it is desired to find an equivalent pencil $(\bar{A}, \bar{B})$, such that the base form $\bar{A}$ shall lack the middle coefficient, choose α and β such that $\alpha b + \beta b' = 0$. This may be done in such a way that α and β shall be relatively prime. Thus if d is the greatest common factor of b and b', the values of α and β are b/d and b'/d. γ and δ may now be found such that $\alpha\delta - \beta\gamma = \pm 1$. Then the α-substitution $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ accomplishes the desired result. As γ and δ are not uniquely determined, an infinite number of such pencils may be found. It is not, however, in general possible to make more than one coefficient vanish. Thus if (γ, δ) are one pair of values such that $\alpha\delta - \beta\gamma = \pm 1$, then the other pairs are $\gamma + k\alpha$ and $\delta + k\beta$, where k is any integer. If now we wish to make the third coefficient, say, of $\bar{B}$, vanish, we must choose k so that

$$(\gamma + k\alpha)c + (\delta + k\beta)c' = 0,$$

whence

$$k = \frac{\gamma c + \delta c'}{\alpha c + \beta c'},$$

which is not generally integral.

14. PROBLEM. *To find a pencil that shall have a given Jacobian form.*

We suppose, without loss of generality by the preceding theorem, that $b=0$. We have then,

$$\begin{aligned} (1) \quad & -ac = D, & (4) \quad & ab' = L, \\ (2) \quad & b'^2 - a'c' = D', & (5) \quad & ac' - a'c = M, \\ (3) \quad & -ac' - a'c = F, & (6) \quad & -b'c = N; \end{aligned}$$

where L, M and N are the coefficients of the given form J (note that M is the entire coefficient of the term xy, not $2M$). From equations (4) and (6) we have $a/c = -L/M$, and this with (1) gives $a^2 = DL/N$ and $c^2 = DN/L$. From (3) and (5) we have $ac' = (M-F)/2$ and $a'c = +(M+F)/2$. Let now d be the greatest common divisor of L and N, so that $L = dL'$ and $N = dN'$, where L' and N' are relatively prime. Since $a^2 = DL/N = DL'/N'$ and $c^2 = DN'/L'$, it is clear that D must be a multiple of L' and of N' and that the multiple must be a square. Put then $D = k^2 L' N'$, and we get:

$$\begin{aligned} a &= kL', \\ c &= -kN', \\ a' &= (M+F)/2kN', \\ b' &= d/k, \\ c' &= (M-F)/2kL'. \end{aligned}$$

From these equations it appears that $M+F$ and $M-F$ are divisible by $2k$ and that therefore k divides both M and F. k also divides d, and since d divides L and M, k is a factor of $R = M^2 - 4LN$.

It also appears that

$$F \equiv M \pmod{2kL'}$$

and

$$F \equiv -M \pmod{2kN'},$$

so that the values of F which are available differ by multiples of $2kL'N'$ or $2D/k$.

The form H which goes with the above pencil is

$$H = k^2 L' N' x^2 + Fxy + [d^2/k^2 + (F^2 - M^2)/4k^2 N' L']\, y^2.$$

If now we take another F, say $F + 2Dg/k$, where g is any integer, we get for the corresponding form H'

$$H' = k^2 N' L' x^2 + (F + 2Dg/k)xy + [d^2/k^2 + (\overline{F + 2Dg}/k^2 - M^2)/4kLN]\, y^2.$$

It is easily verified that H' may be obtained from H by the substitution $\left(\begin{smallmatrix}1 & \frac{g}{k}\\ 0 & 1\end{smallmatrix}\right)$, so that if g/k is an integer H and H' are properly equivalent, also that the two pencils that correspond to these two values of F are connected by the α-substitution $\left(\begin{smallmatrix}1 & 0\\ \frac{g}{k} & 1\end{smallmatrix}\right)$. It follows readily that if two values of g differ by a multiple of k, the corresponding pencils are equivalent, and that therefore for any value of k which is a divisor of d there are k classes of non-equivalent pencils having the same Jacobian form J. It is easily seen also that two pencils obtained from the distinct positive values of k are not equivalent. Indeed one goes into the other by the α-substitution $\left(\begin{smallmatrix}\frac{k'}{k} & 0\\ 0 & \frac{k}{k'}\end{smallmatrix}\right)$, which is not integral in the coefficients unless $k = \pm k'$. It follows therefore that:

The number of non-equivalent pencils having a given Jacobian form is $\Sigma_d k$, *where* k *runs through all of the positive divisors of* d, d *being the greatest common divisor of* L *and* N.

In particular, if J is primitive, $k = 1$ and there is only one class of pencils having J for a Jacobian form. From the above solution of the problem it also appears that:

15. THEOREM. *Two pencils having the same Jacobian form are equivalent, if their harmonic forms are equivalent, and if the Jacobian form is primitive the two pencils are always equivalent.*

16. As an illustration of the solution of the problem of the preceding paragraphs let it be required to find all pencils having the Jacobian form

$$J = 10x^2 + 20xy + 100y^2.$$

Here $d = 10$, $L' = 1$, $N' = 10$, $k = 1, 2, 5$ and 10. The total number of non-equivalent classes of pencils will therefore be $1 + 2 + 5 + 10 = 18$.

The general base forms for the pencils will be

$$A = (k,\ 0,\ -10k), \qquad B = (\overline{20 - F}/2k,\ 10/k,\ \overline{20 - F}/2k).$$

When $k = 1$, $F \equiv 0 \pmod{20}$, giving the class represented by the pencil

$$A = (1,\ 0,\ -10), \qquad B = (1,\ 10,\ 10).$$

When $k = 2$, we have the two classes represented by the pencils:

$A = (2,\ 0,\ 20)$, $B = (1,\ 5,\ 0)$, where $F \equiv 20 \pmod{80}$;
$A = (2,\ 0,\ -20)$, $B = (2,\ 5,\ -10)$, where $F \equiv 60 \pmod{80}$.

When k equals 5, we have the five classes corresponding to the values $F \equiv 80, 180, 280, 380, 480 \pmod{500}$, and represented by the pencils:

$$\begin{array}{ll} (5, 0, -50), & (1, 2, -6), \\ (5, 0, -50), & (2, 2, -16), \\ (5, 0, -50), & (3, 2, -26), \\ (5, 0, -50), & (4, 2, -36), \\ (5, 0, -50), & (5, 2, -46). \end{array}$$

When k equals 10, we have ten classes corresponding to the values $F \equiv 180, 380, 580, 780, 980, 1180, 1380, 1580, 1780, 1980 \pmod{2000}$, and represented by the pencils $A = (10, 0, -100)$ and B the following forms:

$$(1, 1, -8), \quad (2, 1, -18), \quad (3, 1, -28), \quad (4, 1, -38), \quad (5, 1, -48),$$
$$(6, 1, -58), \quad (7, 1, -68), \quad (8, 1, -78), \quad (9, 1, -88), \quad (10, 1, -98).$$

17. THEOREM. *The harmonic form is the duplicate of the Jacobian form.*

This theorem follows from the identity:

$$\begin{aligned}[(ab' - a'b)x^2 + (ac' - a'c)xy + (bc' - b'c)y^2][(ab' - a'b)x'^2 + (ac' - a'c)x'y' \\ + (bc' - b'c)y'^2] = (b^2 - ac)[a'xx' + b'(xy' + x'y) + c'yy']^2 \\ - (2bb' - ac' - a'c) \cdot [a'xx' + b'(xy' + x'y) + c'yy'][axx' + b(xy' + x'y) \\ + cyy'] + (b'^2 - a'c')[axx' + b(xy' + x'y) + cyy']^2.\end{aligned}$$

This indicates that if in the form H we make the bilinear substitution,

$$X = a'xx' + b'xy' + b'x'y + c'yy',$$
$$Y = -axx' - bxy' - bx'y - cyy',$$

we get $J(xy) \cdot J(x'y')$ as above.

18. THEOREM. *If the Jacobian form is primitive, then the harmonic form belongs to the principal genus.*

This follows from the preceding theorem by well-known principles in the composition of classes (cf. Mathews, "Theory of Numbers," p. 171).

19. THEOREM. *If H belongs to the principal genus, it will serve as the harmonic form of some pencil.*

For by a theorem, due to Gauss, every such form is obtainable as the duplicate of some form J. The pencil with this form J for its Jacobian form will have the given form H for a harmonic form.

20. THEOREM. *If the form M is contained in a pencil, the determinant of the form is representable by the harmonic form of pencil.*

For if $M = \alpha A + \beta B$, the determinant of M is found (see paragraph 6) to be

$$D\alpha^2 + F\alpha\beta + D'\beta^2.$$

21. From the preceding theorems, it appears that the determinant of every form contained in a pencil is a quadratic residue of the resultant R of the pencil, and that conversely the resultant R is a quadratic residue of the determinant of every form contained in the pencil. Further, since a quadratic residue of a number is a quadratic residue of every divisor of that number, it follows that every factor of R is a residue of every factor of the determinant of every form contained in the pencil; and conversely, every factor of the determinant of any form contained in the pencil is a residue of every factor of R, the pencil always being supposed primitive. As a special theorem, if R is divisible by a prime of the form $4n-1$, then every odd prime divisor of the determinant of any form contained in the pencil must be of the form $4n+1$. For by the law of quadratic reciprocity there are no two primes both of the form $4n-1$ each of which is a quadratic residue of the other.

22. THEOREM. *If a form $M = (l, m, n)$ is contained in a pencil $A = (a, b, c)$, $B = (a', b', c')$, then $(ab' - a'b)(bn - cm) = (bc' - b'c)(am - bl)$, and conversely.*

For if $M = \alpha A + \beta B$, then

$$\begin{aligned} l &= \alpha a + \beta a', \\ m &= \alpha b + \beta b', \\ n &= \alpha c + \beta c'; \end{aligned}$$

and putting these values in the equation which is to be verified, we have the identity:

$$\begin{aligned} &(ab' - a'b)(\alpha bc + \beta bc' - \alpha cb - \beta b'c) \\ &\qquad = (bc' - b'c)(\alpha ab + \beta ab' - \alpha ab - \beta a'b). \end{aligned}$$

Starting from this identity the converse is easily established.

23. THEOREM. *Given any two pencils, there is one form primitively represented by both.* If there is more than one, then one form is contained in the other and there are an infinite number of such forms.*

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* A form $M = \alpha A + \beta B$ is *primitively* represented by the pencil (A, B) when the parameters α and β are prime to each other.

Let the two pencils be (A, B) and (M, N), where $A=(a, b, c)$, $B=(a', b', c')$, $M=(l, m, n)$, $N=(l', m', n')$. From the equation $\alpha A+\beta B=\gamma M+\delta N$ we have .

$$\begin{aligned}\alpha a+\beta a' &= \gamma l+\delta l',\\ \alpha b+\beta b' &= \gamma m+\delta m',\\ \alpha c+\beta c' &= \gamma n+\delta n'.\end{aligned}$$

Eliminating α, we have

$$\begin{aligned}\beta(ab'-a'b) &= \gamma(am-bl)+\delta(am'-bl'),\\ \beta(bc'-b'c) &= \gamma(bn-cm)+\delta(bn'-cm').\end{aligned}$$

Eliminating β, we have

$$\begin{aligned}&\gamma[(ab'-a'b)(bn-cm)-(bc'-b'c)(am-bl)]\\ &\quad+\delta[(ab'-a'b)(bn'-cm')-(bc'-b'c)(am'-bl')]=0.\end{aligned}$$

If now the forms M and N are not contained in the pencil (A, B), then the coefficients of γ and δ are not zero, by the preceding theorem. Dividing out extraneous factors, we have, for determining γ and δ, an equation $P\gamma+Q\delta=0$, where P and Q are relatively prime to each other. From this $\gamma=kQ$ and $\delta=-kP$, where k is an arbitrary integer, which may be so chosen as to make β and α integral. Different values of k will lead to forms which are identical except for a constant factor.

The theorem still holds when one of the forms, M say, is contained in the pencil (A, B). In that case the coefficient of γ vanishes. Thus $\delta=$ zero and α, β, γ are to be obtained from the equations

$$\begin{aligned}\alpha a+\beta a' &= \gamma l,\\ \alpha b+\beta b' &= \gamma m,\\ \alpha c+\beta c' &= \gamma n,\end{aligned}$$

whence eliminating α, we have

$$\begin{aligned}\beta(ab'-a'b) &= \gamma(am-bl),\\ \beta(bc'-b'c) &= \gamma(bn-cm),\\ \beta(ac'-a'c) &= \gamma(an-cl).\end{aligned}$$

These equations will serve to determine β and γ uniquely, and so also α uniquely except for a constant factor, unless the coefficients of β and γ all vanish, in which case the pencil (A, B) is of a trivial sort, the base forms being identical except for a constant factor. The values of α, β and γ manifestly give the representation of M by the pencil (A, B).

In case both forms belong to the pencil (A, B), it is clear that the pencil (M, N) is contained in the pencil (A, B); and so in this case there are an infinite number of forms primitively representable by both pencils.

24. Applying the above theorem to two similar pencils which are related by the λ-transformation $\begin{pmatrix}\lambda & \mu\\ \nu & \rho\end{pmatrix}$, we have the theorem:

THEOREM. *There are two and only two forms primitively representable by a pencil which are equivalent under a given transformation.*

For let ϕ be that one form primitively representable by the pencil P and by the pencil P' similar to it. Now every form of P is equivalent to some form of P' by the transformation $\begin{pmatrix}\lambda & \mu\\ \nu & \rho\end{pmatrix}$. Some form of P, f say, is therefore equivalent to ϕ by this transformation.

This theorem does not imply that every form of a pencil is equivalent to one and only one other form in that pencil, but that a given transformation reveals one and only one such pair of equivalent forms.

25. An exception to the above theorem arises when the λ-transformation is an automorph of J (see paragraph 19). In that case there are an infinite number of forms primitively representable by the pencil which are equivalent under the transformation. In fact every form of the pencil has a form in the pencil equivalent to it for every automorphic transformation of J. If now R is positive, the number of automorphic transformations of J is infinite, and the number of forms in the pencil equivalent to any given form in the pencil is also infinite. If, however, the number of automorphs of J is finite, as in the case when R is negative, then the number of forms in a pencil which are equivalent to any given form of the pencil is finite. In any case the number is not greater than the number of representations of the determinant of the given form by the harmonic form of the pencil.

On the Principle of Duality in the Geometry of the Sphere.

By Arthur Ranum.

1. In the geometry of the surface of the sphere, or what is practically the same thing, in two-dimensional Riemannian geometry, it has long been known that the principle of duality applies not only to projective theorems, but to metrical ones as well.* But although this principle has been used in spherical trigonometry and in the theory of inscribed polyedra, it does not seem to have been applied to the theory of spherical curves or to the kinematics of the sphere.

In this paper a few of the simpler applications to these subjects will be made. The history of mathematics has shown that before the application of the principle of duality to a given domain is recognized, one of two dual theorems is apt to be discovered long before the other, perhaps because of its closer connection with intuition. In the projective geometry of the plane a case in point is that of Pascal's and Brianchon's theorems. In the metrical geometry of the sphere, similarly, it turns out, as I shall try to show, that certain well-known theorems give rise, by the mere use of duality, to theorems that have an unfamiliar and even startling aspect.

2. The fundamental notion is that of two *polar* configurations, which are defined as two configurations in which every point P of one has a corresponding great circle p of the other, which is its polar with respect to the absolute conic; that is, p is situated at a quadrant's distance from P. We assume the radius of the sphere to be unity. By examining the correspondences that exist between two polar configurations, we easily arrive at all the results of this paper.

To begin with, however, we see that in order to make the duality precise, we must introduce the concept of a *directed great circle*; that is, a circle having

* For a recent statement of this fact see R. Bonola: "Questioni Riguardanti La Geometria Elementare," edited by F. Enriques (1900), p. 217.

a definite direction of description or rotation. By its aid the following dual relations become obvious:

Point of the sphere.	Directed great circle of the sphere.
Pair of diametrically opposite points.	Great circle with both directions.
Any two pairs of diametrically opposite points determine one great circle.	Any two great circles determine one pair of diametrically opposite points.
Spherical *distance* between two points.	*Angle* between two directed great circles.

In a spherical polygon, therefore, the sides are not dual to the *interior* angles; in the special case in which the sides and interior angles are all less than π, and the sides are all directed alike with respect to a point describing the perimeter, the sides are dual to the *exterior* angles.

A circle* is the locus of a point situated at a constant distance from a fixed point called the *center*.	A circle* is the envelope of a directed great circle making a constant angle with a fixed (directed) great circle called the *axis*.

The Duality of Arc and Area.

3. By considering a polygon with an infinite number of sides and passing to the limit, we see that the length of the path, or the linear displacement, of a point moving along a curve from one position P to another Q is equal to the angular displacement of its polar great circle in enveloping the polar curve from the position of the tangent p to that of the tangent q. Here the term angular displacement means the limit of the sum of the infinitesimal angles between successive tangents, which is, of course, not equal, in general, to the angle between the initial and terminal great circles p and q. We can therefore introduce the following duality:

The *linear* displacement of a moving point.	The *angular* displacement of a moving great circle.

4. Since the area of a lune is equal to twice the angle between its bounding great circles, therefore the area swept out by a moving great circle is twice its angular displacement. Here it will be understood that the area swept out by a

* Cf. Coolidge: "Non-Euclidean Geometry" (1909), p. 131.

moving great circle means the area swept out by that semicircle which extends in the positive direction from the point of tangency with the envelope to the diametrically opposite point. The duality just introduced may now be written as follows:

Twice the length of the *arc* described by a moving point.	The *area* swept out by a moving great circle.

5. The theorem implied by this may be given still another statement, as follows: The area included between a curve C and its opposite curve $\overline{C}$, and bounded by two of their common tangents p and q, is equal to twice the length of the corresponding arc of its polar curve C' bounded by the points P and Q, the poles of p and q, respectively.

Here the area bounded by p and q must be understood to mean the area swept out by the tangent in moving from p to q. If an inflectional tangent r lies between p and q, then it will be convenient, although by no means absolutely necessary, to consider the area swept out between p and r and the area swept out between r and q as being of opposite algebraic signs; if so, then in the polar curve, where the point R is a cusp lying between P and Q, the arc PR and the arc RQ must be given opposite algebraic signs. Hence we have the duality:

Twice the length of the arc of a curve bounded by two of its points.	The area between a curve and its opposite curve and bounded by two of their common tangents.
The curve is assumed to have no *point-discontinuity*; that is, to be the locus of a continuously moving point.	The curve is assumed to have no *tangential discontinuity*; that is, to be the envelope of a continuously turning great circle.

Finally, the area between a curve and its opposite curve can be regarded as the *area exterior* to the curve. Hence:

Twice the length of the arc of a curve between two of its points.	The area exterior to a curve and lying between two of its tangents.

6. Now suppose the curves to be closed. In that case the entire area between C and $\overline{C}$ is twice the length of the entire arc of C'; and since the area of the sphere, which is 4π, is made up of the area between C and $\overline{C}$ and twice the area inside of C, therefore the area inside of C plus the length of C' is equal

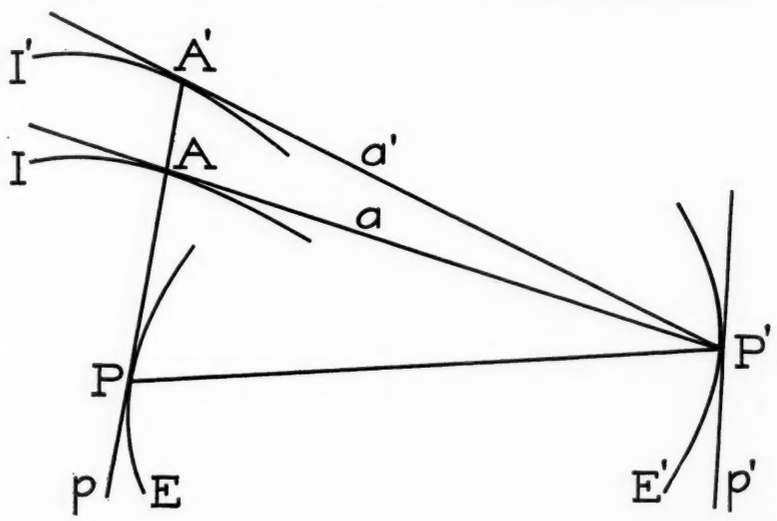
to 2π. So we have proved the known* theorem that the sum of the length of a closed spherical curve having no point-discontinuities and the area of its polar curve is equal to 2π.

Evolutes and Involutes.

7. We shall now consider the relation between spherical evolutes and involutes and its dual relation. In all that follows, in order to avoid circumlocution, we shall not usually distinguish between two diametrically opposite points or curves. We define the evolute of a curve as the envelope of its normals; the more precise definition and its dual are as follows:

The envelope of a great circle p passing though a variable point A of a curve I and making an angle $\frac{\pi}{2}$ with the tangent a at that point is a curve E called the *evolute* of I; I is then an *involute* of E.

The locus of a point P' on a variable tangent a' to a curve I' and situated at a distance $\frac{\pi}{2}$ from the point of tangency A' is a curve E' called the *polar evolute* of I'; I' is then a *polar involute* of E'.



In this figure the straight lines represent great circles; E and E' are polar curves; PP', of length $\frac{\pi}{2}$, is normal to both; p' and p are the polars of P and P', respectively.

* Cf. Böklen, *Grunert's Archiv*, Vol. XLIII (1865), p. 18, and Maxwell's "Electricity and Magnetism," 3rd ed. (1892), Vol. II, p. 40.

It is evident that if I coincides with I', E and E' are polar curves; that is, the polar evolute of a curve is the polar of its evolute; whence the name polar evolute. Similarly, every polar involute of a curve is an involute of its polar (not the polar of an involute).

8. The relation between parallel curves (geodetic parallels on the sphere), on the other hand, is a self-dual one. Thus if I and I' are parallel curves (see figure), on which A and A' are corresponding points, and a and a' are the tangents at these points, respectively, then the great circle AA' is normal to both curves and the distance between A and A' is constant; dually, the point aa' $(=P')$ is at a distance $\frac{\pi}{2}$ from both A and A', and the angle between a and a' is constant.

Every curve has a singly infinite number of parallel curves. In a family of parallel curves the polar of every curve of the family is also included in the family. Every curve has an infinite number of involutes forming a family of parallel curves, and also an infinite number of polar involutes forming another family of parallel curves. Conversely, all the members of a family of parallel curves have the same evolute and the same polar evolute. Hence, any two polar curves have the same evolute and the same polar evolute.

9. The following dual theorems are evident from the figure:

The evolute E of a curve I is the locus of the center P of the circle of curvature* at a variable point A of I.

On a tangent p to a curve E any two of its involutes I and I' cut off a segment AA' of constant length (independent of the position of the tangent).

The polar evolute E' of a curve I' is the envelope of the axis p' of the circle of curvature at a variable point A' of I'.

The angle between the tangents a and a' that can be drawn from a point P' of a curve E' to any two of its polar involutes I and I' is constant (independent of the position of the point).

10. By means of the duality between arc and area we easily obtain the dual of the well-known theorem† that the length of an arc of the evolute is equal

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*The circle of curvature, or osculating circle, of a curve lying on a sphere is a circle of the sphere and can obviously be given a purely spherical definition. *E. g.*, see P. Serret, "Lignes à Double Courbure" (1860), pp. 29–35, where spherical evolutes are also considered, but not polar evolutes.

†See Serret, *loc. cit.*, p. 35.

to the difference between the corresponding radii of curvature of the involute. The following statement of the dual theorems is partly illustrated by the figure:

*If P and Q are points on a curve E and if the tangents at these points meet an involute I of the curve in the points A and B, respectively, then the length of the arc PQ of E is equal to the difference between the distances PA and QB.	*If p' and q' are tangents to a curve E' and if through their respective points of contact great circles a' and b' are drawn touching a polar involute I' of the curve, then the area exterior to E' between p' and q' is equal to the difference between the areas of the lunes $(p'a')$ and $(q'b')$. In other words, the angular displacement of a tangent to E' in moving from p' to q' is equal to the difference between the angles $(p'a')$ and $(q'b')$.

11. Since the semicircle extending along a common tangent a to two diametrically opposite curves I and $\bar{I}$ from one point of contact A to the other is bisected by the corresponding point P' of their polar evolute E', therefore the area between I and $\bar{I}$, bounded by two common tangents, is divided into two equal parts by E'.† This fact makes it possible to state the theorem of § 5 in the following form:

The area included between a curve I and its polar evolute and bounded by two great circles a_1 and a_2, which are tangents to I, is equal to the length of the corresponding arc of its polar curve bounded by the points A_1 and A_2, the poles of a_1 and a_2, respectively. Hence the duality:

The area between a curve and its polar evolute and bounded by two of its tangents.	The length of the arc of a curve bounded by two of its points.

* A more precise statement of the limits within which these theorems are applicable is easily inferred from the analogous statement for the geometry of the plane given by Mangoldt in the *Encyclopädie der Mathematischen Wissenschaften*, Vol. III 3, p. 35.

† If E' is a closed curve, it evidently also divides the entire area of the sphere into two equal parts. See Serret, *loc. cit.*, pp. 90–91, where this theorem is ascribed to Jacobi.

Continuous Spherical Movements.

12. The kinematics of the sphere, or what is the same thing, the kinematics of the bundle whose vertex is the center of the sphere, deals with the movements or displacements which leave the sphere as a whole fixed, or which have one fixed point, the center of the sphere. We shall consider a continuous movement of the sphere; that is, the limit of the sum of an infinite number of infinitesimal rotations.

If throughout the entire movement a point P of the spherical surface (and its opposite point) remains fixed, then the movement becomes a simple rotation about P. Since p, the polar of P, also remains fixed, the movement can be regarded as a translation along p, provided a translation is defined as a spherical movement which leaves a great circle fixed, by analogy with a plane translation, which leaves a straight line fixed.

A *rotation* about a point P through an angle α; P is called the center of the rotation.	A *translation* along a great circle p through a distance α; p is called the axis of the translation.

Notice that a rotation and a translation are here the same movement considered from dual standpoints, and not, as in the kinematics of the plane, essentially distinct movements.

13. In the general case of any continuous spherical movement it is well-known* that the movement can be regarded as due to the rolling (without sliding) of a movable curve C_1 on a fixed curve C, and that C_1 and C are uniquely determined by the movement; at any instant the point of contact P of the two curves is the instantaneous center of rotation.

Let C_1' and C' be the polar curves of C_1 and C, respectively, and let p' be the polar of the point P; then C_1' and C' touch each other and their common tangent at the point of contact is p'. The instantaneous movement evidently consists of a translation along p'; and the entire movement may be regarded as due to a sliding of the movable curve C_1' on the fixed curve C', namely, the limit of the sum of an infinite number of infinitesimal translations. At any instant the common tangent p' is the instantaneous axis of translation.

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* Cf. Ziwet's "Theoretical Mechanics" (1893), Vol. I, p. 19, where the theorem is stated substantially as follows: Any continuous movement about a fixed point O can be produced by the rolling of one cone on another, both cones having their vertices at O. See also Koenigs, "Leçons de Cinématique" (1875), p. 187, theorems 3 and 4.

Consequently, the most general continuous movement of the sphere can be produced by the sliding (without rolling) of a movable curve C_1' on a fixed curve C'. C_1' and C' are uniquely determined by the movement.

Notice that a corresponding theorem does not hold in the kinematics of the plane, for there a sliding or translatory movement is of a very special character.

A movable curve C_1 *rolling* on a fixed curve C; their common point of contact P is the instantaneous *center of rotation.*	A movable curve C_1' *sliding* on a fixed curve C'; their common tangent p' at the point of contact is the instantaneous *axis of translation.*
If one of the two curves degenerates into a point, the other does, and the movement becomes a simple rotation about that point.	If one of the two curves degenerates into a great circle, the other does, and the movement becomes a simple translation along that great circle.
One of the two curves may degenerate into a great circle, but not both.	One of the two curves may degenerate into a point, but not both.

This special case will be considered in §§ 16–20.

14. In all that follows we shall assume the movement to be of such a nature that the curves C_1 and C, one of which rolls on the other, have no point-discontinuities, and therefore the polar curves C_1' and C', one of which slides on the other, have no tangential discontinuities. We shall now apply to these four curves the duality between arc and area.

It is intuitively evident that when one curve rolls on another, the lengths of the corresponding arcs of the two curves are equal. This fact and the one dual to it may be made explicit as follows:

If a curve C_1 rolls on a fixed curve C, and if to the points P and Q of C correspond, respectively, the points P_1 and Q_1 of C_1, then the length of the arc P_1Q_1 is equal to the length of the arc PQ.	If a curve C_1' slides on a fixed curve C', and if to p' and q', tangents to C', correspond, respectively, p_1' and q_1', tangents to C_1', then the area exterior to C_1' between p_1' and q_1' is equal to the area exterior to C' between p' and q'.

Corresponding points and corresponding tangents are, of course, those which

at some instant of the movement become common points of contact and common tangents at those points, respectively.

15. It is also well-known* that a continuous movement of the sphere can be represented by the path-curves of two of its points.

Therefore, a continuous movement of the sphere can be represented by the envelopes of two of its great circles.

Moreover, a continuous movement can be represented self-dually by the path-curve I_1 of a moving point A and the envelope I_2 of a moving great circle a.

Line-Element Movements.

16. In particular, A and a can always be chosen with a passing through A. If so, it may happen, in a certain type of movement, that I_1 and I_2 coincide; that is, that a fixed curve I exists which is at the same time the locus of A and the envelope of a, and moreover that throughout the movement the tangent to I at the point A is precisely the great circle a. This means that the line-element† (A, a) of the curve I is carried along the curve, remaining a line-element of I throughout the movement. We shall call such a movement a *line-element movement* and I a line-element curve of the movement.

17. The analogous case in the kinematics of the plane has been considered by Schoenflies‡ and the conclusion he draws holds also for the sphere. Namely, a line-element movement along the curve I (figure, § 7) is precisely the same as a movement produced by rolling its normal great circle p on its evolute E. Thus the instantaneous center of rotation is always at the center of curvature P of I.

This agrees with the usual derivation of the involute I from its evolute E; for the unwinding of a string amounts to the same thing as the rolling of a great circle.

By the principle of duality, now, we draw a further conclusion which does not hold for the plane. Namely, a line-element movement along I is the same as the movement produced by sliding the point P' on the polar evolute E'. It is to be observed that the sliding of P' on E' is a determinate movement, whereas the mere fact that E' is a path-curve of P' does not fix the movement.

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* Cf. Schoenflies, "Geometrie der Bewegung" (1886), Chap. 1, § 4, Art. 1, and Chap. 2, § 3, Art. 1.

† Cf. Lie-Scheffers, "Geometrie der Berührungstransformationen" (1896), p. 11.

‡ *Loc. cit.*, Chap. 1, § 6, Art. 5.

18. These results may be summed up as follows: Every line-element movement, by which a line-element (A, a) is carried along a curve I (figure, § 7), is identical with

the rolling of a great circle p on a fixed curve E; E is the evolute of I, and p is a normal to I.	the sliding of a point P' on a fixed curve E'; E' is the polar evolute of I, and P' is a point lying on a tangent to I at a distance $\frac{\pi}{2}$ from the point of tangency.
When a great circle p rolls on a curve E, the locus of every point A of p is an involute I of E and is also a line-element curve of the movement.	When a point P' slides on a curve E', the envelope of every great circle a passing through P' is a polar involute I of E', and is also a line-element curve of the movement.
The tangent to the involute I at the point A remains perpendicular to the rolling great circle p.	The point of tangency of the great circle a with the envelope I remains at a distance $\frac{\pi}{2}$ from the sliding point P'.
Every involute I of E is the locus of a point A lying on the rolling great circle p.	Every polar involute I of E' is the envelope of a great circle a passing through the sliding point P'.

Every line-element movement has an infinite number of line-element curves, which form a family of parallel curves.

19. As corollaries to the dual theorems of § 14 we have the following:

If a great circle p rolls on a fixed curve E, and if to the points P and Q of E correspond, respectively, the points P_1 and Q_1 of p, then the distance P_1Q_1 is equal to the length of the arc PQ.	If a point P' slides on a fixed curve E', and if to p' and q', tangents to E', correspond, respectively, p_1' and q_1', great circles passing through P', then the area of the lune $(p_1' q_1')$ is equal to the area exterior to E' between p' and q'. In other words, the angle $(p_1' q_1')$ is equal to the angular displacement of a tangent to E' in moving from p' to q'.

This dual result is also a special case of that obtained in § 10.

Inverse Line-Element Movements.

20. Any spherical movement can be regarded as a displacement of a movable sphere S' with respect to a fixed concentric sphere S. To an observer on S', however, S becomes movable and S' fixed; the resulting displacement of S with respect to S' is called the inverse movement.*

In the inverse movement the rôles of fixed and moving curves described in § 13 are interchanged; C rolls on C_1 and C' slides on C_1'.

It is to be noticed that the inverse movement is not in general the same as the original movement taken backwards. Thus in the original movement let C_1 roll on C from the position C_{10} to the position C_{11}; then in the backward movement C_1 will roll on C from the position C_{11} to the position C_{10}, while in the inverse movement C will roll on C_1. If, however, the original movement is a simple rotation, then the backward movement and the inverse movement are evidently identical.

An inverse line-element movement is therefore the same as the rolling of a curve E on a fixed great circle p, or dually the sliding of a curve E' through a fixed point P'. In the figure (§ 7) P will move along p and p' will revolve around P', as the curves E and E' roll and slide, respectively. Moreover, the line-elements (A, a), (A', a'), etc., infinite in number, whose points lie on the fixed great circle p and whose great circles pass through the fixed point P', will remain fixed, while the curves I, I', etc., will move in such a way as to continue to have these line-elements; *e. g.*, I will continue to touch the fixed great circle a at the fixed point A.

Closed Curves.

21. By a *closed curve* we shall mean one that is dually closed; that is, not only the locus of a continuously moving point which returns to its original position, but also the envelope of a continuously turning great circle which returns to its original position.

If two movements carry a sphere from the same initial position to the same final position, we shall call them *equivalent* movements. Every movement is therefore equivalent to a simple rotation about a point (translation along a great circle).

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* Cf. Schell, "Theorie der Bewegung und der Kräfte" (1879), Vol. I, p. 224; Schoenflies, *loc. cit.*, Chap. 1, § 2, Arts. 3–5, and "Encyclopädie," IV 1, p. 198. The *dualism* between a movement and its inverse, which is considered by Schell and Schoenflies, must not be confused with the *duality* which I am considering in this paper.

Let us now take up the case in which one of the two curves C_1 or C (and therefore also its polar curve C_1' or C') is a closed curve, and in which the movement makes one or more complete circuits of this curve.

First suppose the moving curve to be closed.

If a closed curve C_1, of length α, rolls on a fixed curve C, and if in its initial and final positions the same point P_1 of C_1 coincides, respectively, with the points P and Q of C, then the movement is equivalent to a line-element movement along C from P to Q. If the point of tangency makes n complete circuits of C_1 from P_1 back to P_1, then the length of the arc PQ of C is equal to $n\alpha$.

If a closed curve C_1', of exterior area α', slides on a fixed curve C', and if in its initial and final positions the same great circle p_1', tangent to C_1', coincides, respectively, with the great circles p' and q', tangents to C', then the movement is equivalent to a line-element movement along C' from p' to q'. If the common tangent makes n complete circuits of C_1' from p_1' back to p_1', then the area exterior to C' between p' and q' is equal to $n\alpha'$.

22. Next consider the inverse movement, in which the fixed curve is closed.

If a curve C rolls on a fixed closed curve C_1, of length α, and if in its initial and final positions the points P and Q of C coincide, respectively, with the same point P_1 of C_1, then (letting p_1 be the tangent to C_1 at P_1) the movement is equivalent to an inverse line-element movement, in which C moves along the fixed line-element (P_1, p_1) from P to Q.

If a curve C' slides on a fixed closed curve C_1', of exterior area α', and if in its initial and final positions the great circles p' and q', tangents to C', coincide, respectively, with the same great circle p_1', tangent to C_1', then (letting P_1' be the point of contact of p_1' with C_1') the movement is equivalent to an inverse line-element movement, in which C' moves along the fixed line-element (P_1', p_1') from p' to q'.

A further conclusion concerning arcs and areas can be drawn here, exactly as in the last section.

23. The general theorems just enunciated (§§ 21 and 22) can now be applied to the case in which C is a great circle and C' a point, while C_1 and C_1' are any

closed polar curves. The original movement is then a special kind of line-element movement or its inverse.

Thus the dual theorems of § 21, in this special case, take the following form (figure, § 7):

Every inverse line-element movement due to the rolling of a closed curve E, of length α, on a great circle p, and such that p makes a complete circuit of E, is equivalent to a simple translation along p through a distance α.	Every inverse line-element movement due to the sliding of a closed curve E', of exterior area $2\alpha'$, through a point P', and such that P' makes a complete circuit of E', is equivalent to a simple rotation about P' through an angle α'.

24. Similarly the theorems of § 22 take the following form (figure, § 7):

Every line-element movement produced by rolling a great circle once around a closed curve E, of length α, from the position p to the same position again is equivalent to a simple translation along p backward through a distance α.	Every line-element movement produced by sliding a point once around a closed curve E', of exterior area $2\alpha'$, from the position P' to the same position again is equivalent to a simple rotation about P' backward through an angle α', or forward through an angle $2\pi - \alpha'$ equal to the interior area of the curve.

The movement is of *finite period*, if and only if

the length of the curve	the area of the curve

is a rational multiple of π. In particular, the movement is of period 2 or equivalent to a

translation through the distance π, if the length of the curve is an odd multiple of *half the length of a great circle;*	rotation through the angle π, if the exterior area of the curve is an odd multiple of *the area of a hemisphere;*

and the movement is equivalent to the *identity*; that is, will carry every point of the sphere into itself,

if the length of the curve is an even multiple of half the length of a great circle.	if the exterior area of the curve is an even multiple of the area of a hemisphere.

25. In the special case in which E and E' are spherical triangles, the movement just considered becomes a succession of three simple rotations (translations); and if these triangles are assumed to have their sides and interior angles all >0 and $<\pi$, the last theorem takes the following form:

A spherical movement that consists of three successive rotations about the vertices A, B, C of a spherical triangle through angles equal respectively to the three corresponding exterior angles of the triangle, and that causes a great circle starting at CA to *roll* once around the triangle, is equivalent to a simple translation along CA backwards through a distance equal to the perimeter of the triangle.	A spherical movement that consists of three successive translations along the sides a, b, c of a spherical triangle through distances equal respectively to the lengths of those sides, and that causes a point starting at the vertex ca to *slide* once around the triangle, is equivalent to a simple rotation about ca backwards through an angle equal to the sum of the exterior angles of the triangle, or forward through an angle equal to the *spherical excess*, that is, the interior area, of the triangle.*
The movement can not be of period 1, but will be of period 2, if the perimeter of the triangle is equal to two *quadrants* of a great circle, and will be of period 4, if the perimeter of the triangle is equal to one or three quadrants of a great circle.	The movement can not be of period 1, but will be of period 2, if the interior area of the triangle is equal to the area of two *octants* of the sphere, and will be of period 4, if the interior area of the triangle is equal to the area of one or three octants of the sphere.

26. In the figure of § 7 let E be a closed curve of length α having no cusps, and therefore E' a closed curve of exterior area 2α having no inflections; also suppose $AA' = \alpha$. Then the line-element movement by which p makes one complete circuit of E and P' one circuit of E' will carry A around into A',

* By a curious chance this theorem, which was the starting-point of the investigation contained in this paper, first came to me as the result of a roundabout non-Euclidean line of argument. Afterwards a direct proof, without the use of duality, was communicated to me by Professor J. L. Coolidge, of Harvard University. Finally, I saw how obvious the dual theorem was.

so that I and I' must be parts of the same involute of E. Further, this involute will itself be a closed curve, if and only if α is a rational multiple of π. If E has cusps, and therefore E' has inflections, this theorem will still hold good, provided the length of E and the exterior area of E' are determined according to the convention laid down in § 5.

Hence a necessary and sufficient condition

that the involutes of a closed curve are also closed curves is that its length be a rational multiple of π.	that the polar involutes of a closed curve are also closed curves is that its area be a rational multiple of π.

Application to the Riemannian Plane.

27. In Riemannian space there are four simple two-dimensional configurations whose geometries are closely analogous to that of the Euclidean (or Riemannian) sphere, namely the plane, the bundle or sheaf, the right paratactic congruence, and the left paratactic congruence, paratactic lines being Clifford parallels.* All the dual theorems of this paper are immediately applicable to each of the four geometries. At present I merely wish to consider briefly the first one, that of the plane.

If the Riemannian space is double, the plane is so perfect a picture of the sphere that the only change needed is to read "straight line" for "great circle." We assume, of course, that the radius of curvature of the space is $+1$.

28. If the space is single, a few other changes will be needed and the duality will become simpler in some respects than before. Thus, diametrically opposite points and curves become coincident points and curves, and the point becomes dual to the undirected straight line; in § 4, instead of having "the area swept out by a great circle" defined in a rather artificial way, we see that the new phrase "the area swept out by a straight line" will mean exactly what it says, since the entire length of the straight line is π.

It is to be noticed that if a tangent to a closed curve in the single Riemannian plane makes one complete circuit of the curve, it will sweep out the entire exterior area of the curve just twice, and therefore that while the area swept out is *twice* the entire length of the polar curve, the exterior area of the original curve

*I have made a study of the relation between these four geometries in a paper which I am preparing for publication elsewhere.

is exactly *equal* to the length of the polar curve. On this account it will be necessary in the right-hand theorems of §§ 21 and 22 to replace "exterior area α'" by "exterior area $\frac{\alpha'}{2}$," and similarly in §§ 23 and 24 to replace "exterior area $2\alpha'$" by "exterior area α'." Moreover, in the right-hand theorem of § 24 "the area of a hemisphere" must be replaced by "half the area of a plane," and in the left-hand theorem of the same section "half the length of a great circle" must be replaced by "the length of a straight line."

When the suggested alterations and the obvious changes of language that go with them have been made, the results of this paper will take their place as a set of dual theorems in the geometry of the single Riemannian plane.

CORNELL UNIVERSITY, *June*, 1910.

On the Fundamental Postulate of Tamisage.

By Raymond K. Morley.

1. *Introduction.*

As is well known, Sylvester and Franklin in their computation of tables of groundforms of binary quantics by means of generating functions made use of the so-called Fundamental Postulate of Tamisage, which is briefly that there can not exist a groundform and a fundamental syzygy of the same degorder. They inferred the truth of this from its holding for all binary quantics of order not greater than six, but they were unable to prove it in general.* In 1882 J. Hammond showed that there is an exception to this postulate in the case of the seventhic of degorder 5.13.† This result was at once confirmed by Cayley‡ and generally accepted by mathematicians, but none the less it seems to have been viewed with suspicion by Sylvester and his followers for some time, or at any rate to have been regarded as a very remarkable and perhaps solitary exception. This view was strengthened by the fact that Sylvester in 1881 had succeeded in proving the non-existence of a groundform of the eighthic of degorder 10.4, where von Gall thought he had found an exception.§ Besides this, the seventhic was regarded‖ as somewhat exceptional anyway, inasmuch as Sylvester and Franklin were in this case alone unable to exhibit the Representative Form of the Generating Function with a finite numerator, owing to their supposing that there was no invariant of order $10k$ (k integer). This assumption was, as a matter of fact, due directly to the Fundamental Postulate, and of course indicated nothing with regard to it.

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* See Sylvester, American Journal, Vol. II, p. 230; Franklin, American Journal, Vol. III, p. 128.

† *Proc. Lond. Math. Soc.*, Vol. XIV, p. 85, and American Journal, Vol. V, p. 218.

‡ *Proc. Lond. Math. Soc.*, Vol. XIV, p. 88.

§ Von Gall, *Math. Ann.*, Vol. XVII, pp. 31, 139, 456; Sylvester, *Comptes Rendus*, Vol. XCIII, pp. 192, 365; also American Journal, Vol. IV, p. 62.

‖ See *e. g.* Elliott, "Algebra of Quantics," pp. 181–182. He seems to be unaware of Hammond's work cited below.

In 1888 von Gall computed the complete system of groundforms of the binary seventhic by the method of Gordan,* completing the work begun by Gordan himself in the well-known "Formensystem Binärer Formen." Gordan's method fails if at all by an excess of groundforms, the only proof of the irreducibility of the forms being the failure of the investigator to reduce them. On the other hand, the effect of the Fundamental Postulate is to make the method of Sylvester fail if at all by defect. Von Gall found a number of cases in which he was unable to make his results agree with Sylvester's, among them the case already pointed out by Hammond of degorder 5.13. Von Gall states (*loc. cit.*, p. 335) that in correspondence with him Sylvester said that he (Sylvester) had succeeded in proving the existence of a groundform of degorder 5.13, and admitted the possibility of other exceptions to the Fundamental Postulate. Hammond in 1890† by a direct method proved the existence of one irreducible invariant of the binary seventhic of order 20 and of one of order 30, in part confirming von Gall's work in those cases. With the aid of this result he reduced the Generating Function of the seventhic to a Representative Form with a finite numerator. This reason for regarding the seventhic as inherently exceptional was thus proven groundless.

There remain a number of cases in which von Gall's results for the seventhic do not agree with Sylvester's besides the three already mentioned in which Hammond has confirmed von Gall. The fact that such experts as Gordan and von Gall have failed to find the forms in question reducible does not of course prove them so (witness the case of the eighthic referred to above), however strong a presumption it may make in that direction. It should perhaps be mentioned here that Stroh‡ has deduced a theoretic criterion for testing reducibility, but it is not always easy to apply in any given case.

In consideration, therefore, of the lack of absolute proof either way in the cases of disagreement between Sylvester's results and von Gall's, it has seemed worth while to discuss some of the simpler of these cases. Further, although of course a single exception disproves the Fundamental Postulate as thoroughly as more from a logical standpoint, yet in view of the possibility that the seventhic might in some way peculiar to itself cause exceptions it seems desirable to point out that exceptions to the Postulate are by no means confined to the seventhic.

* *Math. Ann.*, Vol. XXXI, p. 318.
† *Math. Ann.*, Vol. XXXVI, p. 255.
‡ *Math. Ann.*, Vol. XXXIII, p. 106.

2. *The Binary Seventhic.*

The present section shows that in the four simplest cases, after that discussed by Hammond, von Gall's results are correct, that Sylvester's table is defective owing to further exceptions to the Fundamental Postulate.

The following table shows the differences between Sylvester's results and von Gall's.* In the squares containing one figure the results agree. Where two figures are given the upper one is von Gall's result, the lower Sylvester's.

ORDER IN VARIABLES.

DEGREE IN COEFFICIENTS.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1								1								
2			1				1				1					
3				1		1		1		1		1				1
4	1				2		1		2		1				1	
5		1		2		2		2		2				$\frac{1}{0}$		
6			3		2		2		2				$\frac{1}{0}$			
7		3		2		4		2				$\frac{1}{0}$				
8	3		3		3		3				$\frac{1}{0}$					
9		3		5		2				$\frac{1}{0}$						
10			4		$\frac{4}{3}$				$\frac{1}{0}$							
11		5		3				$\frac{1}{0}$								
12	6		6				$\frac{1}{0}$									
13		7		$\frac{1}{0}$		$\frac{1}{0}$										
14	4				$\frac{2}{0}$											
15		3		$\frac{1}{0}$												
16	2		$\frac{3}{0}$													
17		2														
18	9		$\frac{1}{0}$													
19		$\frac{2}{0}$														
20	$\frac{2}{0}$															
22	$\frac{3}{1}$															
23		$\frac{1}{0}$														
25		$\frac{1}{0}$														
26	$\frac{2}{0}$															
30	$\frac{1}{0}$															

* Von Gall, *Math. Ann.*, Vol. XXI, p. 336; Sylvester, AMERICAN JOURNAL, Vol. II, p. 230.

Before taking up the cases of disagreement it will be convenient to collect certain abbreviations. The system of groundforms of the binary seventhic adopted and the abbreviations for them are those used by von Gall.

Groundforms of Binary Seventhic.

$f = a_x^7$;
$l = l_x^2 = (ff)^6$; $k = k_x^6 = (ff)^4$; $H = H_x^{10} = (ff)^2$;
$r = r_x^3 = (fk)^5$; $\varepsilon = \varepsilon_x^9 = (fk)^2$; $\gamma = \gamma_x^{11} = (fk)$; $T = T_x^{15} = (fH)$;
$C = (ll)^2$; $p = p_x^4 = (kk)^4$; $\Delta = \Delta_x^8 = (kk)^2$; $\zeta = \zeta_x^{14} = (Hk)$;
$g = g_x^5 = (rk)^2$; $\eta = \eta_x^7 = (rk)$;
$\tau = \tau_x^2 = (rr)^2$; $\beta = \beta_x^8 = (kp)$;
$\alpha = \alpha_x = (pr)^8$;
$\lambda = \lambda_x^6 = (\tau k)$; $\mu = \mu_x^4 = (\tau k)^2$.

As is usual the index scheme $\binom{f_1\ f_2\ f_3}{\alpha_1\ \alpha_2\ \alpha_3}$ will be used as abbreviation for the well-known Gordan series,*

$$\sum_i \frac{\binom{n_2-\alpha_1-\alpha_3}{i}\binom{\alpha_2}{i}}{\binom{n_1+n_2-2\alpha_3-i+1}{i}} \left((f_1 f_2)^{\alpha_3+i} f_3\right)^{\alpha_1+\alpha_2-i}$$

$$= (-1)^{\alpha_1} \sum_i \frac{\binom{n_3-\alpha_1-\alpha_2}{i}\binom{\alpha_3}{i}}{\binom{n_1+n_2-2\alpha_3-i+1}{i}} \left((f_1 f_3)^{\alpha_2+i} f_2\right)^{\alpha_1+\alpha_3-i},$$

where $n_1\ n_2\ n_3$ are the orders of $f_1\ f_2\ f_3$ respectively, and either

$$\alpha_1 = 0, \quad \alpha_2 + \alpha_3 \overline{\overline{<}} n_1, \quad \alpha_3 \overline{\overline{<}} n_2, \quad \alpha_2 \overline{\overline{<}} n_3;$$

or

$$\alpha_2 + \alpha_3 = n_1, \quad \alpha_1 + \alpha_3 \overline{\overline{<}} n_2, \quad \alpha_1 + \alpha_2 \overline{\overline{<}} n_3.$$

Certain basic or elementary syzygies are of the greatest use in the investigation. All but the last two of them are taken from Stroh,† though the simpler ones were known previously.

$f_1\ f_2\ f_3\ f_4$ denote forms of order as great as the weight of the syzygy in which they occur, unless otherwise stated. t is a cubic form. q and q' are quadratic forms.

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* Gordan, "Formensystem," p. 11, III.

† Stroh, *Math. Ann.*, Vol. XXXVI, p. 271.

WEIGHT 1.

$$(f_1 f_2 f_3) = f_1(f_2 f_3) + f_2(f_3 f_1) + f_3(f_1 f_2) = 0.$$

WEIGHT 2.

$$[f_1 f_2 f_3 f_4]_2 = 2(f_1 f_3)(f_2 f_4) + f_1 f_2 (f_3 f_4)^2 + (f_1 f_2)^2 f_3 f_4 - f_1 f_4 (f_2 f_3)^2 - (f_1 f_4)^2 f_2 f_3 = 0.$$

(The sign of the third term as printed in Stroh, *loc. cit.*, is incorrect.)

WEIGHT 3.

$$[f_1 f_2 f_3 q]_3 = 2(f_1 f_2)(f_3 q)^2 - 2(f_1 q)(f_2 f_3)^2 + 2(f_2 q)(f_1 f_3)^2 - f_3((f_1 q)^2 f_2) + f_3((f_2 q)^2 f_1) + f_3 q (f_1 f_2)^3 = 0.$$

This syzygy holds also when f_3 is a quadratic form.

WEIGHT 4.

$$[f_1 f_2 q q']_4 = 2(f_1 f_2)^2 (q q')^2 - 2(f_1 f_2)^3 (q q') + (f_1 f_2)^4 q q' - (f_1 q)^2 (f_2 q')^2 - (f_1 q')^2 (f_1 q)^2 - 2q((f_2 q')^2 f_1)^2 - 2q'((f_1 q)^2 f_2)^2 + f_1 (f_2, q q')^4 + f_2 (f_1, q q')^4 = 0.$$

The two following syzygies also of weight 4 are not given by Stroh:

$$[f_1 f_2 t q]_4 = -3 f_1((f_2 q)^2 t)^2 - 3 f_2((f_1 q)^2 t)^2 - 2 t q (f_1 f_2)^4 - 6(f_1 f_2)^2 (t q)^2 + 3(f_1 t)^2 (f_2 q)^2 + 3(f_1 q)^2 (f_2 t)^2 + 3 t((f_1 q)^2 f_2)^2 + 3 t((f_2 q)^2 f_1)^2 + 2q((f_1 t)^3 f_2) + 2q((f_2 t)^3 f_1) = 0,$$

$$\{f_1 f_2 t q\}_4 = 2 f_2((f_1 t)^3 q) - 3 f_2((f_1 q)^2 t)^2 - t q (f_1 f_2)^4 - 3(f_1 f_2)^2 (t q)^2 + 3(f_1 t)^2 (f_2 q)^2 - 2(f_1 q)(f_2 t)^3 + 2 t((f_1 q)^2 f_2)^2 + t((f_2 q)^2 f_1)^2 = 0.$$

These syzygies are most easily proven by a method due to Stroh.* Let the forms f_1, f_2, t, q be represented as powers of linear functions,

$$f_1 = (x + a y)^{n_1}, \quad f_2 = (x + b y)^{n_2}, \quad t = (x + c y)^3, \quad q = (x + d y)^2.$$

Then the $(i+1)$th coefficient of f_1, say a_i, is represented by a^i; of f_2, b_i, by b^i; and so on. Form the transvectants of these forms and take the first coefficient, which will be denoted by a subscript 0 on the transvectant. Then

$$(f_1 f_2)^\lambda_0 = \sum_i (-1)^i \binom{\lambda}{i} a_i b_{\lambda - i}$$

may be represented by

$$\sum_i (-1)^i \binom{\lambda}{i} a^i b^{\lambda - i} = (a - b)^\lambda.$$

* Stroh, *Math. Ann.*, Vol. XXXVI, p. 273.

Further, if the index of the transvectant is the same as the order of the second form within it, it is clear that the second coefficient of the transvectant is obtained from the first merely by increasing the subscript of every a involved by unity. A similar process gives the third from the second and so on. For instance, if the transvectant with the subscript j denotes the $(j+1)$th coefficient of the transvectant,

$$(f_1 t)^3_j = \sum_i (-1)^i \binom{3}{i} a_{j+i} c_{3-i}.$$

Then the first coefficient of a transvectant like $((f_1 t)^3 f_2)^\mu$,

$$((f_1 t)^3 f_2)^\mu_0 = \sum_j (-1)^j \binom{\mu}{j} (f_1 t)^3_j b_{\mu-j}$$
$$= \sum_j (-1)^j \binom{\mu}{j} b_{\mu-j} \sum_i (-1)^i \binom{3}{i} a_{j+i} c_{3-i},$$

may be represented by

$$\sum_j \sum_i (-1)^{j+i} \binom{\mu}{j} \binom{3}{i} a^{j+i} b^{\mu-j} c^{3-i} = (a-c)^3 (a-b)^\mu.$$

So also for other transvectants of similar form.

Then the syzygy $[f_1 f_2 tq]_4 = 0$ may be represented thus:

$$-3(b-d)^2(b-c)^2 - 3(a-d)^2(a-c)^2 - 2(a-b)^4 - 6(a-b)^2(c-d)^2$$
$$+3(a-c)^2(b-d)^2 + 3(a-d)^2(b-c)^2 + 3(a-d)^2(a-b)^2$$
$$+3(b-d)^2(b-a)^2 + 2(a-c)^3(a-b) + 2(b-c)^3(b-a) = 0.$$

Now put

$$a-b = -b', \quad a-c = -c', \quad a-d = -d';$$

then

$$b-c = b'-c', \quad b-d = b'-d', \quad c-d = c'-d'.$$

After making these substitutions, which serve to express the syzygant as a function of three variables, drop the primes and the result is exactly what would result if $a=0$ in the original expression. It becomes, therefore,

$$-3(b-d)^2(b-c)^2 - 3c^2d^2 - 2b^4 - 6b^2(c-d)^2 + 3c^2(b-d)^2$$
$$+3d^2(b-c)^2 + 3b^2d^2 + 3b^2(b-d)^2 + 2bc^3 + 2b(b-c)^3 = 0,$$

which is easily verified.

$\{f_1 f_2 tq\}_4 = 0$ is proven in a precisely similar way.

Take up now the cases of disagreement between Sylvester's results and von Gall's. The simplest case is that in which Hammond has proven that there is an exception to the Fundamental Postulate, namely of degorder 5.13. A brief analysis of it will be included here for completeness.

Von Gall shows one irreducible covariant of this degorder, namely $(Tl)^2$. There are four reducible covariants, (1.7).(4.6),* (2.2).(3.11), (2.6).(3.7), (2.10).(3.3), which are

$$f.(kl), \quad l.\gamma, \quad k.(fl), \quad H.r.$$

A computation of $\Delta(7, 5; 13)$† shows that there are four linearly independent covariants of degorder 5.13. If, therefore, there exists a syzygy among these four reducible covariants, there must be a groundform. Further, von Gall's system is certainly complete; therefore if there is a groundform, the one he exhibits must be irreducible. There is such a syzygy, namely,

$$(fkl) = f(kl) + k(lf) + l(fk) = 0;$$

or

$$f(kl) - k(fl) + l.\gamma = 0.$$

There are then in this case both a fundamental syzygy and a groundform, and consequently an exception to the Fundamental Postulate.

The next case in which the two sets of results differ is that of degorder 6.12. Von Gall exhibits one irreducible covariant, namely $(\zeta l)^2$, and it is easy to find that there are the following seven reducible covariants:

$$(1.7)(5.5)[2], \quad (2.2)(4.10), \quad (2.6)(4.6), \quad (3.3)(3.9), \quad (3.5)(3.7), \quad (1.7)(2.2)(3.3),$$

which are:

$$f.g, \quad f.(fl^2)^3, \quad l.(Hl), \quad k.(kl), \quad r.\varepsilon, \quad (fl)^2.(fl), \quad f.l.r.$$

A computation of $\Delta(7, 6; 12)$ shows that there are seven linearly independent covariants of degorder 6.12. If, therefore, there exists a syzygy of this degorder, there must be a groundform, and by the same reasoning as above it may be taken as the form $(\zeta l)^2$. There is such a syzygy, namely,

$$f.(fl^2)^3 - l.(Hl) - (fl)^2.(fl) = 0.$$

For from $(f, (fl), l) = 0$ it follows that

$$f.((fl)^2 l) + (fl)^2(lf) + l.(f, (fl)^2) = 0.$$

But the series $\begin{pmatrix} l & f & l \\ 1 & 0 & 2 \end{pmatrix}$ shows at once that $((fl)^2 l) = -(l^2 f)^3 = (fl^2)^3$, and the series $\begin{pmatrix} f & l & f \\ 0.1 & & 2 \end{pmatrix}$ gives $((fl)^2 f) = ((ff)^2 l) = (Hl)$. Therefore

$$f.(fl^2)^3 - (fl)^2.(fl) - l(Hl) = 0.$$

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* $(i.m)$ denotes a covariant of degree i, order m.

† The symbol $\Delta(j, i; m)$ is used in the same sense as in Franklin, AMERICAN JOURNAL, Vol. III, p. 129.

In this case also, then, von Gall's table is correct and Sylvester's is defective by one groundform, owing to another exception to the Fundamental Postulate.

The case of next higher degree is of degorder 7.11. Von Gall shows one irreducible covariant $(Tl^2)^4$. There are 15 reducible covariants, namely

(1.7)(6.4)[2], (2.2)(5.9)[2], (2.6)(5.5)[2], (2.10)(5.1), (3.3)(4.8)[2], (3.5)(4.6), (3.7)(4.4)[2], (3.11)(4.0), $(2.2)^2(3.7)$, (2.2)(2.6)(3.3),

which are :

$$f.\begin{Bmatrix}(kl^2)^3\\(pl)\end{Bmatrix},\quad l.\begin{Bmatrix}(\gamma l)^2\\(\varepsilon l)\end{Bmatrix},\quad k.\begin{Bmatrix}g\\(fl^2)^3\end{Bmatrix},\quad H.(rl)^2,\quad r.\begin{Bmatrix}\Delta\\(Hl)\end{Bmatrix},$$

$$(fl)^2.(kl),\quad (fl).\begin{Bmatrix}p\\(kl)^2\end{Bmatrix},\quad \gamma.C,\quad l^2.(fl),\quad l.k.r.$$

The value of $\Delta(7, 7; 11)$ is 11, the number of linearly independent covariants of this degorder. There must therefore be certainly four independent syzygies among these reducible covariants, and five if $(Tl^2)^4$ is irreducible Conversely, $(Tl^2)^4$ is irreducible if there are five syzygies.

Perhaps the simplest proof of the irreducibility of $(Tl^2)^4$ is the following: If it be reducible it must satisfy a relation of the form $(Tl^2)^4 = \Sigma\, m\, K$, where K is a reducible covariant of degorder 7.11, and the m's are constants. This relation, for the same m's, must hold for all values of the coefficients of f. Take two sets of values for them:

I. $a_1 = a_2 = a_3 = a_5 = a_6 = 0, \quad a_0 \neq 0,\ a_4 \neq 0,\ a_7 \neq 0.$

II. $a_1 = a_2 = a_5 = a_6 = a_7 = 0, \quad a_0 \neq 0,\ a_3 \neq 0,\ a_4 \neq 0.$

On the basis of these suppositions the computations are not difficult. Of course only the first term of each covariant of degorder 7.11 is necessary. The supposition I gives, omitting forms and parts of forms that can not affect the result:

$$l = \binom{2}{1} a_0 a_7 xy + \binom{2}{2}(-20\, a_4^2)\, y^2,$$

$$l^2 = \binom{4}{2} 2/3\, a_0^2 a_7^2 x^2 y^2 + \binom{4}{3}(-20\, a_0 a_4^2 a_7)\, x y^3 + \binom{4}{4} 400\, a_4^4 y^4,$$

$$T = \binom{15}{1} 4/15\, a_0^2 a_4\, x^{14} y + \binom{15}{4} 1/1365\, a_0^2 a_7\, x^{11} y^4 + \text{etc.},$$

$$k = 2\, a_0 a_4\, x^6 + \binom{6}{3} 1/10\, a_0 a_7\, x^3 y^3 + \text{etc.},$$

$$p = 8\, a_0 a_4^3\, x^4,$$

$$g = -\,2\, a_0^2 a_4^2 a_7\, x^5 + \text{etc.},$$

$$(fl)^2 = -\,20\, a_0 a_4^2\, x^5 + \text{etc.},$$

$$(fl) = a_0^2 a_7\, x^7 + \text{etc.}$$

The supposition II gives:

$$l = -20\,a_3^2 x^2 + \binom{2}{1}(-5\,a_3 a_4)\,xy + \binom{2}{2}(-20\,a_4^2)\,y^2,$$

$$l^2 = 400\,a_3^4 x^4 + \binom{4}{1} 100\,a_3^3 a_4\, x^3 y + \binom{4}{2} 150\,a_3^2 a_4^2\, x^2 y^2 + \binom{4}{3} 100\,a_3 a_4^3\, x y^3 + \binom{4}{4} 400\,a_4^4\, y^4,$$

$$T = a_0^2 a_3\, x^{15} + \binom{15}{1} 4/15\,a_0^2 a_4\, x^{14} y + \binom{15}{3}(-31/91\,a_0 a_3^2)\, x^{12} y^3 + \binom{15}{4}(-80/273\,a_0 a_3 a_4)\, x^{11} y^4 + \text{etc.},$$

$$k = 2\,a_0 a_4\, x^6 + \binom{6}{2} 2\,a_3^2 x^4 y^2 + \binom{6}{3} 3/2\,a_3 a_4\, x^3 y^3 + \binom{6}{4} 2\,a_4^2 x^2 y^4,$$

$$p = (8\,a_0 a_4^3 + 24\,a_3^4)\, x^4 + \binom{4}{1} 6\,a_3^3 a_4\, x^3 y + \text{etc.},$$

$$\gamma = \binom{11}{1} 10/11\,a_0 a_3^2 x^{10} y + \binom{11}{2}(-3/11\,a_0 a_3 a_4)\, x^9 y^2 + \text{etc.},$$

$$\varepsilon = 2\,a_0 a_3^2 x^9 + \binom{9}{1} 16/9\,a_0 a_3 a_4\, x^8 y + \text{etc.},$$

$$r = -20\,a_3^3 x^3 + \binom{3}{1}(-5\,a_3^2 a_4)\, x^2 y + \binom{3}{2} 5\,a_3 a_4^2\, x y^2 + 20\,a_4^3 y^3,$$

$$(fl)^2 = -20\,a_0 a_4^2 x^5 + \text{etc.},$$

$$(fl) = -5\,a_0 a_3 a_4\, x^7 + \text{etc.},$$

$$H = \binom{10}{1} a_0 a_3\, x^9 y + \binom{10}{2} 4/9\,a_0 a_4\, x^8 y^2 + \text{etc.},$$

$$g = 10\,a_0 a_3 a_4^3\, x^5 + \text{etc.}$$

From these may be obtained completely the numerical coefficients of the following terms of the first coefficients of all the covariants of degorder 7.11: $a_0^3 a_4^3 a_7$, $a_0^2 a_3 a_4^4$, $a_0 a_3^5 a_4$. Supposition I gives the first, II the others. The first two are sufficient for the purpose in hand. They are most plainly exhibited in a table.

	$(Tl^2)^4$	$f(kl^2)^3$	$f(pl)$	$l(\gamma l)^2$	$l(\varepsilon l)$	kg	$k(fl^2)^3$	$H(rl)^2$	$r\Delta$
$a_0^3 a_4^3 a_7$	$\frac{64}{3}$	−40	8	0	0	−4	−40	0	0
$a_0^2 a_3 a_4^4$	$\frac{880}{3}$	200	−40	0	0	20	200	0	0

	$r(Hl)^2$	$(fl)^2(kl)$	$(fl)p$	$(fl)(kl)^2$	$\gamma . C$	$l^2.(fl)$	lkr
$a_0^3 a_4^3 a_7$	0	−40	8	−40	0	0	0
$a_0^2 a_3 a_4^4$	0	200	−400	200	0	0	0

Here, except for the first form $(Tl^2)^4$, the second row is -5 times the first. It follows that in any expression $\Sigma m K$ the coefficient of $a_0^2 a_3 a_4^4$ will be -5 times that of $a_0^3 a_4^3 a_7$; but this is not true of $(Tl^2)^4$. Hence $(Tl^2)^4 \neq \Sigma m K$ and $(Tl^2)^4$ is irreducible. In this case again, therefore, an exception to the Fundamental Postulate has made Sylvester's table incomplete.

A second proof consists in showing that there are five independent syzygies of degorder 7.11, which as already pointed out involves the irreducibility of $(Tl^2)^4$. These five syzygies will now be obtained.

In the syzygy $(f_1 f_2 f_3) = 0$ put $f_1 = (fl)^2$, $f_2 = k$, $f_3 = l$, and it becomes

$$(fl)^2 (kl) - k((fl)^2 l) + l((fl)^2 k) = 0.$$

But from $\begin{pmatrix} l & f & l \\ 1 & 0 & 2 \end{pmatrix}$ follows $((fl)^2 l) = (fl^2)^3$, and from $\begin{pmatrix} f & f & f \\ 0 & 3 & 4 \end{pmatrix}$ comes

$$(fk)^3 = 1/10\,(fl), \tag{1}$$

which, taken with $\begin{pmatrix} f & l & k \\ 0 & 1 & 2 \end{pmatrix}$, gives

$$((fl)^2 k) = (\gamma l)^2 + 10/11\,(\varepsilon l) + 1/45\,(fl)\,l.$$

These give the syzygy

$$l(\gamma l)^2 + 10/11\,l(\varepsilon l) - k(fl^2)^3 + (fl)^2(kl) + 1/45\,l^2(fl) = 0.$$

Again, if in the same syzygy $f_1 = (kl)^2$, $f_2 = f$, $f_3 = l$, the result is

$$(kl)^2 (fl) - f((kl)^2 l) + l((kl)^2 f) = 0.$$

Here $\begin{pmatrix} l & k & l \\ 1 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} k & l & f \\ 0 & 1 & 2 \end{pmatrix}$, together with (1), give

$$((kl)^2 l) = (kl^2)^3, \text{ and } ((kl)^2 f) = -(\gamma l)^2 + 12/11\,(\varepsilon l) - 1/30\,(fl)\,l.$$

These values make the syzygy become

$$-f(kl^2)^3 - l(\gamma l)^2 + 12/11\,l(\varepsilon l) + (fl)(kl)^2 - 1/30\,l^2(fl) = 0.$$

Another syzygy results from the basic syzygy $[f_1 f_2 f_3 q]_3 = 0$ on putting $f_1 = f$, $f_2 = k$, $f_3 = q = l$, namely,

$$2(fk)(ll)^2 - 2(fl)(kl)^2 + 2(kl)(fl)^2 - l((fl)^2 k) + l((kl)^2 f) + l^2(fk)^3 = 0,$$

which, on putting in the values of unknown transvectants already obtained, becomes

$$-l(\gamma l)^2 + 1/11\,l(\varepsilon l) + (fl)^2(kl) - (fl)(kl)^2 + \gamma C + 1/45\,l^2(fl) = 0.$$

A fourth syzygy comes from $(f_1 f_2 f_3) = 0$ on putting $f_1 = f$, $f_2 = (fr)^3$, $f_3 = l$. It is

$$f((fr)^3 l) - (fr)^3 (fl) - l((fr)^3 f) = 0.$$

From $\binom{f\,f\,f}{1\,3\,4}$ it is easy to find

$$(fl)^2 = -5(fk)^4, \tag{2}$$

and this, taken with $\binom{f\,f\,l}{0\,2\,4}$ and $\binom{f\,k\,f}{1\,3\,4}$, yields by elimination

$$(fr)^3 = 5/2\,p - 1/10\,l^2. \tag{3}$$

Moreover, on this account,

$$((fr)^3 l) = 5/2\,(pl), \tag{4}$$

since $(l^2, l) = 0$.

From $\binom{f\,k\,f}{0\,4\,3}$, $\binom{k\,f\,f}{1\,3\,3}$, $\binom{f\,k\,f}{0\,3\,4}$ and $\binom{f\,f\,f}{3\,3\,4}$ [the last shows

$$(fk)^6 = 0 \tag{5}]$$

elimination gives

$$(fr)^2 = 1/2\,(lk). \tag{6}$$

But this, in the result of elimination between $\binom{k\,l\,f}{0\,2\,1}$ and $\binom{k\,l\,f}{0\,1\,2}$, together with (1), gives

$$((fr)^2 f)^2 = -1/6\,(\gamma l)^2 - 7/22\,(\varepsilon l) + 1/45\,(fl)\,.\,l.$$

Now from $\binom{f\,r\,f}{0\,2\,2}$ and $\binom{f\,r\,f}{0\,1\,3}$ it is easy to obtain

$$((fr)^3 f) = 3((fr)^2 f)^2 - 1/2\,k\,.\,r,$$

which gives at once

$$((fr)^3 f) = -1/2\,(\gamma l)^2 - 21/22\,(\varepsilon l) + 1/15\,(fl)\,.\,l - 1/2\,k\,.\,r.$$

The syzygy then becomes

$$5/2 f(pl) - 5/2\,p(fl) + 1/30\,l^2(fl) + 1/2\,l(\gamma l)^2 + 21/22\,l(\varepsilon l) + 1/2\,k\,.\,r\,.\,l = 0.$$

The fifth syzygy is obtained from $\{f_1 f_2 tq\}_4 = 0$ by putting $f_1 = f_2 = f$, $t = r$, $q = l$. It is

$$2f((fr)^3 l) - 3f((fl)^2 r)^2 - rl(ff)^4 - 3(ff)^2 (rl)^2 + 3(fr)^2 (fl)^2 - 2(fl)(fr)^3 + 3r((fl)^2 f)^2 = 0.$$

From $\binom{f\ l\ r}{0\ 2\ 2}$ comes $((fl)^2 r)^2 = ((fr)^2 l)^2 + 1/3\,((fr)^3 l)$, and from $\binom{l\ l\ k}{1\ 2\ 0}$, $\binom{l\ k\ l}{1\ 1\ 1}$ and (6) comes

$$((fr)^2 l)^2 = -1/3\,(kl^2)^3. \tag{7}$$

Therefore $((fl)^2 r)^2 = -1/3\,(kl^2)^3 + 1/3\,((fr)^3 l)$.

Again, $\binom{f\ f\ l}{0\ 2\ 2}$ gives $((fl)^2 f)^2 = (Hl)^2 + 5/18\,k.l.$

These values, together with some of those previously obtained, give the fifth syzygy,

$$5/2\,f(pl) + f(kl^2)^3 - 5\,p(fl) - 3\,H(rl)^2 + 3\,r(Hl)^2 - 3/2\,(kl)\,(fl)^2 - 1/6\,k.l.r + 1/5\,l^2(fl) = 0.$$

That these five syzygies are linearly independent may be ascertained by the usual tests or briefly thus: The terms $k(fl^2)^3$, $\gamma.C$, $H(rl)^2$ appear in the first, third and last syzygies respectively, and in no others. These syzygies, then, could enter into no linear relation connecting the five. The only remaining possibility is that the second and fourth syzygies should be related, which is clearly not the case.

There are then five linearly independent syzygies of degorder 7.11, and as already noted this involves the existence of a groundform and therefore of an exception to the Fundamental Postulate.

The next case is that of degorder 8.10. Von Gall shows one irreducible covariant $(\zeta l^2)^4$. There are 20 reducible covariants, namely:

(1.7)(7.3)[2], (2.2)(6.8)[2], (2.6)(6.4)[2], (3.3)(5.7)[2], (3.5)(5.5)[2], (3.7)(5.3)[2], (3.9)(5.1), (4.0)(4.10), (4.4)(4.6)[2], (1.7)(3.3)(4.0), (1.7)(2.2)(5.1), (2.2)(3.3)(3.5), $(2.2)^2(4.6)$.

These are:

$$f\left\{\begin{matrix}(fl^3)^5\\(gl)^2\end{matrix}\right.,\ l\left\{\begin{matrix}\beta\\(Hl^2)^3\end{matrix}\right.,\ k\left\{\begin{matrix}(kl^2)^3\\(pl)\end{matrix}\right.,\ r\left\{\begin{matrix}\eta\\(\varepsilon l)^2\end{matrix}\right.,\ (fl)^2\left\{\begin{matrix}g\\(fl^2)^3\end{matrix}\right.,\ (fl)\left\{\begin{matrix}(rl)\\(fl^2)^4\end{matrix}\right.,$$

$$\varepsilon.(rl)^2,\ C.(Hl),\ \left.\begin{matrix}p\\(kl)^2\end{matrix}\right\}(kl),\ f.r.C,\ f.l.(rl)^2,\ l.r.(fl)^2,\ l^2(kl).$$

A computation of $\Delta(7, 8; 10)$ shows that there are fourteen linearly independent covariants of degorder 8.10. There must then certainly be six linearly independent syzygies of this degorder, and if there are seven there must be a

groundform. As von Gall's system is certainly complete, it may be taken as the form $(\zeta l^2)^4$. There are seven such syzygies, as will now be shown.

In $(f_1 f_2 f_3) = 0$ let $f_1 = f$, $f_2 = (kr)^3$, $f_3 = l$; then,

$$f((kr)^3 l) - (kr)^3 . (fl) - l . ((kr)^3 f) = 0.$$

From $\binom{f\,f\,r}{0\,3\,4}$, $\binom{f\,f\,r}{0\,2\,5}$, $\binom{k\,l\,f}{0\,5\,1}$, $\binom{k\,l\,f}{0\,4\,2}$ and $\binom{l\,l\,f}{2\,2\,0}$, together with (5) and (6), elimination gives

$$(kr)^3 = 1/10\,(fl^2)^4 - 3/10\,(rl). \tag{8}$$

Therefore

$$((kr)^3 l) = 1/10\,((fl^2)^4 l) - 3/10\,((rl)\,l),$$

which, by the aid of $\binom{l\,r\,l}{0\,1\,1}$ and $\binom{l^2 f\,l}{1\,0\,4}$, becomes

$$((kr)^3 l) = 1/10\,(fl^3)^5 - 1/5\,l . (rl)^2 + 3/20\,r . C. \tag{9}$$

Also

$$((kr)^3 f) = 1/10\,((fl^2)^4 f) - 3/10\,((rl) f),$$

which, by $\binom{f\,l^2\,f}{0\,1\,4}$ and $\binom{l\,l\,k}{0\,1\,0}$ and an elimination between $\binom{r\,l\,f}{0\,1\,1}$, $\binom{l\,r\,f}{0\,1\,1}$ and $\binom{f\,r\,l}{0\,1\,1}$, becomes

$$((kr)^3 f) = -1/20\,(rl)^2 . f + 17/120\,(kl) . l + 3/20\,(fl)^2 . r + 1/5\,(Hl^2)^3. \tag{10}$$

The syzygy becomes, then,

$$3/10\,(rl) . (fl) - 1/10\,(fl^2)^4 . (fl) - 3/20\,f . l . (rl)^2 - 17/120\,(kl) . l^2$$
$$- 3/20\,(fl)^2 . r . l - 1/5\,l . (Hl^2)^3 + 1/10\,f(fl^3)^5 + 3/20\,f . r . C = 0.$$

The second syzygy comes directly from $(f_1 f_2 f_3) = 0$ on putting $f_1 = p$, $f_2 = k$, $f_3 = l$. That is,

$$p . (kl) + k . (lp) + l . (pk) = 0,$$

or

$$p . (kl) - k . (pl) - l . \beta = 0.$$

The third syzygy is found from $(f_1 f_2 f_3) = 0$, when $f_1 = k$, $f_2 = (kl)^2$, $f_3 = l$. It is

$$k . ((kl)^2 l) - (kl)^2 . (kl) - l . ((kl)^2 k) = 0.$$

$((kl)^2 l)$ is evaluated at once by $\binom{l\,k\,l}{1\,0\,2}$, which gives $((kl)^2 l) = (kl^2)^3$. $((kl)^2 k)$ is more complicated. First from $\binom{k\,l\,k}{0\,1\,2}$ comes $((kl)^2 k) = ((kk)^2 l)$.

Next express the groundform $(kk)^2$ differently. Namely $\binom{f\,f\,k}{0\,3\,3}$, $\binom{f\,l\,f}{0\,2\,2}$, $\binom{f\,l\,f}{0\,3\,1}$, together with (1), (2) and (5), give

$$(kk)^2 = -7/45\, k.l - 1/5\,(Hl)^2 - 2/5.(fr).$$

Then

$$((kk)^2\, l) = -7/45\,(kl, l) - 1/5\,((Hl)^2\, l) - 2/5\,((fr)\, l).$$

This, by the aid of $\binom{l\,k\,l}{0\,1\,0}$, $\binom{l\,H\,l}{1\,0\,2}$ and an elimination between $\binom{r\,l\,f}{0\,1\,1}$, $\binom{l\,r\,f}{0\,1\,1}$ and $\binom{f\,r\,l}{0\,1\,1}$, gives finally

$$((kl)^2\, k) = ((kk)^2\, l) = -1/15\, l.(kl) - 1/5\,(Hl^2)^3 + 1/5\, f.(rl)^2 - 1/5\, r.(fl)^2,$$

and the third syzygy is

$$k.(kl^2)^3 - (kl)^2.(kl) + 1/15\, l^2.(kl) + 1/5\, l.(Hl^2)^3 - 1/5\, f.l.(rl)^2 + 1/5\, r.l.(fl)^2 = 0.$$

The fourth syzygy results from $[f_1 f_2 f_3 f_4]_2 = 0$ by placing $f_1 = f$, $f_2 = r$, $f_3 = f_4 = l$. Namely,

$$2(fl)(rl) + f.r.(ll)^2 + (fr)^2.l^2 - f.l.(rl)^2 - (fl)^2.r.l = 0.$$

Here it is only necessary to put $(fr)^2 = -1/2\,(kl)$ from (6) to get as the fourth syzygy

$$2(fl).(rl) + f.r.C - 1/2\, l^2.(kl) - f.l.(rl)^2 - r.l.(fl)^2 = 0.$$

For the fifth sygyzy, in $[f_1 f_2 f_3 q]_3 = 0$ put $f_1 = f$, $f_2 = (fl)^2$, $f_3 = q = l$. There results

$$2(f, (fl)^2)(ll)^2 - 2(fl).((fl)^2\, l)^2 + 2((fl)^2\, l)(fl)^2 - l.((fl)^2, (fl)^2) + l(((fl)^2\, l)^2 f) + l^2 (f, (fl)^2)^3 = 0.$$

The unknown transvectants here are evaluated thus: From $\binom{f\,l\,f}{0\,1\,2}$ comes $(f, (fl)^2) = -(Hl)$. From $\binom{l\,f\,l}{0\,2\,2}$ is found $((fl)^2\, l)^2 = (fl^2)^4$. $\binom{l\,f\,l}{1\,0\,2}$ gives $((fl)^2\, l) = (fl^2)^3$. Obviously $((fl)^2, (fl)^2) = 0$. $(((fl)^2\, l)^2 f) = ((fl^2)^4 f)$ by means of $((fl)^2\, l)^2 = (fl^2)^4$ just obtained, and this with the aid of $\binom{f\,l^2\,f}{0\,1\,4}$ and $\binom{l\,l\,k}{0\,1\,0}$ is easily changed to $(((fl)^2\, l)^2 f) = 2(Hl^2)^3 + 2/3\, l.(kl)$.

$\binom{f\,l\,f}{0\,3\,2}$ at once shows that $(f, (fl)^2)^3 = -(kl)$.

These give for the fifth syzygy

$$2l.(Hl^2)^3 - 1/3\,l^2.(kl) + 2(fl)^2.(fl^2)^3 - 2(fl).(fl^2)^4 - 2C.(Hl) = 0.$$

The sixth syzygy is obtained from $[f_1 f_2 tq]_4 = 0$ by making $f_1 = f$, $f_2 = k$, $t = r$, $q = l$; namely,

$$-3f((kl)^2 r)^2 - 3k((fl)^2 r)^2 - 2r.l.(fk)^4 - 6(fk)^2.(rl)^2 + 3(fr)^2.(kl)^2 + 3(fl)^2.(kr)^2 + 3r((fl)^2 k)^2 + 3r.((kl)^2 f)^2 + 2l.((fr)^3 k) + 2.l.((kr)^3 f) = 0.$$

To get $((kl)^2 r)^2$ in terms of known transvectants use $\begin{pmatrix} k & l & r \\ 0 & 2 & 2 \end{pmatrix}$. It gives, with the aid of (9),

$$((kl)^2 r)^2 = ((rk)^2 l)^2 + 1/25\,(fl^3)^5 - 2/25\,l.(rl)^2 + 3/50\,r.C.$$

From $\begin{pmatrix} f & l & r \\ 0 & 2 & 2 \end{pmatrix}$ and (3) and (7) is obtained the expression

$$((fl)^2 r)^2 = -1/3\,(kl^2)^3 + 5/6\,(pl).$$

$(fk)^4$ has already been found in (2), namely $(fk)^4 = -1/5\,(fl)^2$, and $(fr)^2$ also in (6) as $(fr)^2 = -1/2\,(kl)$.

From $\begin{pmatrix} f & l & k \\ 0 & 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} l & f & l \\ 0 & 1 & 1 \end{pmatrix}$, together with (1) and (2), comes

$$((fl)^2 k)^2 = (\varepsilon l)^2 + 1/30\,(fl)^2.l - 2/45\,f.C.$$

To evaluate $((kl)^2 f)^2$ use $\begin{pmatrix} k & l & f \\ 0 & 2 & 2 \end{pmatrix}$, $\begin{pmatrix} l & f & l \\ 0 & 1 & 1 \end{pmatrix}$ and (1). Thus is found

$$((kl)^2 f)^2 = ((fk)^2 l)^2 - 1/6\,(fl)^2 l + 1/18\,C.f.$$

From (3), $((fr)^3 k) = 5/2\,pk - 1/10\,l^2 k$, which by use of $\begin{pmatrix} l & l & k \\ 0 & 1 & 0 \end{pmatrix}$ becomes

$$((fr)^3 k) = 5/2\,(pk) + 1/10\,l.(kl),$$

and from (10)

$$((kr)^3 f) = -1/20\,(rl)^2 f + 17/120\,(kl).l + 3/20\,(fl)^2.r + 1/5\,(Hl^2)^3.$$

These substituted in the syzygy above serve to express it in transvectants of the complete system thus:

$$-3/25\,f.(fl^3)^5 - 3f.(gl)^2 - 11/75\,f.r.C + 7/50\,f.l.(rl)^2 - 5\,l.\beta + 2/5\,l.(Hl^2)^3 + 3/10\,l.r.(fl)^2 + 29/60\,l^2(kl) + k.(kl^2)^3 - 5/2\,k.(pl) + 6r.(\varepsilon l)^2 + 3(fl)^2.g - 6\varepsilon.(rl)^2 - 3/2\,(kl)^2.(kl) = 0.$$

The seventh syzygy comes from making the same substitution, $f_1 = f$, $f_2 = k$, $t = r$, $q = l$, in $\{f_1 f_2 tq\}_4 = 0$. That makes it

$$2k.((fr)^3 l) - 3k.((fl)^2 r)^2 - r.l.(fk)^4 - 3(fk)^2.(rl)^2 + 3(fr)^2.(kl)^2 - 2(fl).(kr)^3 + 2r.((fl)^2 k)^2 + r.((kl)^2 f)^2 = 0.$$

The value $((fr)^3 l) = 5/2(pl)$ was found in (4), and from (8)

$$(kr)^3 = 1/10(fl^2)^4 - 3/10(rl).$$

The other unknown transvectants were all evaluated for the preceding syzygy. Putting in the values there obtained and these two gives, for the final form of this syzygy,

$$-1/30 f.r.C + 1/10 l.r.(fl)^2 + k(kl^2)^3 + 5/2 k(pl) + 3r.(\varepsilon l)^2 + 3/5(fl)(rl) - 1/5(fl)(fl^2)^4 - 3\varepsilon.(rl)^2 - 3/2(kl)^2.(kl) = 0.$$

That these seven syzygies are independent appears from the fact that the determinant of the coefficients of the terms in $f(gl)^2$, $r.(\varepsilon l)^2$, $p.(kl)$, $(fl)^2(fl^2)^3$, $(kl)^2(kl)$, $f.(fl^3)^5$, $f.l.(rl)^2$ does not vanish. Other sets of terms are also easily found that give a non-vanishing determinant. In this case also, therefore, Sylvester's table is defective, owing to another exception to the Fundamental Postulate.

The last case of disagreement which will be taken up is that of degorder 10.4. Owing to its occurrence in a sense relatively early in the table of groundforms, it is comparatively simple. Sylvester shows here three irreducible covariants, while von Gall shows four. These are $(Hl^4)^7$, $(\beta l^2)^4$, $(\lambda l)^2$, (μl). There are the following ten reducible covariants of this degorder:

(2.2)(8.2) [3], (3.3)(7.1) [3], (4.0)(6.4) [2], (5.1)(5.3) [2],

which are

$$l\left\{\begin{matrix}(kl^3)^5\\(pl^2)^3\\(\tau l)\end{matrix}\right. \quad r.\left|\begin{matrix}\alpha\\(fl^3)^6\\(rl^2)^3\end{matrix}\right. \quad C.\left\{\begin{matrix}(kl^2)^3\\(pl)\end{matrix}\right. \quad (rl)^2.\left\{\begin{matrix}(rl)\\(fl^2)^4\end{matrix}\right.$$

The value of $\Delta(7, 10; 4)$ is 13, which is therefore the number of linearly independent covariants of degorder 10.4. If then there exists any syzygy of this degorder among the ten reducible covariants enumerated, there must be $13 - 10 + 1 = 4$ groundforms, and as before the completeness of von Gall's system shows that the forms given may be taken as the four groundforms.

There is one such syzygy, namely,

$$r.(rl^2)^3 - l.(\tau l) - (rl)^2(rl) = 0.$$

For in $(f_1 f_2 f_4) = 0$ put $f_1 = (rl)^2$, $f_2 = r$, $f_3 = l$. Then

$$(rl)^2(rl) - r((rl)^2 l) + l((rl)^2 r) = 0.$$

But from $\begin{pmatrix} l & r & l \\ 1 & 0 & 2 \end{pmatrix}$ follows $((rl)^2 l) = (rl^2)^3$, and from $\begin{pmatrix} r & l & r \\ 0 & 1 & 2 \end{pmatrix}$, $((rl)^2 r) = ((rr)^2 l) = (\tau l)$, which give the above syzygy at once.

Here also, therefore, another exception to the Fundamental Postulate makes Sylvester's result incorrect.

3. *Some Exceptions to the Fundamental Postulate for Quantics of Order Higher than the Seventh.*

As already pointed out there has existed a suspicion that the exceptions to the Fundamental Postulate might possibly be confined to the seventhic. Such is not the case, as will be shown in the present section. In fact the exceptions in the cases of higher orders are fairly numerous.

Every quantic of odd order as great as 5 has at least one syzygy of degorder 8.8, at least two of degorder 9.7, at least one of degorder 9.9, and at least one of degorder 10.4. These syzygies produce exceptions to the Fundamental Postulate in the cases of the ninethic degorders 8.8, 9.7, 10.4; eleventhic degorders 8.8, 9.7, 9.9, 10.4; thirteenthic degorders 8.8, 9.7, 9.9, 10.4.

These two facts are proven thus. Every quantic of odd order as great as 3 has a groundform of degorder 2.2, and one of degorder 3.3. For the quadratic has an invariant (2.0), and it follows from Hermite's Law of Reciprocity that every quantic of order $2m+1$ has a covariant of degorder 2.2, which is evidently irreducible for $m > 0$. Again, the cubic has an invariant (4.0) and a covariant (3.3). Then, by Hermite's Law, any quantic of order $4m+1$ or $4m+3$, *i. e.*, $2m+1$, has a covariant of degorder 3.3, which is also evidently irreducible for $m > 0$. Further, from the fact that the quadratic has a quadratic invariant it follows by Hermite's Law that every quantic of order $2m+3$ has a covariant of degorder 2.6, which is easily seen to be irreducible if $m > 0$. Again, the cubic has a (reducible) covariant of degorder 3.5, one of degorder 5.5 and a quartic invariant. Therefore a quantic of order $4m+3$ or $4m+5$, *i. e.*, $4m+1$, or in general $2m+1$, has a covariant of degorder 3.5, and this is irreducible if the order of the quantic is as great as 5.

For convenience, for a quantic of odd order in general, denote the ground-form of degorder 2 2 by L, that of degorder 3.3 by R, that of degorder 2.6 by F, and that of degorder 3.5 by G.

If now, in $[f_1 f_2 qq']_4 = 0$, $f_1 = f_2 = F$, $q = q' = L$, it becomes a syzygy of degorder 8.8 for the quantic of odd order in general; namely,

$$2(FF)^2(LL)^2 + (FF)^4 L^2 - 2[(FL)^2]^2 - 4L((FL)^2 F)^2 + 2F.(FL^2)^4 = 0,$$

or, expressing the degorders involved,

$$2(4.8)(4.0) + (4.4)'(2.2)^2 - 2(4.4)^2 - 4(2.2)(6.6) + 2(2.6)(6.2) = 0,$$

which is a syzygy of degorder 8.8.

Two syzygies of degorder 9.7 are obtained thus. In $[f_1 f_2 f_3 q]_3 = 0$ put $f_1 = F$, $f_2 = R$, $f_3 = q = L$. This gives the syzygy

$$2(FR)(LL)^2 - 2(FL)(RL)^2 + 2(RL)(FL)^2 - L((FL)^2 R) + L((RL)^2 F) + L^2(FR)^3 = 0,$$

or

$$2(5.7)(4.0) - 2(4.6)(5.1) + 2(5.3)(4.4) - (2.2)(7.5) + (2.2)(7.5)' + (2.2)^2(5.3) = 0,$$

which is a syzygy of degorder 9.7. Again, in $[f_1 f_2 qq']_4 = 0$, the substitution $f_1 = F$, $f_2 = G$, $q = q' = L$ gives

$$2(FG)^2(LL)^2 + (FG)^4 L^2 - 2(FL)^2(GL)^2 - 2L((GL)^2 F)^2 - 2L((FL)^2 G)^2 + F(GL^2)^4 + G(FL^2)^4 = 0,$$

or

$$2(5.7)(4.0) + (2.2)^2(5.3) - 2(4.4)(5.3) - 2(2.2)(7.5) - 2(2.2)(7.5)' + (2.6)(7.1) + (3.5)(6.2) = 0.$$

This is evidently independent of the preceding and makes therefore two syzygies of degorder 9.7.

To get the syzygy of degorder 9.9 put $f_1 = F$, $f_2 = G$, $f_3 = q = L$ in $[f_1 f_2 f_3 q]_3 = 0$. This gives

$$2(FG)(LL)^2 - 2(FL)(GL)^2 + 2(GL)(FL)^2 - L((FL)^2 G) + L((GL)^2 F) + L^2(FG)^3 = 0,$$

or

$$2(5.9)(4.0) - 2(4.6)(5.3) + 2(5.5)(4.4) - (2.2)(7.7) + (2.2)(7.7)' + (2.2)^2(5.5) = 0.$$

The syzygy of degorder 10.4 may be obtained from $(f_1 f_2 f_3) = 0$ by making $f_1 = (RL)^2$, $f_2 = R$, $f_3 = L$, or directly from the syzygy found for the seventhic case 10.4. For the result found there depends only on the degorders of l and r, and not at all on the order of the quantic. But the degorders of L and R are precisely those of l and r. Then it gives a syzygy true for the general quantic of odd order to replace l by L and r by R. This gives

$$R(RL^2)^3 - L((RR)^2 L) - (RL)^2 (RL) = 0,$$

or

$$(3.3)(7.1) - (2.2)(8.2) - (5.1)(5.3) = 0,$$

which is the general syzygy of degorder 10.4.

It now remains to show that these syzygies produce exceptions to the Fundamental Postulate. Sylvester's table for the ninethic shows two groundforms of degorder 8.8, two of degorder 9.7, and fourteen of degorder 10.4, and these numbers are certainly minima. In these three cases therefore, since syzygies of these degorders have just been found, there exist together groundforms and fundamental syzygies, *i. e.*, exceptions to the Fundamental Postulate.

Further, a computation* of the groundforms of the eleventhic and the thirteentic by the method of tamisage shows a positive number of them for degorders 8.8, 9.7, 9.9 and 10.4 in both cases. There are therefore exceptions to the Fundamental Postulate in all these cases, in fact double exceptions for degorder 9.7.

So far all exceptions to the Fundamental Postulate have been confined to quantics of odd order. It is the purpose of the present paragraph to show that

* As the author is not aware where such results as these are to be found if they have been previously computed, he ventures to append them as far as he has found it desirable to carry them. They were calculated by "crude" or "brute" tamisage, which for a limited field costs less labor than the more complete reduction of the Generating Function to the Representative Form. The minus (written above for compactness) indicates syzygies.

TABLES OF GROUNDFORMS, AS GIVEN BY TAMISAGE, TO DEGREE 10, ORDER 10.

Eleventhic.

	0	1	2	3	4	5	6	7	8	9	10
1											
2			1				1				1
3				1		1		1		2	
4	2				3		2		4		3
5		2		4		5		7		7	
6			7		7		11		12		11
7		8		12		18		18		17	
8	10		17		25		28		19		16
9		21		37		37		24		7	
10	13		45		52		19		$\overline{8}$		$\overline{104}$

Thirteenthic.

	0	1	2	3	4	5	6	7	8	9	10
1											
2			1				1				1
3				1		1		1		2	
4	2				4		2		5		4
5		3		5		7		9		11	
6			11		11		18		20		20
7		14		21		32		36		39	
8	19		33		51		62		54		58
9		45		87		94		84		62	
10	33		112		145		100		57		$\overline{100}$

Fourteenthic.

	0	2	4	6	8	10
1						
2	1		1		1	
3		1		2	1	2
4	2		4	2	5	4
5		5	6	9	10	12
6	7	6	15	19	22	25
7	4	22	31	37	47	42
8	18	36	58	74	67	75
9	23	73	108	119	112	67
10	52	129	179	175	72	$\overline{40}$

there are exceptions to the Postulate in the case of quantics of even order. It will be shown first, that there is in general a syzygy of degorder 10.6 for every quantic of order $4m+2$ $[m>0]$, and second, that this produces exceptions to the Fundamental Postulate in the cases of the tenthic and of the fourteenthic. The cubic has an invariant of degree 4 and a covariant of degorder 2.2. It follows by Hermite's Law that any quantic of order $4m+2$ has a covariant (3.2), and this is necessarily irreducible if $m>0$. Similarly, since the quadratic has a quadratic invariant, every quantic of even order has a covariant (2.4). Denote the covariant (3.2) by Q and (2.4) by P. Now in $(f_1 f_2 f_3)=0$ put $f_1=(PQ)^2$, $f_2=P$, $f_3=Q$. Then

$$(PQ)^2(PQ)-P((PQ)^2Q)+Q((PQ)^2P)=0;$$

or, since from $\begin{pmatrix} Q & P & Q \\ 1 & 0 & 2 \end{pmatrix}$, $((PQ)^2Q)=(PQ^2)^3$, and from $\begin{pmatrix} P & Q & P \\ 0 & 1 & 2 \end{pmatrix}$, $((PQ)^2P)=((PP)^2Q)$, this may be written, perhaps a little more neatly, as

$$(PQ)^2(PQ)-P(PQ^2)^3+Q((PP)^2Q)=0,$$

or, expressing the degorders involved,

$$(5.2)(5.4)-(2.4)(8.2)+(3.2)(7.4)=0,$$

which is a syzygy of degorder 10.6.

The tenthic is a special case of this. Sylvester's table for the tenthic shows ten groundforms of degorder 10.6. There is therefore an exception to the Fundamental Postulate in this case. Again, the method of tamisage applied to the fourteenthic gives 175 groundforms of degorder 10.6 (see foot-note, p. 65), so that here too is an exception.

It is easy to show that the twelfthic has at least one exception to the Fundamental Postulate. For it has groundforms (2.4) and (2.8), as may be readily ascertained by Hermite's Law or by reference to Sylvester's table for the twelfthic.* Let the quantic be represented by f, the covariant (2.4) by u and (2.8) by v. Then make $f_1=f$, $f_2=u$, $f_3=v$, in $(f_1 f_2 f_3)=0$. That is,

$$f(uv)+u(vf)+v(fu)=0,$$

or

$$(1.12)(4.10)+(2.4)(3.18)+(2.8)(3.14)=0,$$

which is a syzygy of degorder 5.22. Now Sylvester's table (*loc. cit.*) shows four groundforms of degorder 5.22. Consequently there is here also an exception to the Postulate.

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* Sylvester, AMERICAN JOURNAL, Vol. IV, p. 48.

With regard to the syzygies in this section, where some forms occur which might be reducible, one precaution is necessary, namely, to be sure that the syzygies are actually such, not merely identities. Take first the syzygy of degorder 8.8 belonging to quantics of odd order. As already pointed out, the forms F and L are irreducible. So also is the form $(LL)^2$, for it is of degorder 4.0 and no quantic of odd order can have a quadratic invariant. Therefore the only way in which this syzygy might reduce to an identity would be

$$2\,(2.2)\,(2.6)\,(4.0) + (4.4)'\,(2.2)^2 - 2\,(4.4)^2 - 4\,(2.2)\,(2.6)\,(4.0) + 2\,(2.6)\,(2.2)\,(4.0) = 0.$$

But this requires

$$(4.4)'\,(2.2)^2 = 2\,(4.4)^2,$$

or, since the right-hand member is a square (barring the numerical constant) and so is the second factor on the left, $(4.4)'$ must be a square. But since there is but one form (2.2), namely L, this gives $(4.4)'$ (and hence (4.4) also) = a numerical multiple of L^2. Referring back to the transvectant values of these symbols, this is

$$(FF)^4 = \lambda_1 \,.\, L^2, \text{ and } (FL)^2 = \lambda_2 \,.\, L^2$$

(where λ_1 and λ_2 are numerical multiples), which are not true if the order of the quantic is as great as 5. The syzygy then can not reduce to an identity.

The syzygies of degorder 9.7 belonging to quantics of odd order are simpler. In the first, the second term (4.6) (5.1) can not be reduced. For the factor (4.6) is irreducible because a quantic of odd order can not have a form (2.4) nor (3.1). (This appears at once from Hermite's Law applied to the quadratic and cubic respectively.) (5.1) is also plainly irreducible. This term then can not go out identically and this is an actual syzygy. Similarly in the second syzygy of this degorder the next to the last term (2.6) (7.1) is plainly irreducible, and therefore the syzygy can not reduce to an identity.

In the syzygy of degorder 9.9 for quantics of odd order the second term (4.6) (5.3) is irreducible into factors of lower degorder, and can not cancel; so the syzygy can not be an identity.

In the same way the last syzygy for odd quantics, of degorder 10.4, is easily seen to be an actual syzygy.

The syzygy proven for quantics of order $4m + 2$ was

$$(5.2)\,(5.4) - (2.4)\,(8.2) + (3.2)\,(7.4) = 0.$$

The quantic of order $4m + 2$ can not have a cubic invariant, for the cubic has only one invariant, degree 4, so by Hermite's Law only a quantic of order $4m$

can have a cubic invariant. Also a quantic of even order can not have a covariant (2.2) nor one of degorder 4.2. The possible decompositions, then, of the above relation are:

$$\left.\begin{matrix}(2.4)\,[(5.0)\,(3.2)]\\ \text{or}\ (2.4)\,[(2.0)\,(6.2)]\end{matrix}\right\} = \left\{\begin{matrix}(3.2)\,[(5.0)\,(2.4)]+[(3.2)\,(2.0)]\,(5.4)\\ \text{or}\ (3.2)\,[(2.0)\,(5.4)]+[(3.2)\,(2.0)]\,(5.4)\end{matrix}\right.$$

In any one of the four possible combinations of the two members it is to be understood that factors included in brackets [] may occur as a single irreducible one, and that a reduction into factors may introduce numerical multiples which are not shown. It may be remarked that the first possibility on each side can occur only if the order of the quantic is $4m+18\lambda\,[m\geqq 0,\ \lambda>0]$. For since the quintic has irreducible invariants of degrees 4, 8, 12, 18, only, to have a quintic invariant a quantic must be of order $4\lambda_1+8\lambda_2+12\lambda_3+18\lambda_4$, *i. e.*, of order $4m+18\lambda$. This number can not coincide with $4m+2$, the order of the quantic under discussion, unless $\lambda>0$. However, in any case the syzygy can not reduce to an identity.

The syzygy found for the twelfthic is also obviously not an identity.

4. *Conclusion.*

It appears then that exceptions to the Fundamental Postulate of Tamisage are much more widespread than the few cases pointed out by Hammond. They are not confined to the seventhic, nor to quantics of odd order. Indeed, if we divide quantics into three classes, according as the orders are $4m$, $4m+2$, or $2m+1$, we may say very roughly that in the tables of groundforms and syzygies, there is a region of syzygies that is nearly fixed; *i. e.*, that changes but little as the order of the quantic is increased within the class. (For odd quantics this region is likely to begin when the sum of the degree and order exceeds 14.) On the other hand, the region of groundforms in the tables appears to extend continuously as the order of the quantic grows greater in its class. When these two regions overlap, exceptions to the Fundamental Postulate occur. This begins to be the case for odd quantics when the order is as great as 7, for even quantics when it is as great as 10. There seems to be no reason for supposing that Sylvester's method will ever become exact again for quantics of a sufficiently high finite order. Sylvester's method remains valuable only as furnishing a minimum number of groundforms. It is likely to be exact for a quantic of high order only when the sum of the degree and the order of the case in question is relatively small.

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Derivative Domains of Rationality.

By S. Epsteen.

Introduction.

In this paper I have attempted to outline a theory of derivative domains of rationality and to give a few applications to linear homogeneous differential equations.

§ 1, 3°, is based on a paper of E. H. Moore; the points of difference being (a) the function theoretic domains may be founded on any of the five precisely defined number domains instead of only one, the domain of rationals; (b) the assumptions regarding the functions are such as to lead to Fuchs' existence theorem for linear homogeneous differential equations; and (c) it is shown how a domain may be enlarged by the adjunction of *operations* as well as *functions*. § 2 is based on the work of Frobenius, Koenigsberger and Loewy. The irreducibility and reducibility definitions of Frobenius and Koenigsberger overlap when the latter are applied to linear homogeneous equations, and this leads to confusion. The discriminating notation of § 2, 1° and 2°, removes the difficulty; in § 2, 3°, this is extended to Loewy's relative irreducibility and relative reducibility, and concrete illustrations are given.

In all cases when theorems due to others are used or restated from the new standpoint, due credit is given in the foot-notes to the respective authors. The remaining portions of the paper (excepting a few points of common knowledge, such as § 1, 1° and 5°) I believe to be new.

It may be well to call the attention of the reader to the analogies with Algebra. Indeed, the Normal Domains (§ 3) and the Index of an Equation (§ 4) were originally suggested in this manner.

§ 1. Construction of the Domains.

1°. *The Numbers of the Domains.* A set of numbers is called a *domain (realm, field) of rationality*, or briefly, a *domain*, when all the sums, differences,

products and quotients (division by zero excepted) are numbers belonging to the set. All rational numbers (the positive and negative integers and fractions) constitute a domain which we shall designate by the black-faced letter **r**. The integers by themselves do not form a domain, because the quotient of two integers may be fractional. The positive numbers by themselves do not form a domain, because the difference of two positives may be negative.

The numbers of the domain **r**, together with $i = \sqrt{-1}$ ($a_1 + ib_1$, where a_1 and b_1 belong to **r**), form a domain **c**.

The totality of real numbers forms a domain **R**, the *domain of reals*. The numbers of **R**, together with $i = \sqrt{-1}$ ($A_1 + iB_1$, where A_1 and B_1 belong to **R**), form a domain **C**, the *complex domain*.

Since the sum, difference, product and quotient of two algebraic numbers* is an algebraic number, the totality of algebraic numbers forms a domain **A**.

The above five domains are given as illustrations; they are not exhaustive, since other domains may be constructed.

2°. *The Functions of the Domains.* Throughout this paper it will be assumed that all the functions of the complex variable $x = x_1 + ix_2$, which we consider, are analytic and meromorphic in a certain common finite region of existence X of the x-plane, the poles being isolated points.† These two assumptions are made in order that we may be able to assert with Fuchs‡ that every linear homogeneous differential equation

$$K(y) \equiv \frac{d^n y}{dx^n} + k_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + k_n(x)\,y = 0,$$

with coefficients in the given domain, has a fundamental system of integrals $y_1 \cdots y_n$, every one of whose elements is holomorphic and analytic in X except at isolated points, and is determined by its value and the values of its first $n-1$ derivatives at any arbitrary regular point $\bar{x}$ of X.

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* An algebraic number is the root of an algebraic equation; *i. e.*, of an equation in integral powers of a single unknown with coefficients belonging to **r**.

† The totality of these singularities does not have to be a system of isolated points; the requirement is merely that the singular points of any one function be isolated.

‡ L. Fuchs: "Zur Theorie der linearen Differentialgleichungen mit veränderlichen Coefficienten," *Journal für reine und angewandte Mathematik*, Vol. LXVI (1866), pp. 121–160.

3°. *Function-Theoretic Domains.** Take any m functions

$$u_j = u_j(x), \qquad (j = 1, \ldots, m),$$

and consider the totality of rational functions of these, the coefficients being numbers of $\mathbf{C}$.† This totality constitutes the *function-theoretic domain of rationality*

$$\mathbf{C}\langle x\rangle \equiv \mathbf{C}[u_1 \ldots u_m].$$

We say that $\mathbf{C}\langle x\rangle$ is obtained from $\mathbf{C}$ by the *adjunction* of the functions $u_1 \ldots u_m$.

Certain functions of a domain, without vanishing formally, that is, for $u_1 \ldots u_m$ as indeterminates, vanish identically as functions of x; these equations are the *identities* of the domain. For example, the expression $1 - u_1^2 - u_2^2$ does not vanish when u_1 and u_2 are regarded as indeterminates, but it becomes identically zero for $u_1 = \sin x$, $u_2 = \cos x$.

If in addition to the m functions u_j of $\mathbf{C}\langle x\rangle$ certain M additional functions $U_J(x)$ are introduced and these $m + M$ functions are utilized for the construction of a new domain $\bar{\mathbf{C}}\langle x\rangle$, then $\bar{\mathbf{C}}\langle x\rangle$ contains $\mathbf{C}\langle x\rangle$ as a *subdomain.* One sees easily that $\mathbf{r}$ is a subdomain of $\mathbf{c}$ and of $\mathbf{R}$; and that $\mathbf{r}$, $\mathbf{c}$ and $\mathbf{R}$ are all subdomains of $\mathbf{C}$.

As before, it is said that $\bar{\mathbf{C}}\langle x\rangle$ is obtained from $\mathbf{C}\langle x\rangle$ by the *adjunction* of the M functions U_J, or, $\bar{\mathbf{C}}\langle x\rangle$ is obtained from $\mathbf{C}$ by the adjunction of the $m + M$ functions u_j, U_J.

If to the domain $\mathbf{C}\langle x\rangle$ the first derivatives

$$\frac{du_j}{dx} = u_j', \qquad (j = 1, \ldots, m),$$

of the functions $u_j(x)$ are adjoined, the *first derivative domain* is obtained,

$$\mathbf{C}^{(1)}\langle x\rangle \equiv \mathbf{C}^{(1)}[u_1 \ldots u_m] \equiv \mathbf{C}[u_1 \ldots u_m, u_1' \ldots u_m'];$$

and similarly, by adjunction of all derivatives of order 1, 2, ..., h, the h-th derivative domain

$$\mathbf{C}^{(h)}\langle x\rangle \equiv \mathbf{C}^{(h)}[u_1 \ldots u_m],$$

and thus an infinite succession of derivative domains ($h = 1, 2, \ldots$) is obtained.

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* E. H. Moore: "Concerning Transcendentally Transcendental Functions," *Mathematische Annalen*, Vol. XLVIII.

† For some investigations the original number domain might well be $\mathbf{r}$, $\mathbf{c}$, $\mathbf{R}$ or $\mathbf{A}$. It is an interesting problem, in any specific investigation, to define the smallest domain in which the results are valid.

If the m first derivatives $u_1' \ldots u_m'$ belong to the original domain $\mathbf{C}\langle x\rangle$, then the derivative of every function of $\mathbf{C}\langle x\rangle$ belongs to $\mathbf{C}\langle x\rangle$ and the original domain is identical with the first derivative domain, and, indeed, with every derivative domain $\mathbf{C}^{(h)}\langle x\rangle$, $(h = 1, 2, \ldots)$. In this case the domain $\mathbf{C}\langle x\rangle$ is said to be *perfect.*

One should distinguish between $\mathbf{C}\langle x\rangle$ and $\mathbf{C}[x]$. The former contains functions of x, but does not necessarily contain the independent variable x itself; the latter is the domain $\mathbf{C}$ with x adjoined, *i. e.*, the domain of rational functions of x with coefficients belonging to $\mathbf{C}$. In a sense, $\mathbf{C}[x]$ may be regarded as a special case of $\mathbf{C}\langle x\rangle$.

The following are examples of perfect domains:

a) $\mathbf{R}$; b) $\mathbf{C}$; c) $\mathbf{C}[x]$; d) $\mathbf{R}[x, e^x]$; e) $\mathbf{C}[x, e^x]$;
f) $\mathbf{R}[x, \log x]$; g) $\mathbf{C}[x, \log x]$.

In the applications we shall consider none but perfect domains. Should it be desired to transform the independent variable of the equation $K(y) = 0$ from x to ξ by means of

$$x = x(\xi), \text{ or } \xi = \xi(x),$$

the perfect domain $\mathbf{C}\langle x\rangle \equiv \mathbf{C}[u_1 \ldots u_m]$ is transformed into

$$\mathbf{C}\langle x(\xi)\rangle \equiv C_1\langle \xi\rangle \equiv \mathbf{C}[g_1(\xi) \ldots g_m(\xi)],$$

where

$$g_j(\xi) = u_j(x(\xi)) = u_j(x).$$

In order that $C_1\langle x\rangle$ shall also be a perfect domain, it is necessary and sufficient that $d\xi/dx$ shall belong to $\mathbf{C}\langle x\rangle$, or, what amounts to the same thing, $dx/d\xi$ belong to $\mathbf{C}\langle \xi\rangle$.

A domain may be enlarged not only by the adjunction of functions, but also by the adjunction of *operations* (*processes*). Thus, having given a domain $\mathbf{C}\langle x\rangle$, the adjunction of the operation of differentiation means that the original functions $u_1 \ldots u_m$ must be such as to possess derivatives, and that the derivative of every function is in the domain.

4°. *Theory of Adjunction.* Consider the quadratic equation $ax^2 + bx + c = 0$ with coefficients in the domain $\mathbf{r}$. For brevity, we say the equation is in $\mathbf{r}$. Ordinarily, the equation is not solvable in this domain. The simplest irrationality that need be adjoined is the solution of $x^2 = \Delta_1$, where Δ_1 is any number in $\mathbf{r}$. In the domain $\mathbf{r}[\Delta_1^{\frac{1}{2}}]$ all quadratic equations are solvable.

Ordinarily, cubic equations with coefficients in $\mathbf{r}$ are not solvable in $\mathbf{r}$ or $\mathbf{r}[\Delta_1^{\frac{1}{3}}]$. The simplest irrationality that need be adjoined is the solution of $x^3 = \Delta_2$, where Δ_2 is any number in $\mathbf{r}[\Delta_1^{\frac{1}{3}}]$. In the domain $\mathbf{r}[\Delta_1^{\frac{1}{3}}, \Delta_2^{\frac{1}{3}}]$ all cubic equations are solvable.

All quartic equations in $\mathbf{r}$ are solvable in $\mathbf{r}[\Delta_1^{\frac{1}{3}}, \Delta_2^{\frac{1}{3}}]$, but quintic equations are not. It is known that the simplest irrationality which need be adjoined to render quintic equations solvable is the icosahedron irrationality.*

The question now arises, how much knowledge of a new irrationality is needed to make it known to the extent that it may be adjoined to (included in) the domain? The reply to this is a relative one, and can be gathered from some concrete illustrations. Let the reader think of a hypothetical person who is thoroughly familiar with the domain of rational numbers $\mathbf{r}$, and who is unacquainted with the extraction of square roots. In order that this person may adjoin to the domain $u = \sqrt{a}$, where a is a positive number in $\mathbf{r}$, he would have to learn (1°) some definition of u, and from this (2°) to expand u in a convergent series $a_0 + a_1/10 + a_2/10^2 + \ldots$. (*i. e.*, learn to approximate it in $\mathbf{r}$ to any assigned degree of accuracy), (3°) to perform on it the operations of $\mathbf{r}$, such as $u + u$, $u \times u$, etc., (4°) to combine it with the numbers $\alpha, \beta, \ldots$, of $\mathbf{r}$, as for instance, αu, $(\alpha u + \beta)/(\gamma u + \delta)$, $\alpha u^2 + \beta u + \gamma$. In time, the person in question would be so familiar with the square root process that he could adjoin it (or the square root of any definite positive number a) to the domain $\mathbf{r}$.

Suppose this hypothetical person is to be taught the trigonometric functions until he can (if he so chooses) adjoin any of them, such as $\sin x$, to the domain. He would have to learn (1°) some definition of each of these functions, and from these (2°) to expand them in infinite series; (3°) the addition theorems; (4°) the algebraic relations between these functions, such as $\sin^2 x + \cos^2 x = 1$; (5°) the differential coefficients of these functions.† In brief, after a full course in Trigonometry, he could adjoin $\sin x$ to the domain of rationality, if he so desired.

Following out this thought, a person familiar with the Elliptic Functions may adjoin them to the domain of rationality in which he is solving a given (algebraic or differential) equation. In general, when we encounter a class of algebraic or differential equations giving rise to a new irrationality, we must familiarize ourselves with this new function sufficiently, and then equations

* F. Klein: "Vorlesungen über das Ikosaeder," Leipzig, Teubner (1884).

† A knowledge of the derivatives of the functions would be necessary on account of our assumption that all the domains under consideration are perfect.

dependent on it may be regarded as solved if the functional dependence is known.

5°. *Irrationality.* Summarizing the foregoing, the derivative domains of rationality used in this paper, and designated frequently by $\mathcal{D}\langle x\rangle$, will consist of some number domain (usually **C**) to which any arbitrary functions, fulfilling the conditions of § 1, 2°, are adjoined. Every sum, difference, product, quotient (excepting the zero-divisor) and derivative will give an element of the domain. When equations are under consideration, the domain is enlarged by adjunction until it contains every coefficient at least.

An algebraic equation or an algebraic differential equation all of whose coefficients are elements of a given (derivative) domain $\mathcal{D}\langle x\rangle$ is said to *belong to, lie in,* or *be an equation in* $\mathcal{D}\langle x\rangle$.

A function (or number) $f(x)$ which lies in $\mathcal{D}\langle x\rangle$ is *rational with respect to this domain*; a function which does not lie in $\mathcal{D}\langle x\rangle$ is *irrational with respect to this domain.* Irrational functions are of two kinds: *algebraic* if they satisfy some algebraic equation in $\mathcal{D}\langle x\rangle$; *transcendental* if they do not satisfy such an equation. Thus, e and π are transcendental (numbers) with respect to the (number) domains **r**, **c**. Transcendental functions are of two kinds: *algebraically transcendental* with respect to* $\mathcal{D}\langle x\rangle$ if they satisfy some algebraic differential equation in this domain, and *transcendentally transcendental* if they do not satisfy such an equation. Thus e^x is algebraically transcendental with respect to $\mathbf{r}[x]$, being a solution of $y' - y = 0$, while the well-known gamma function, $\Gamma(x)$, is transcendentally transcendental with respect to the same domain.†

§ 2. The Concepts of Irreducibility and Reducibility in Prescribed Domains.

1°. *Frobenius' Irreducibility and Reducibility.* If a linear homogeneous differential equation in a domain $\mathcal{D}\langle x\rangle$ of order n, $K(y) = 0$, does not have an integral in common with another linear homogeneous differential equation in $\mathcal{D}\langle x\rangle$ of order $n_1 < n$, we shall call it *f-irreducible* (*f-i*) *in* $\mathcal{D}\langle x\rangle$. An equation which is not *f-i* in $\mathcal{D}\langle x\rangle$ is *f-reducible*‡ (*f-r*).

That the irreducibility or reducibility of an equation depends on the domain of rationality is easily illustrated. Thus, $y'' + y = 0$ is reducible in the domain **C**,

* E. H. Moore, *loc. cit.*, p. 52.

† O. Hölder: "Ueber eine Eigenschaft der Gammafunction keiner algebraischen Differentialgleichung zu genügen," *Mathematische Annalen*, Vol. XXVIII (1887), pp. 1–13.

‡ After Frobenius, *Crelle's Journal für reine und angewandte Mathematik*, Vol. LXXVI.

because the integral $y_1 = e^{ix}$ satisfies the equation (in **C**) $y' - iy = 0$; however, the equation is irreducible in **R**.

There are many theorems on reducible differential equations analogous to well-known theorems on reducible algebraic equations, but occasionally the analogy breaks down. For instance, the fact that every root of a (reducible) algebraic equation satisfies *some* irreducible algebraic equation of lower degree has no counterpart. An integral of a reducible differential equation does not necessarily satisfy some irreducible equation of lower order. Thus, the equation $y''' - 4y'' + 5y' - 2y = 0$ is reducible in **R**; its integrals $y_1 = e^{2x}$, $y_2 = e^x$ satisfy the irreducible equations $y' - 2y = 0$, $y' - y = 0$, but the integral $y_3 = xe^x$ does not satisfy any equation irreducible in **R** and of order lower than 3. Indeed, the equation of lowest order which it satisfies is a reducible one:

$$y'' - 2y' + y = 0.$$

A differential equation with the special property of possessing a fundamental system of integrals $y_1 \ldots . y_n$ each of which satisfies an irreducible equation, has been called *completely reducible.**

2°. *Koenigsberger's Irreducibility and Reducibility.* We will call an algebraic differential equation of order n and degree d,

$$E_{nd} \equiv e_0(x\,y\,y' \ldots . y^{(n-1)})(y^{(n)})^d + e_1(x\,y\,y' \ldots . y^{(n-1)})(y^{(n)})^{d-1} + \ldots . + e_d(x\,y\,y' \ldots . y^{(n-1)}) = 0,$$

prime in $\mathfrak{D}\langle x\rangle$, if, considered as an integral function of $y\,y' \ldots . y^{(n)}$, it is algebraically irreducible in $\mathfrak{D}\langle x\rangle$; that is, if it is not the product of integral functions of the same arguments with coefficients in $\mathfrak{D}\langle x\rangle$. The same thought may be restated in the following equivalent manner:

a) Since each integral function of $y, y', \ldots . , y^{(n-1)}$ contained in E_{nd} must be a common divisor of $e_0, e_1, \ldots . , e_d$, the coefficients $e_0, \ldots . , e_d$ can not have a common factor; and

b) E_{nd} considered as a polynomial in $y^{(n)}$ must be algebraically irreducible.

An algebraic differential equation $E_{nd} = 0$ in $\mathfrak{D}\langle x\rangle$, which is prime and does not have an integral in common with another algebraic differential equation of order $n_1 < n$ in $\mathfrak{D}\langle x\rangle$ is said to be *k-irreducible*† (*k-i*) *in* $\mathfrak{D}\langle x\rangle$. An equation which is not *k-i* in $\mathfrak{D}\langle x\rangle$ is *k-reducible* (*k-r*) in this domain.

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*This concept is due to A. Loewy, *Mathematische Annalen*, Vol. LXII, who uses the term *vollständig reduzibel*. This memoir contains a number of important propositions on completely reducible equations.

†After Koenigsberger: "Allgemeine Untersuchungen aus der Theorie der Differentialgleichungen," Leipzig, Teubner (1882).

Koenigsberger's definition of reducibility, applied to a linear homogeneous differential equation, becomes: a linear homogeneous differential equation of order n is k-r if it has an integral in common with any algebraic differential equation of order $n_1 < n$, both equations lying in $\mathfrak{D}\langle x\rangle$. In particular, an equation having an algebraic integral is reducible, because the algebraic functions satisfy a differential equation of zero order (*i. e.*, an algebraic equation).

A linear homogeneous equation may be f-i and k-r at the same time. Thus, the equation $y'' + y = 0$ in the domain $\mathbf{R}$ is f-i but it is k-r, having the integral $y_1 = \sin x$ in common with $(y')^2 + y^2 - 1 = 0$.

An equation which is f-r is also k-r; an equation which is k-i is surely f-i.

It should be noticed that f-r and f-i never apply to non-linear equations,* but k-r and k-i apply to both linear and non-linear equations.

When dealing with linear homogeneous equations, we shall say, briefly, that they are irreducible or reducible, meaning thereby f-i and f-r. When k-i and k-r are intended in connection with such equations, it will be specifically stated.

3°. *Relative Irreducibility and Reducibility.*† An algebraic differential equation $E_{nd}(y) = 0$ in $\mathfrak{D}\langle x\rangle$ is said to be *k-irreducible (k-i) relative to any one of its integrals* y_1 if it is prime in the domain and if y_1 does not satisfy an algebraic equation in $\mathfrak{D}\langle x\rangle$ of order $n_1 < n$. A linear homogeneous equation is said to be *f-irreducible (f-i) relative to any one of its integrals* y_1 if y_1 does not satisfy a linear homogeneous equation of lower order and in the same domain. In the contrary cases, we say, respectively, that the equations are k-r or f-r relative to the integral y_1.

In the domain $\mathbf{R}$ the equation

$$y'' - 2y' + y = 0$$

is f-r relative to $y_1 = e^x$ but it is f-i relative to $y_2 = xe^x$. The equation

$$(1 + x^2)\,y'' + 2xy' - 2y = 0$$

in the domain $\mathbf{R}[x]$ is f-r relative to $y_1 = x$, since this integral satisfies $xy' - y = 0$; it is f-i relative to $y_2 = 1 + x \arctan x$; however, it is k-r relative to y_2, since this function satisfies the non-homogeneous equation

$$x(1 + x^2)\,y' - (1 + x^2)\,y + 1 = 0.$$

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* One could, of course, call a non-linear equation f-r if it has an integral in common with a linear homogeneous equation of lower order; otherwise f-i. But I know of no application of this extension of the concept.

† A. Loewy: "Algebraische und gruppentheoretische Untersuchungen aus dem Gebiete der Differentialgleichungen," *Monatshefte für Mathematik und Physik*, Vol. VIII (1897), pp. 227–234.

A non-linear equation of the first order, which has an algebraic and a transcendental integral, is reducible relative to the former, which satisfies an algebraic equation (a differential equation of order zero), and is irreducible relative to the transcendental integral.

If $E_{nd}(y) = 0$ in $\mathcal{D}\langle x\rangle$ is k-i relative to an integral y_1, we say that y_1 is a *superior integral** of the equation in this domain. An integral which satisfies

$$\frac{\partial E}{\partial y^{(n)}} = 0$$

is called *singular*. A non-singular integral which is not superior is called an *inferior* integral.

4°. *Koenigsberger's Theorem.*† The following theorem, due to Koenigsberger, is restated without proof in the terminology of this section. An application will be made in § 3.

THEOREM I. *If a superior integral of an equation in* $\mathcal{D}\langle x\rangle$, $E_{nd}(y) = 0$, *satisfies any algebraic differential equation also in* $\mathcal{D}\langle x\rangle$, $F_{vd_1}(y) = 0$, *where* $v > n$, *then all the non-singular integrals of* $E = 0$ *satisfy* $F = 0$.

§ 3. NORMAL DOMAINS.

If the linear homogeneous differential equation

$$K(y) \equiv y^{(n)} + k_1(x) \cdot y^{(n-1)} + \ldots + k_n(x) \cdot y = 0$$

is irreducible in $\mathcal{D}\langle x\rangle$, it follows that none of its integrals belong to $\mathcal{D}\langle x\rangle$. For, if an integral y_1 belongs to $\mathcal{D}\langle x\rangle$, then $\eta = y_1'/y_1$ also belongs to this domain, and y_1 satisfies an equation in $\mathcal{D}\langle x\rangle$,

$$H(y) \equiv \frac{dy}{dx} - \eta y = 0,$$

of the first order. This is impossible if $K(y) = 0$ is irreducible.

In case an integral y_1 does belong to $\mathcal{D}\langle x\rangle$, and, therefore, satisfies the equation $H(y) = 0$, the equation $K = 0$ is reducible, and can be factored (symbolically in the sense of Boole) into

$$K(y) \equiv G\{H(y)\}$$

---

* Superior integrals correspond to primitive roots in Algebra. I prefer this term in connection with differential equations, since "primitive" is used in § 3 in a different sense.

† L. Koenigsberger, *loc. cit.* Also A. Loewy, "Algebraische und Gruppentheoretische Untersuchungen," etc., *loc. cit.*, p. 231.

(written for brevity $GH(y)$), where

$$G \equiv \frac{d^{n-1}}{dx^{n-1}} + g_1 \frac{d^{n-2}}{dx^{n-2}} + \cdots + g_{n-1},$$

the $g_1 \ldots g_{n-1}$ belonging to $\mathcal{D}\langle x\rangle$. This is easily proved by calculation from the identity $K(y) = GH(y)$,

$$k_1 = g_1 - \eta, \qquad k_2 = g_2 - g_1\eta - (n-1)\eta',$$

$$k_3 = g_3 - g_2\eta - (n-2)g_1\eta' - \frac{(n-1)(n-2)}{2!}\eta'',$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

whence

$$g_1 = k_1 + \eta, \qquad g_2 = k_2 + k_1\eta + \eta^2 - (n-1)\eta',$$

$$g_3 = k_3 + k_2\eta + k_1(\eta^2 - \eta') + \eta^3 + n\eta\eta' + \frac{(n-1)(n-2)}{2!}\eta^3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

and it is thus seen that $g_1, g_2, \ldots, g_{n-1}$, being rational in terms of $k_1 \ldots k_n$, $\eta, \eta', \eta'', \ldots$, must lie in $\mathcal{D}\langle x\rangle$.

In general, if $K(y)$ is reducible, it can be factored symbolically into at least two factors $K = L(M)$,

$$L(y) \equiv \frac{d^{n_1}y}{dx^{n_1}} + l_1\frac{d^{n_1-1}y}{dx^{n_1-1}} + \cdots + l_{n_1}y,$$

$$M(y) \equiv \frac{d^{n-n_1}y}{dx^{n-n_1}} + m_1\frac{d^{n-n_1-1}y}{dx^{n-n_1-1}} + \cdots + m_{n-n_1}y,$$

where $l_1 \ldots l_{n_1}$, $m_1 \ldots m_{n-n_1}$ are functions in $\mathcal{D}\langle x\rangle$ and $n_1 < n$.*

The domain $\mathcal{D}\langle x, y_1\rangle$ will be said to be of the n-th order when it is obtained by adjoining an integral y_1 of the equation of n-th order $K(y) = 0$, irreducible in $\mathcal{D}\langle x\rangle$.

If n particular integrals $y_1 \ldots y_n$ are adjoined to $\mathcal{D}\langle x\rangle$, we obtain n domains

$$\mathcal{D}\langle x, y_1\rangle \ldots \mathcal{D}\langle x, y_n\rangle,$$

which are said to be *conjugate to each other.* The domains may be all different; some, or all of them, may be identical. Thus, the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0,$$

* Schlesinger, "Handbuch der Theorie der linearen Differentialgleichungen," Vol. I, pp. 42–46, 81–87.

of which $y_1 = e^{\sin^{-1} x}$, $y_2 = e^{\cos^{-1} x}$ are integrals, is irreducible (*f-i*) in the domain $\mathbf{r}[x]$. The conjugate domains

$$\mathbf{r}[x, e^{\sin^{-1} x}] \text{ and } \mathbf{r}[x, e^{\cos^{-1} x}]$$

are different. However, starting with $\mathbf{R}[x]$, the conjugate domains

$$\mathbf{R}[x, e^{\sin^{-1} x}] \text{ and } \mathbf{R}[x, e^{\cos^{-1} x}]$$

are identical, since $1/y_1 = y_2 e^{-\frac{\pi}{2}}$. The equation

$$\frac{d^2 y}{dx^2} + y = 0,$$

of which $y_1 = \sin x$ and $y_2 = \cos x$ are integrals, is irreducible in $\mathbf{R}$, but the domains

$$\mathbf{R}[y_1] \text{ and } \mathbf{R}[y_2]$$

are identical, since $dy_1/dx = y_2$ and $dy_2/dx = -y_1$.

A domain $\mathfrak{D}\langle y_2\rangle$ may be equivalent to $\mathfrak{D}\langle y_1\rangle$ while $\mathfrak{D}\langle y_1\rangle$ may be inequivalent to $\mathfrak{D}\langle y_2\rangle$. For instance, if $y_2 = y_1^2$, then $\mathfrak{D}\langle y_2\rangle$ is equivalent to $\mathfrak{D}\langle y_1\rangle$, but the inverse is not true. Thus, equivalence of domains is not reciprocal; if $\mathfrak{D}_1 \equiv \mathfrak{D}_2$, it does not follow that $\mathfrak{D}_2 \equiv \mathfrak{D}_1$.

If all the conjugates of a given domain $\mathfrak{D}\langle y_1\rangle$ are equivalent to it, then $\mathfrak{D}\langle y_1\rangle$ is said to be a *normal domain.*

If $\mathfrak{D}\langle x, y_1\rangle$ is a normal domain, it follows that y_2 is in this domain, and therefore that y_2 is a rational differential function of y_1,

$$y_2 = \phi_2(y_1, y_1', \ldots).$$

Similarly,

$$y_3 = \phi_3(y_1, y_1', \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots,$$
$$y_n = \phi_n(y_1, y_1', \ldots),$$

the ϕ's denoting rational functions of the indicated arguments.

THEOREM II. *Let $K(y) = 0$ be an equation in $\mathfrak{D}\langle x\rangle$ of order n, such that the adjunction of a certain particular integral y_1 produces a normal domain $\mathfrak{D}\langle x, y_1\rangle$; then the general integral of $K(y) = 0$ may be expressed as a rational differential function of a single particular integral (and n parameters).*

The general integral of $K(y)=0$ being

$$y=\alpha_1 y_1+\alpha_2 y_2+\ldots\ldots+\alpha_n y_n,$$

the α's denoting parameters, we have

$$\begin{aligned} y&=\alpha_1 y_1+\alpha_2\phi(y_1, y_1'\ldots\ldots)+\ldots\ldots+\alpha_n\phi_n(y_1, y_1'\ldots\ldots)\\ &=\psi(y_1, y_1'\ldots\ldots|\alpha_1\ldots\ldots\alpha_n).\end{aligned}$$

THEOREM III. *Let*

$$A(y)\equiv\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+a_n y=0$$

be an equation with constant coefficients, and let the domain of rationality be C. *If the auxiliary equation of* $A(y)=0$,

$$u^n+a_1 u^{n-1}+\ldots\ldots+a_n=0,$$

has distinct roots, then the domain C $[V]$, *where*

$$V=y_1+y_2+\ldots\ldots+y_n,$$

is normal.

We designate the roots of the auxiliary equation by

$$u_1,\ u_2,\ \ldots\ldots,\ u_n;$$

then

$$y_i=e^{u_i x},\qquad (i=1,\ \ldots\ldots,\ n),$$

and

$$V=e^{u_1 x}+e^{u_2 x}+\ldots\ldots+e^{u_n x}.$$

By differentiation

$$\begin{aligned} V'&=u_1 e^{u_1 x}+\ldots\ldots+u_n e^{u_n x},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ V^{(n-1)}&=u_1^{n-1}e^{u_1 x}+\ldots\ldots+u_n^{n-1}e^{u_n x}.\end{aligned}$$

The determinant

$$\begin{vmatrix} 1 & 1 & \ldots\ldots & 1\\ u_1 & u_2 & \ldots\ldots & u_n\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ u_1^{n-1} & u_2^{n-1} & \ldots\ldots & u_n^{n-1}\end{vmatrix}$$

can not vanish, the u_i being unequal. We can therefore find the y_i in terms of V,

$$y_i=b_0 V+b_1 V'+\ldots\ldots+b_{n-1}V^{(n-1)};$$

the $b_1\ldots\ldots b_{n-1}$ denoting numbers in C. Since V is itself an integral of $A(y)=0$, the domain C (V) must be a normal one.

THEOREM IV. *If two roots of the auxiliary equation of* $A(y) = 0$ *are equal,* $u_{n-1} = u_n$, *then* $\mathbf{C}[x, V_{n-1}]$ *is a normal domain,*

$$V_{n-1} = y_1 + y_2 + \cdots + y_{n-1}.$$

The proof of this theorem and the generalization to the case where m roots of the auxiliary equation are equal is similar to the proof of Theorem III, and will present no special difficulty to the reader.

An equation of the n-th order (linear or non-linear) whose general integral is a rational differential function in $\mathfrak{D}\langle x\rangle$ (with n parameters) of a single particular superior integral, is called a *normal equation.*

The following is an immediate consequence of this definition:

THEOREM V. *By adjoining a certain particular integral* y_1 *of a normal equation to* $\mathfrak{D}\langle x\rangle$, *the enlarged domain* $\mathfrak{D}\langle x, y_1\rangle$ *is normal.*

THEOREM VI. *Let* $E(y) = 0$ *be a normal equation,* y_1 *a superior integral and* $y = \phi(y_1, y_1', \ldots)$ *the general integral. If* y_2 *is any other integral of* $E(y) = 0$, *then* $\phi(y_2, y_2', \ldots)$ *is also an integral of* $E(y) = 0$.

Consider the equation

$$E(\phi(z, z', \ldots)) = 0.$$

The equation $E(y) = 0$ has the superior integral y_1 in common with it; therefore (Theorem I) every integral, as y_2, of $E(y) = 0$ satisfies it also. From this we see that

$$E(\phi(y_2, y_2', \ldots)) = 0,$$

and thus

$$\phi(y_2, y_2', \ldots)$$

is also an integral of $E(y) = 0$.*

A rational integral differential function in $\mathfrak{D}\langle x\rangle$ of the fundamental system $y_1 \ldots y_n$ of $K(y) = 0$,

$$V = \pi(y_1 \ldots y_n, y_1' \ldots y_n', \ldots),$$

written for brevity

$$V = \pi(y),$$

is said to be *primitive*† in this domain when the y's are also rational integral functions in $\mathfrak{D}\langle x\rangle$ of $V, V', \ldots,$

$$y_i = X_i(V, V', V'', \ldots), \qquad (i = 1, \ldots, n).$$

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* E. Vessiot, *Annals de l'École Normale Supérieure*, Vol. IX (1892), p. 227. See also A. Loewy, "Algebraische und Gruppentheoretische Untersuchungen," etc., *loc. cit.*, § 4.

† A. Loewy, "Die Rationalitätsgruppe einer linearen homogenen Differentialgleichung," *Mathematische Annalen*, Vol. LXV (1908), p. 136.

If we differentiate $V=\pi(y)$ for x n^2 times, there arise

$$\begin{aligned} V &= \pi(y_1 \cdots y_n,\ y_1' \cdots y_n',\ \cdots), \\ \frac{dV}{dx} &= \pi_1(y_1 \cdots y_n,\ y_1' \cdots y_n',\ \cdots), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \frac{d^{n^2}V}{dx^{n^2}} &= \pi_{n^2}(y_1 \cdots y_n,\ y_1' \cdots y_n',\ \cdots). \end{aligned}$$

By means of the differential equation $K(y)=0$ derivatives of $y_1 \cdots y_n$ of order higher than $n-1$ can be eliminated, giving us then n^2+1 equations in the n^2 functions $y_1 \cdots y_n,\ y_1' \cdots y_n',\ \cdots,\ y_1^{(n-1)} \cdots y_n^{(n-1)}$. Suppose that these functions can be eliminated from the first $s+1$ of the equations ($s \leq n^2$), giving rise to the resolvent

$$M\left(V, \frac{dV}{dx}, \cdots, \frac{d^sV}{dx^s}\right) = 0.$$

The equation $M=0$ is not necessarily k-irreducible relative to V. Among all the possible equations $M=0$ which V satisfies, we shall consider one which is irreducible in $\mathfrak{D}\langle x\rangle$ relative to V; that is, which has V for a superior integral. Let this equation be

$$N\left(V, \frac{dV}{dx}, \cdots, \frac{d^rV}{dx^r}\right) = 0.$$

The following theorem is a statement in the terminology of this section of a result due to Loewy.*

THEOREM VII. *The resolvent $N=0$ is a normal equation.*

§ 4. The Index of a Linear Homogeneous Differential Equation.

Denote the general linear homogeneous group in n variables and n^2 parameters by

$$L:\ z_{i_1} = \sum_{i_2} a_{i_1 i_2} y_{i_2},\ |a_{i_1 i_2}| \neq 0, \qquad (i = 1, \cdots, n).$$

Let $y_1 \cdots y_n$ be a fundamental system of the equation $K(y)=0$ in $\mathfrak{D}\langle x\rangle$, then a rational differential function of these integrals with coefficients in $\mathfrak{D}\langle x\rangle$,

$$t(y) \equiv t(y_1 \cdots y_n,\ y_1' \cdots y_n',\ \cdots),$$

is called a *natural transcendental of the domain.*†

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* A. Loewy, "Die Rationalitätsgruppe einer linearen homogenen Differentialgleichung," *Mathematische Annalen*, Vol. LXV (1908), p. 160.

† A. Loewy, "Algebraische und Gruppentheoretische Untersuchungen," etc., *loc. cit.*, p. 254.

By the rationality group G, of the equation $K=0$, is meant the totality of substitutions (taken from L) possessing the following double property:

a) Every natural transcendental invariant under the substitutions of G has its numerical value in $\mathcal{D}\langle x\rangle$; and b) every natural transcendental whose numerical value lies in $\mathcal{D}\langle x\rangle$ is numerically invariant under each substitution of G.

If r is the order of the normal resolvent (and therefore the number of parameters in the rationality group G of the equation $K=0$*), the integer

$$\delta = n^2 - r$$

is called the *index of the equation* $K(y)=0$.

The lower the order of $N=0$, the greater is the index of $K=0$. When $\delta=n^2$, the integrals $y_1 \ldots . y_n$ are all in $\mathcal{D}\langle x\rangle$, and the equation $K=0$ is solved (requiring at most the adjunction of an algebraic irrationality). For, when $\delta=n^2$, we have $r=0$, and $N=0$ is an algebraic equation (differential equation of order zero). The solution V of this equation is at most an algebraic irrationality, and the equations

$$y_i = X_i(V, V', \ldots .), \qquad (i=1, \ldots ., n),$$

give us a fundamental system of $K=0$.

When the adjunction of a new irrationality to $\mathcal{D}\langle x\rangle$ makes $N=0$ k-reducible, a new normal resolvent, of lower order, is obtained, and the index is thus increased. From this point of view, the problem of solving a differential equation consists in adjoining irrationalities to $\mathcal{D}\langle x\rangle$ until the index becomes a maximum, $\delta=n^2$.

A differential equation $K(y)=0$ is said to be *special in* $\mathcal{D}\langle x\rangle$ if the numerical value $a(x)$ of a natural transcendental $t(y)$ lies in $\mathcal{D}\langle x\rangle$ for certain fundamental systems, but not for every one of the ∞^{n^2} fundamental systems defined by $y_1 \ldots . y_n$ and L.

THEOREM VIII. *If* $K(y)=0$ *is a special equation, its rationality group* G *is the largest subgroup of* L *which leaves* $t(y)=a(x)$ *numerically invariant.*

Suppose that A and B are two substitutions leaving t numerically invariant,

$$t_A = t, \quad t_B = t.$$

* L. Schlesinger, "Bericht über die Entwicklung der Theorie der linearen Differentialgleichungen seit 1865," *Jahresbericht der deutschen Mathematischen Vereinigung*, Vol. XVIII (1909), p. 151.

Applying B to $t_A - t = 0$, we have $t_{AB} - t_B = 0$. From this, $t_{AB} = t_B = t$, and therefore the totality of substitutions leaving t (formally and) numerically invariant form a group G. From the foregoing definition of the rationality group, any substitution of L which does not leave $t(y)$ numerically invariant can not belong to G. On the other hand, every substitution of L which leaves $t(y)$ numerically invariant does belong to G. For, every natural transcendental $f(y)$ whose value is in the domain must be of the form

$$f(y_1 \ldots y_n,\ y_1' \ldots y_n',\ \ldots) = \beta(x,\ t,\ t',\ \ldots),$$

where the function β lies in $\mathfrak{D}\langle x\rangle$; and every substitution of L leaving t numerically invariant leaves f numerically invariant and belongs to G.

This proof of the theorem involves the assumption that there exists only one natural transcendental $t(y)$ whose numerical value $\alpha(x)$ lies in the domain for some, but not for all the ∞^{n^2} fundamental systems defined by L. If there should exist a such independent natural transcendentals, we have the

COROLLARY: *If $t_1(y) = \alpha_1(x) \ldots t_a(y) = \alpha_a(x)$ are natural transcendentals whose numerical values lie in $\mathfrak{D}\langle x\rangle$, then the rationality group G of $K(y) = 0$ is that subgroup of L which leaves all these natural transcendentals numerically invariant. That is to say, it is the greatest common subgroup of the groups to which $t_1 \ldots t_a$ belong.*

The following theorem will be needed in the application of § 5. We refer the reader to the proof in Loewy's paper, "Algebraische und Gruppentheoretische Untersuchungen aus dem Gebiete der Differentialgleichungen," *Monatshefte für Mathematik und Physik*, Vol. VIII (1897), p. 254.*

THEOREM IX. *If the rationality group of $K = 0$ is G, the adjunction of the numerical value $\alpha(x)$ of a natural transcendental $S(y)$ lowers it to the largest subgroup of G which leaves S numerically invariant.*

We now prove†

THEOREM X. *For a special equation $K(y) = 0$, the index is greater than zero,*

$$\delta > 0.$$

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* See also E. Vessiot, *Annales de l'École Normale Supérieure*, Vol. IX (1892), p. 235.

† Theorems X and XI are closely related to those in Picard's Traité d'Analyse, Vol. III, 2d edition (1908), p. 571. There are two elements of difference. First, in this paper they are stated from the new standpoint of the index of an equation; and, secondly, the proof is based on the general primitive functions V of Loewy instead of on the special one (known as the Picard resolvent) $u_1 y_1 + \ldots + u_n y_n$.

As before, let

$$V = \pi(y_1 \dots y_n,\ y_1' \dots y_n',\ \dots)$$

be a primitive function, which, solved for $y_1 \dots y_n$, gives

$$y_i = X_i(V, V', \dots), \qquad (i = 1, \dots, n).$$

Since $\delta = n^2 - r$, it will suffice to show that for a special equation the order r of the normal resolvent $N(V) = 0$ is less than n^2,

$$r < n^2.$$

The equation $K(y) = 0$ being special, there exists a natural transcendental $t(y)$ whose numerical value $\alpha(x)$ lies in $\mathcal{D}\langle x\rangle$,

$$t(y_1 \dots y_n,\ y_1' \dots y_n',\ \dots) = \alpha(x).$$

Substituting the values of the y_i,

$$t[X_1(V, V', \dots) \dots X_n(V, V', \dots),\ \dots] - \alpha(x) = 0$$

may be regarded as a differential equation for V. The order s of this equation must be less than n^2; for, to each one of its integrals corresponds a fundamental system $y_1 \dots y_n$ of K which satisfies $t(y) = \alpha(x)$; and if the order of $t[X_1 \dots X_n, \dots] - \alpha(x) = 0$ is n^2, then $t(y)$ would equal $\alpha(x)$ for every one of the ∞^{n^2} fundamental systems of $K = 0$; but this is contrary to the hypothesis that $K = 0$ is a special equation. $N(V) = 0$ has the superior integral V in common with $t_1[X(V, V', \dots) \dots] - \alpha(x) = 0$; therefore (Theorem I) every integral of $N = 0$ satisfies this equation, and

$$r \leqq s < n^2.$$

THEOREM XI. *If the index of $K(y) = 0$ is greater than zero,*

$$\delta > 0,$$

the equation is a special one.

From $\delta = n^2 - r > 0$, *we have, for the order of the resolvent* $N = 0$,

$$r < n^2.$$

Since $N(V) = 0$ is of order $r < n^2$, the corresponding fundamental systems $y_1 \dots y_n$,

$$y_i = X_i(V, V', \dots), \qquad (i = 1, \dots, n),$$

are ∞^r in number (and not ∞^{n^2}). If $r < n$, we can eliminate V from the n equations $y_i = X_i(V, V', \dots)$ and thus obtain a relation between the y_i

satisfied by some, but not by all fundamental systems. If $n \leqq r < 2n$, consider the $2n$ equations

$$y_i = X_i(V, V', \ldots.),$$
$$y_i' = \bar{X}_i(V, V', \ldots.).$$

Remove derivatives of V beyond order r by means of $N=0$. If $r < 2n$, we can eliminate as before, and obtain a relation between $y_1 \ldots . y_n$, $y_1' \ldots . y_n'$ true for some, but not for all fundamental systems of $K=0$. If $2n \leqq r < 3n$, the same process will give us a relation between $y_1 \ldots . y_n$, $y_1' \ldots . y_n'$, $y_1'' \ldots . y_n''$. Continuing in this manner, it is shown that $K=0$ is a special equation.

§ 5. Integration of a Differential Equation by Chains of Resolvents.

There are two methods of integrating linear differential equations by chains of resolvents. The first one (although second in historical order) is due to Loewy* and may be called the *Method of Normal Resolvents.* This method has the advantage that but one particular integral of each resolvent need be computed to give the general integral; its disadvantage (as compared with the second method) is that the resolvents are not of lowest possible order. The second method (first in historical order) is due to Vessiot,† and may be called the *Method of Minimum Resolvents.* This method has the advantage that the resolvents to be integrated are of the lowest possible order; its disadvantage (as compared with the first method) is that more than one particular integral of each resolvent must be determined to give the general integral.

In this section these two methods are restated (in modified form) as applications of the concepts of § 4.

1°. Let G be the rationality group of $K(y)=0$ in $\mathfrak{D}\langle x\rangle$, and by adjoining an algebraic irrationality‡, f, to $\mathfrak{D}\langle x\rangle$ we lower it to the maximal continuous subgroup of G, denoted by Γ, and generated by infinitesimal transformations. Let $\Gamma^{(1)}$ be a maximal self-conjugate subgroup of $\Gamma^{(0)} \equiv \Gamma$, let $\Gamma^{(2)}$ be a maximal self-conjugate subgroup of $\Gamma^{(1)}$, and so forth. By continuing this process we finally reach a simple group $\Gamma^{(m-1)}$, and the next one in the list, $\Gamma^{(m)}$, must be the identity, I. The set of groups

$$\Gamma^{(0)}, \Gamma^{(1)}, \ldots., \Gamma^{(m-1)}, \Gamma^{(m)}$$

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* A. Loewy, "Algebraische und Gruppentheoretische Untersuchungen," etc., *loc. cit.*, p. 264.

† E. Vessiot, *loc. cit.*, p. 239.

‡ L. Schlesinger, "Handbuch der Theorie der linearen Differentialgleichungen," II, 1, p. 80.

is called a *composition series* of Γ. The number of parameters in $\Gamma^{(k)}$ is designated by r_k, and we let

$$\rho_k = r_{k-1} - r_k, \qquad (k = 1, \ldots, m),$$

where $r_0 = r$, $r_m = 0$.

Let $t_1 \equiv t_1(y) \equiv t_1(y_1 \ldots y_n, y_1' \ldots y_n', \ldots)$ be a natural transcendental belonging to $\Gamma^{(1)}$; that is, an invariant of $\Gamma^{(1)}$ which is altered in value by any substitution outside of $\Gamma^{(1)}$. The function t_1, when transformed by Γ, contains ρ_1 effective parameters,* and it is a solution of a normal resolvent

$$\Phi_1(t_1) = 0,$$

of order ρ_1. Any other solution of this equation, $\bar{t}_1$, belongs to the same group $\Gamma^{(1)}$; the same is true of the general integral T_1, and we have †

$$T_1 = A_1(t_1 t_1' \ldots | p_1 \ldots p_{\rho_1}),$$

A_1 denoting an algebraic function, $p_1 \ldots p_{\rho_1}$ parameters.

If we determine one integral t_1 of $\Phi_1 = 0$ and adjoin it to $\mathfrak{D}\langle x, f\rangle$, the rationality group of $K(y) = 0$ in $\mathfrak{D}\langle x, f, t_1\rangle$ falls to $\Gamma^{(1)}$ (Theorem IX). The index δ of $K(y) = 0$ is increased,

$$\delta_1 = \delta + \rho_1.$$

Let t_2 be a natural transcendental belonging to $\Gamma^{(2)}$; when transformed by $\Gamma^{(1)}$ it takes ρ_2 effective parameters and it is a solution of a normal resolvent in $\mathfrak{D}\langle x, f, t_1\rangle$,

$$\Phi_2(t_2) = 0,$$

of order ρ_2. In the domain $\mathfrak{D}\langle x, f, t_1, t_2\rangle$ the group of $K = 0$ is $\Gamma^{(2)}$ and the index

$$\delta_2 = \delta_1 + \rho_2 = \delta + \rho_1 + \rho_2.$$

Similarly, in the domain $\mathfrak{D}\langle x, f, t_1, t_2, t_3\rangle$ the rationality group of $K = 0$ is $\Gamma^{(3)}$ and the index

$$\delta_3 = \delta + \rho_1 + \rho_2 + \rho_3.$$

Continuing this process, the solution of $K(y) = 0$ is accomplished by the integration of the normal resolvents

$$\Phi_1 = 0, \ \Phi_2 = 0, \ \ldots, \ \Phi_m = 0,$$

of orders

$$\rho_1, \ \rho_2, \ \ldots, \ \rho_m,$$

respectively. When an integral of the last normal resolvent is adjoined, and the domain becomes

$$\mathfrak{D}\langle x, f, t_1, \ldots, t_m\rangle,$$

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* Vessiot, *loc. cit.*, p. 208.

† Vessiot, *loc. cit.*, p. 223.

the rationality group falls to the identical transformation. The index δ_m becomes

$$\delta_m = \delta + \rho_1 + \rho_2 + \cdots + \rho_m;$$

and, since

$$\delta = n^2 - r,$$

$$\rho_1+\rho_2+\cdots+\rho_m=(r-r_1)+(r_1-r_2)+\cdots+(r_{m-2}-r_{m-1})+(r_{m-1}-0)=r,$$

we have

$$\delta_m = n^2,$$

and the differential equation is integrated.

2°. Between each pair of consecutive groups $\Gamma^{(k-1)}$ and $\Gamma^{(k)}$ of the composition series

$$\Gamma^{(0)},\ \Gamma^{(1)},\ \ldots,\ \Gamma^{(m-1)},\ \Gamma^{(m)},$$

we insert a group $\Lambda^{(k)}$, of σ_k parameters, which is the maximal subgroup of $\Gamma^{(k-1)}$ containing $\Gamma^{(k)}$. Let u_k be a natural transcendental belonging to $\Lambda^{(k)}$. The function u_k, when transformed by $\Gamma^{(k-1)}$, contains $r_{k-1} - \sigma_k$ effective parameters, and therefore satisfies a minimal resolvent

$$\Psi_k(u) = 0, \qquad (k = 1 \ \ldots \ m),$$

of order $r_{k-1} - \sigma_k$.

The adjunction of the general integral U_1 of $\Psi_1 = 0$ to the domain $\mathcal{D}\langle x, f\rangle$ lowers the rationality group of $K = 0$ from $\Gamma^{(0)}$ to $\Gamma^{(1)}$, the largest subgroup of $\Lambda^{(1)}$ which is self-conjugate in $\Gamma^{(0)}$.* The index of the equation $K = 0$ is therefore increased from δ to $\delta + \rho_1$.

To the domain $\mathcal{D}\langle x, f, U_1\rangle$ we adjoin U_2, the general integral of $\Psi_2 = 0$, and the index of the equation $K = 0$ is therefore increased from $\delta + \rho_1$ to $\delta + \rho_1 + \rho_2$.

By continuing this process m times, it is seen that in the domain $\mathcal{D}\langle x, f, U_1, \ldots, U_m\rangle$ the group of the equation becomes the identical transformation; its index rises to

$$\delta + \rho_1 + \rho_2 + \cdots + \rho_m = n^2,$$

and $K(y) = 0$ is integrated.

THE UNIVERSITY OF COLORADO, *March*, 1911.

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* E. Vessiot, *loc. cit.*, p. 238.

Critical Revision of de Haan's Tables of Definite Integrals.

By E. W. Sheldon.

§ 1.

In 1858, D. Bierens de Haan published as "Vierde Deel" of the *Verhandelingen der Koninklijke Akademie van Wetenschappen* (Amsterdam) his "Tables d'Intégrales Définies." In this volume there are 7300 formulas, and these are accompanied by complete bibliographical references. De Haan then undertook the revision of these formulas, and the consideration of the underlying theory of Definite Integrals. Accordingly, in 1862, his "Exposé de la Théorie des Propriétés, des Formules de Transformation, et des Méthodes d'Évaluation des Intégrales Définies" was issued as Volume VIII in the same *Verhandelingen*. In this is included all his critical work of the intervening four years.

"Nouvelles Tables d'Intégrales Définies par D. Bierens de Haan" were published in 1867 (Leide). These tables contain 8339 formulas, of which 4200 come from the original tables, 2620 from the "Exposé" and the remainder from notes published at various times.

In this paper it is proposed to examine from the modern rigorous standpoint certain evaluations in the "Nouvelles Tables," and the theorems of the "Exposé" on which they depend. In nearly every case the formulas here considered will involve the principal value integral.

§ 2. *Definitions.*

De Haan first considers the function $f(x)$ defined over the interval $(a < b)$* and possessing the continuous primitive function $F(x)$. He employs Cauchy's† definition for the integral of a continuous function:

$$\int_a^b f(x)\,dx = \lim_{\delta=0} \sum_{\iota=1}^{n} (x_\iota - x_{\iota-1}) f(x_{\iota-1}),$$

* Here and elsewhere, unless the contrary is explicitly stated, all letters are finite.

† Cauchy, "Œuv. Comp.," Sér. 2, Tome 4, p. 122.

and gives also the definition $F(b) - F(a)$, which is dismissed as being equivalent to the other for the functions under consideration. The assumption that $f(x)$ is integrable in $(a < b)$ is not explicitly stated; this is important, for the function defined by Volterra* shows that $f(x)$ may satisfy the conditions here given and yet not be integrable.

Apparently de Haan means to include here not only proper integrals, but also certain improper integrals with infinite integrands; viz., those whose integrands have continuous primitives. However, Cauchy's definition will not apply to unlimited functions, for as δ approaches zero, the sum Σ can always be made to approach infinity.

Improper integrals with infinite fields are not separately defined in the "Exposé."

Improper integrals with infinite integrands: we shall compare the modern definition with that of de Haan, first of all.

1°. *De Haan's Definition.*

Let $f(x)$ possess the primitive function $F(x)$, in $(a < b)$.

Let $F(x)$ be continuous in $(a < b)$ except for an infinite discontinuity at c, $a < c < b$. Then we define

$$\int_a^b f(x)\,dx = \lim_{\theta=0} \left\{ \int_a^{c-\mu\theta} f(x)\,dx + \int_{c+\nu\theta}^b f(x)\,dx \right\}.$$

It is not explicitly stated, but from the context it is clear that μ, ν are any positive constants. It is also implied that for each choice of μ and ν the limit must exist, and have the same value for all such choices. This is the "general value" of the integral; when $\mu = \nu = 1$, we have the definition of the "principal value" (Cauchy).

2°. *The Modern Definition for this Case — One Singularity.*

Let $f(x)$ be regular† in $(a < b)$ except at c. Then we define

$$\int_a^b f(x)\,dx = \lim_{\theta=0} \int_a^{c-\theta} f(x)\,dx + \lim_{\theta=0} \int_{c+\theta}^b f(x)\,dx, \text{ if these exist;}$$

or

$$= \lim_{\theta_1=0,\,\theta_2=0} \left\{ \int_a^{c-\theta_1} f(x)\,dx + \int_{c+\theta_2}^b f(x)\,dx \right\}, \text{ if it exists.}$$

* *Giorn. di Battaglini*, Vol. XIX (1881).

† Lect. 1, p. 400, § 572.

3°. *The Principal Value Definition is Adopted by de Haan.*

There is no good reason, it is stated by de Haan,* for taking μ and ν other than unity. Accordingly it is decided to use the principal value integral throughout. This statement and decision are based upon the consideration of a single example, viz., $\int_0^\infty \frac{1}{1-x^2}\,dx$. His course of reasoning is as follows:

The primitive function of $\frac{1}{1-x^2}$, $F(x) = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| = \frac{1}{2}\log\left|\frac{1+1/x}{1-1/x}\right|$.

Now,

$$\begin{aligned}\int_0^\infty \frac{1}{1-x^2}\,dx &= \int_0^1 \frac{1}{1-x^2}\,dx + \int_1^\infty \frac{1}{1-x^2}\,dx \\ &= \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\Big]_0^1 + \frac{1}{2}\log\left|\frac{1+1/x}{1-1/x}\right|\Big]_1^\infty \\ &= \frac{1}{2}\log\frac{2}{0} - 0 + 0 - \frac{1}{2}\log\frac{2}{0} \\ &= 0,\end{aligned}$$

which is the integral's true value.

The "general value" is

$$\lim_{\theta=0}\left\{\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\Big]_0^{1-\mu\theta} + \frac{1}{2}\log\left|\frac{1+1/x}{1-1/x}\right|\Big]_{1+\nu\theta}^\infty\right\} = \frac{1}{2}\log\frac{\nu}{\mu}.$$

The principal value, therefore, $=0$, which is the "true" value obtained above. But, $\frac{1}{2}\log\frac{2}{0}$ is *not a number*, and it is not possible to state that $\frac{1}{2}\log\frac{2}{0} - \frac{1}{2}\log\frac{2}{0} = 0$. Nor will it do to make so important an inference from the investigation of a *single* example.

Accordingly we may expect to find in the "Exposé" and in the "Nouvelles Tables" many integrals that exist as principal values only. In this paper these will be clearly indicated by the use of the symbol Pf, with suitable subscripts in each case.

4°. *Finite Number of Singularities: Cauchy's Definition of the Principal Value.*†

Let $f(x)$ become infinite at $c_1, c_2, \ldots, c_s$; $a < c_\iota < b$, $\iota = 1, 2, \ldots, s$. Then the principal value integrals are defined thus:

$$\int_a^b f(x)\,dx = \lim_{\theta=0}\left\{\int_a^{c_1-\theta} + \int_{c_1+\theta}^{c_2-\theta} + \ldots + \int_{c_s+\theta}^b\right\},$$

and

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\theta=0}\left\{\int_{-1/\theta}^{c_1-\theta} + \int_{c_1+\theta}^{c_2-\theta} + \ldots + \int_{c_s+\theta}^{1/\theta}\right\}.$$

* "Exposé," p. 7.

† "Œuv. Comp.", Sér. 2, Tome 4, p. 143—Résumé des leçons calc. inf. (1823).

According to these definitions, $f(x)$ may possess a principal value integral in $(a<b)$, and yet *not* possess a principal value integral in $(a<x_0)$, $x_0 \neq c_\iota$; or in $(x_1<x_2)$, $a<x_1$, $x_2<b$ and $x_1, x_2 \neq c_\iota$. Similarly in the case of the integral for $(-\infty, \infty)$.

5°. *Finite Number of Singularities: G. H. Hardy's Definition.**

Let $f(x)$ become infinite at $c_1, c_2, \ldots, c_s$; $a<c_\iota<b$, $\iota=1, 2, \ldots, s$. Then Hardy defines

$$P\int_a^b f(x)\,dx = \underset{\theta_1=0,\,\theta_2=0,\,\ldots,\,\theta_s=0}{\text{limit}} \left\{\int_a^{c_1-\theta_1} + \int_{c_1+\theta_1}^{c_2-\theta_2} + \ldots + \int_{c_s+\theta_s}^{b}\right\}.$$

According to this definition, if $f(x)$ possesses a principal value integral in $(a<b)$, it also possesses a principal value integral in any $(x_1<x_2)$; $a<x_1$, $x_2<b$; $x_1, x_2 \neq c_\iota$.

We may add the remark that, consistently with the above, if $f(x)$ has the singularities $c_1, c_2, \ldots, c_s$, and we take the principal value integral $\int_{-\infty}^{\infty} f(x)\,dx$, its convergence implies the convergence of each of the following principal value integrals:

$\left\{\int_{-\infty}^{x_1} + \int_{x_2}^{\infty}\right\}$, where $-\infty < x_1 < c_1$ and $c_s < x_2 < \infty$;

and

$\int_{x_1}^{x_2}$, where $x_1, x_2 \neq c_\iota$, but include, between themselves, at least one of the c_ι.

This definition* appears in the first of Hardy's four important papers on "The Theory of Cauchy's Principal Values," in which he establishes under fairly broad conditions many useful theorems on Infinite Series, Continuity, Inversion of Integration, Differentiation under the Integral Sign, etc. Frequent references we here made to these papers.

6°. *The Various Definitions Required here.*

(a) $$P_c\int_a^b f(x)\,dx = \lim_{\theta=0}\left\{\int_a^{c-\theta} + \int_{c+\theta}^{b}\right\}, \quad \text{where } a<c<b.$$

See § 7, (4), (5) and (6).

* *Proc. Lond. Math. Soc.*, Vol. XXXIV (1901), p. 18.

(b) $$P_{\pm\infty}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\Theta=\infty}\int_{-\Theta}^{\Theta} f(x)\,dx.$$

See § 8, (5).

(c) $$P_{\pm c}\int_{-c}^{c} f(x)\,dx = \lim_{\theta=0}\int_{-c+\theta}^{c-\theta} f(x)\,dx.$$

See § 6, (3), (5), (7) and (9).

(d) $$P_{\pm c}\int_{a}^{b} f(x)\,dx = \lim_{\theta=0}\int_{-c+\theta}^{c-\theta} f(x)\,dx + \lim_{\theta=0}\left\{\int_{a}^{-c-\theta} + \int_{c+\theta}^{b}\right\},$$

where $a < -c < c < b$.

See § 6, (4); cf. (8).

(e) $$P_{\{c_\iota\}}\int_{a}^{b} f(x)\,dx = \lim_{\theta=0}\left\{\int_{a}^{c_1-\theta} + \int_{c_1+\theta}^{d_1}\right\} + \lim_{\theta=0}\left\{\int_{d_1}^{c_2-\theta} + \int_{c_2+\theta}^{d_2}\right\} + \ldots + \lim_{\theta=0}\left\{\int_{d_{s-1}}^{c_s-\theta} + \int_{c_s+\theta}^{b}\right\},$$

where $c_{\iota-1} < d_{\iota-1} < c_\iota$,

or $$P_{\{c_\iota\}} = \sum_{\iota=1}^{s} P_{c_\iota},$$ to put it briefly.

See § 4, (25) and (27); § 6, (8).

This last agrees with Hardy's definition given in 5° above. The meaning of such symbols as $P_{c,\pm\infty}$, $P_{-d,\,e,\,\pm f,\,\pm\infty}$, etc., is at once evident.

§ 3. *Abbreviations, etc.*

VIII: Verhandelingen der Koninklijke Akademie van Wetenschappen, Deel VIII, 1862. Exposé de la théorie, des proprietés, etc.

Nouv. Tab.: Nouvelles Tables d'Intégrales Définies, par D. B. de Haan, 1867.

T 172, F 4: Table 172, Formula 4, in the Nouvelles Tables.

Lect. I: Lectures on the Theory of Functions of Real Variables, Volume I, by James Pierpont, 1905.

si (u): $\int_0^u \frac{\sin x}{x}dx$, u in the domain $(-\infty, \infty)$.

§ 4.

$$\int_0^\infty \frac{\sin px\,dx}{(q^2-x^2)\,.\,x} = \frac{\pi}{2q^2}(1-\cos pq), \quad \text{Nouv. Tab., T 172, F 4.} \tag{1}$$

We shall prove the following:

$$\left.\begin{aligned} P_q\int_0^\infty \frac{\sin px\,dx}{(q^2-x^2)\,.\,x} &= \frac{\pi}{2q^2}(1-\cos pq); \quad p \gtreqless 0,\ q>0, \\ \text{or} \qquad &= -\frac{\pi}{2q^2}(1-\cos pq); \quad p<0,\ q>0. \end{aligned}\right\} \tag{2}$$

At $x=0$, set the integrand, $f(x,p) = p/q^2$.

We shall show that we may differentiate twice under the integral sign. Consider the integral, I, in three parts:

$$I_1 = \int^a, \quad I_2 = P_q \int_a^b, \quad I_3 = \int_b^\infty; \quad \text{where } 0 < a < q < b < \infty.$$

1°. $$\frac{d^2 I_1}{dp^2} = \int_0^a \frac{\partial^2 f}{\partial p^2}\, dx. \tag{3}$$

In $R_1 = (0 \leqq x \leqq a;\ 0 \leqq p \leqq p_1)$ both $f(x, p)$ and the derivative $f'_p(x, p)$ are continuous. Therefore,

$$\frac{d I_1}{dp} = \int_0^a \frac{\partial f}{\partial p}\, dx. \tag{4}$$

Again, in R_1, both $f'_p(x, p)$ and its derivative $f''_{pp}(x, p)$ are continuous Hence from (4) we get (3), by differentiating again.

2°. $$\frac{d^2 I_2}{dp^2} = P_q \int_a^b \frac{\partial^2 f}{\partial p^2}\, dx. \tag{5}$$

$$R_2 = (a \leqq x \leqq b;\ 0 \leqq p \leqq p_1).$$

Let $f(x, p) = \dfrac{1}{q - x} \cdot \Theta(x)$, where $\Theta(x) = \dfrac{\sin px}{x(q + x)}$.

The derivatives of $\Theta(x)$ are

$$\Theta'_p = \frac{\cos px}{q + x} \quad \text{and} \quad \Theta'_x = \frac{px(q + x)\cos px - \sin px(q + 2x)}{x^2(q + x)^2},$$

both of which are continuous in R_2. Moreover, q is a constant independent of p. A corollary of Hardy's* justifies differentiation under the principal value integral sign:

$$\frac{d I_2}{dp} = P_q \int_a^b \frac{\partial f}{\partial p}\, dx = P_q \int_a^b \frac{\cos px}{q^2 - x^2}\, dx. \tag{6}$$

We may apply this same course of reasoning again, having now:

$$f(x) = \frac{\cos px}{q^2 - x^2}, \qquad \Theta(x) = \frac{\cos px}{q + x},$$

$$\Theta'_p = \frac{-x \sin px}{q + x} \quad \text{and} \quad \Theta'_x = \frac{p(q + x)\sin px - \cos px}{(q + x)^2}.$$

We obtain (5).

3°. $$\frac{d^2 I_3}{dp^2} = \int_b^\infty \frac{\partial^2 f}{\partial p^2}\, dx. \tag{7}$$

$$R''_3 = (b \leqq x < \infty, \quad 0 < p_0 \leqq p \leqq p_1) = (x \text{ in } \mathfrak{A}_3,\ p \text{ in } \mathfrak{B}''_3).\dagger$$

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* *Proc. Lond. Math. Soc.*, Vol. XXXV, p. 84.

† v. Lect. I, p. 387.

$f(x, p)$ is regular in $R_3' = (b, \infty\,;\ 0, p_1)$ and continuous in p for each x in $\mathfrak{A}_3$, $f_x'(x, p)$ is regular in R_3' and uniformly continuous in p in any rectangle $(b, B;\ 0, p_1)$. Since $|f_p'| \leqq \frac{1}{q^2 - x^2}$, $\int_0^\infty f_p'$ is uniformly convergent in $\mathfrak{B}_3'$. Therefore, we have:*

$$\frac{dI_3}{dp} = \int_0^\infty \frac{\partial f}{\partial p}\,dx \text{ in } (0, p_1). \tag{8}$$

We may differentiate under the integral sign again, for p in $(0, p_1)$. In this case, the derivative $f_{pp}'' = \frac{-x \sin px}{q^2 - x^2}$ is integrable in (b, ∞) for each p in $\mathfrak{B}_3''$, since $\int_b^x \sin px$ is limited in $\mathfrak{A}_3$, $\frac{-x}{q^2 - x^2}$ is monotone in $V(\infty)$ and $\doteq 0$ as $x \doteq \infty$.† And $\int_b^\infty dx$ converges uniformly in $\mathfrak{B}_3'' = (p_0, p_1)$, since we have

$$\int_b^\infty dx = \int_b^\infty \frac{\sin px}{x} \cdot \frac{1}{1 - q^2/x^2}\,dx,$$

where $\int_b^\infty \frac{\sin px}{x}\,dx$ converges uniformly in $\mathfrak{B}_3''$ and $\frac{1}{1 - q^2/x^2}$ is monotone.‡

Hence§ we may write:

$$\frac{d^2 I_3}{dp^2} = \int_0^\infty \frac{\partial^2 f}{\partial p^2}\,dx \quad \text{in } (p_0, p_1). \tag{7}$$

4°. Bringing together the results of 1°, 2° and 3°:

$$\frac{d^2 I}{dp^2} = P_q \int_0^\infty \frac{\partial^2 f}{\partial p^2}\,dx \quad \text{in } (p_0, p_1), \tag{9}$$

also

$$\frac{dI}{dp} = P_q \int_0^\infty \frac{\partial f}{\partial p}\,dx \quad \text{in } (0, p_1). \tag{10}$$

(9) gives

$$\frac{d^2 I}{dp^2} = \int_0^\infty \frac{\sin px\,dx}{x} - q^2 P_q \int_0^\infty \frac{\sin px\,dx}{x(q^2 - x^2)} \tag{11}$$

$$= \frac{\pi}{2} - q^2 I \quad \text{if } p > 0, \tag{12}$$

or

$$= 0 \qquad \text{if } p = 0.$$

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* Lect. I, p. 493, § 683, 2. This theorem holds true if f_y' is uniformly continuous in y in any arbitrary interval $(a < b)$, except on the lines $x = a_1$. Cf. Lect. I, §§ 666, 669.

† Lect. I, p. 453, § 643.

‡ Lect. I, p. 470; p. 468, § 661, 2.

§ Lect. I, p. 493, § 683, 2.

Now, multiplying (12) by $2\frac{dI}{dp}$ and integrating, we have

$$\left(\frac{dI}{dp}\right)^2 = \pi I - 2q^2 I^2 + C.$$

Setting $p=0$ and using T 17, F 1 (see § 4):

$$C=0 \quad \text{and} \quad \frac{dI}{dp} = \pm\sqrt{\pi I - 2q^2 I}.$$

Integrating again, setting $p=0$, we have $I=\frac{\pi}{2q^2}(1-\cos pq)$; $p \gtrless 0$. (2a)

Similarly, making the proper changes in (11) we get the result for $p<0$. (2b)

Criticism.

(a) For T 172, F 4 to be true, it is necessary that $p \gtrless 0$. This is shown clearly at equation (12), § 4, where the first integral of (11) is $-\frac{\pi}{2}$ if $p<0$.

(b) The condition on q given in (2) is necessary. For if $q=0$, the integral is divergent in any interval $(0<a)$. If $q<0$ we must consider $P_{|q|}$.

(c) In accordance with de Haan's definition of the improper integral, the notation in T 172, F 4 conveys no hint of the fact that the integral exists only as a principal value.

(d) The method of proof used here is that of de Haan in VIII Méth. 25, N 3, p. 522; where, however, differentiation under the integral sign is employed twice without any justification.

$$\int_0^\infty \frac{\cos px\,dx}{q^2-x^2} = \frac{\pi}{2q}\sin pq, \qquad \text{Nouv. Tab., T 161, F 5.} \tag{13}$$

This becomes:

$$\left.\begin{aligned} P_q\int_0^\infty \frac{\cos px\,dx}{q^2-x^2} &= \frac{\pi}{2q}\sin pq; \quad p \gtrless 0,\ q>0, \\ \text{or} \qquad &= -\frac{\pi}{2q}\sin pq; \quad p<0,\ q>0. \end{aligned}\right\} \tag{14}$$

According to the preceding proof, equation (2) may be differentiated under its integral sign, giving (14). See, again, VIII Méth. 25, N 3, p. 522. This integral is evaluated in one of Hardy's papers* also.

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* *Quart. Journ. P. A. Math.*, Vol. XXXII (1901): "On Differentiation and Integration under the Integral Sign."

$$\int_0^\infty \frac{x \sin px\, dx}{q^2 - x^2} = -\frac{\pi}{2} \cos pq, \qquad \text{Nouv. Tab., T 161, F 4} \qquad (15)$$

(see "Corrections" also).

This becomes:

$$\left.\begin{aligned} P_q \int_0^\infty \frac{x \sin px\, dx}{q^2 - x^2} &= -\frac{\pi}{2} \cos pq; \quad p > 0,\ q > 0, \\ \text{or} \qquad &= \frac{\pi}{2} \cos pq; \quad p < 0,\ q > 0. \end{aligned}\right\} \qquad (16)$$

As before, by differentiating under the integral sign in (13), we get (15). See again the references given under (13).

$$\left.\begin{aligned} \int_0^\infty \frac{\sin px \sin rx\, dx}{q^2 - x^2} &= -\frac{\pi}{2q} \cos pq \,.\, \sin qr; \quad p > r, \\ \text{or} \qquad &= -\frac{\pi}{4q} \sin 2pq; \quad p = r, \\ \text{or} \qquad &= -\frac{\pi}{2q} \sin pq \,.\, \cos qr; \quad p < r, \end{aligned}\right\} \qquad (17)$$

Nouv. Tab., T 166, F 1.

These become:

$$\left.\begin{aligned} P_q \int_0^\infty \frac{\sin px \sin rx}{q^2 - x^2}\, dx &= -\frac{\pi}{2q} \cos pq \,.\, \sin qr; \quad p \gtreqless r \gtreqless 0,\ q > 0, \\ \text{or} \qquad &= -\frac{\pi}{2q} \sin pq \,.\, \cos qr; \quad r \gtreqless p \gtreqless 0,\ q > 0. \end{aligned}\right\} \qquad (18)$$

In the proof use (14) = T 161, F 5, writing

$$\sin px \,.\, \sin rx = \tfrac{1}{2} \cos (p - r)x - \tfrac{1}{2} \cos (p + r)x.$$

And similarly T 166, F 2 and F 3.

$$\int_0^\infty \frac{x \sin px\, dx}{(q^2 - x^2)^2} = -\frac{p\pi}{4q} \sin pq, \qquad \text{Nouv. Tab., T 171, F 1.} \qquad (19)$$

$$\int_0^\infty \frac{x^3 \sin px\, dx}{(q^2 - x^2)^2} = \frac{\pi}{4} (2 \cos pq - pq \sin pq), \qquad \text{Nouv. Tab., T 171, F 2.} \qquad (20)$$

$$\int_0^\infty \frac{\cos px\, dx}{(q^2 - x^2)^2} = \frac{\pi}{4q^3} (\sin pq - pq \cos pq), \qquad \text{Nouv. Tab., T 171, F 3.} \qquad (21)$$

$$\int_0^\infty \frac{x^2 \cos px\, dx}{(q^2 - x^2)^2} = \frac{-\pi}{4q} (\sin pq + pq \cos pq), \qquad \text{Nouv. Tab., T 171, F 4.} \qquad (22)$$

But these four do not even exist, if $q>0$, either as integrals or as principal value integrals, except in the degenerate case of (20), where $p>0$ and $q=0$ give the integral used above at (12).

De Haan derives these in VIII at p. 565, where they all depend upon the differentiation under the integral sign of (14) = T 161, F 5 and (15) = T 161, F 4.

$$\int_0^\infty \frac{x \sin px\, dx}{(q^2-x^2)(r^2-x^2)} = \frac{\pi}{2(q^2-r^2)}(\cos pq - \cos pr), \tag{23}$$

Nouv. Tab., T 174, F 3.

$$\int_0^\infty \frac{x^3 \sin px\, dx}{(q^2-x^2)(r^2-x^2)} = \frac{\pi}{2(q^2-r^2)}(r^2\cos pr - q^2\cos pq), \tag{24}$$

Nouv. Tab., T 174, F 4.

The first of these becomes:

$$P_{q,r}\int_0^\infty \frac{x \sin px\, dx}{(q^2-x^2)(r^2-x^2)} = \frac{\pi}{2(q^2-r^2)}(\cos pq - \cos pr); \tag{25}$$

$p>0,\ q>r>0.$

And exactly the same revision is necessary in (24). (26)

In VIII, p. 331, de Haan decomposes these integrands into partial fractions and uses T 161, F 4 in both cases.

Similarly, to T 175, F 3 and F 4 the principal value sign and conditions on the constants must be added. Then we have:

$$P_{q,r}\int_0^\infty \frac{\cos px\, dx}{(q^2-x^2)(r^2-x^2)} = \frac{\pi}{2qr(q^2-r^2)}(q\sin pr - r\sin pq); \tag{27}$$

$p \gtrless 0,\ q>r>0.$

$$P_{q,r}\int_0^\infty \frac{x^2 \cos px\, dx}{(q^2-x^2)(r^2-x^2)} = \frac{\pi}{2(q^2-r^2)}(r\sin pr - q\sin pq); \tag{28}$$

$p \gtrless 0,\ q>r>0.$

As in the last group, we use T 161, F 5, as at VIII, p. 331, where, however, the evaluations given in F (484) and F (485) are wrong.

$$\int_0^\infty \arctan\frac{x}{q}\cdot\frac{dx}{(p-x)^2} = \frac{1}{p^2+q^2}\left(q\log\frac{q}{p} - \frac{1}{2}p\pi\right), \tag{29}$$

Nouv. Tab., T 249, F 2.

$$\int_0^\infty \arctan\frac{x}{q}\cdot\frac{x\,dx}{(p^2-x^2)^2} = \frac{-\pi}{4(p^2+q^2)}, \tag{30}$$

Nouv. Tab., T 249, F 4.

$$\int_0^\infty \operatorname{arc\,cot}\frac{x}{q}\cdot\frac{dx}{(p-x)^2} = \frac{q}{p(p^2+q^2)}\left(p\log\frac{p}{q}+\frac{1}{2}q\pi\right), \tag{31}$$

Nouv. Tab., T 249, F 9.

$$\int_0^\infty \operatorname{arc\,cot}\frac{x}{q}\cdot\frac{x\,dx}{(p^2-x^2)^2} = \frac{-\pi q^2}{4p^2(p^2+q^2)}, \quad \text{Nouv. Tab., T 249, F 11.} \tag{32}$$

But, these four are not even convergent at $x=p$, $p>0$, since $f(x)$ has a pole of order 2 there, and $\frac{1}{(p-x)^2}>0$ on each side of that point.

De Haan derives these at VIII, p. 595, from VIII, p. 230, F (207) by the method of integration by parts.

$$\int_0^\infty \frac{dx}{q^2-x^2} = 0, \quad \text{Nouv. Tab., T 17, F 1.} \tag{33}$$

This becomes:

$$P_{|q|}\int_0^\infty \frac{dx}{q^2-x^2} = 0; \quad q \neq 0.$$

$$\int_0^\infty \frac{dx}{(q^2-x^2)^2} = 0 \text{ (v. T 17, F 1)}, \quad \text{Nouv. Tab., T 17, F 17.} \tag{34}$$

This integral does not exist. De Haan derives it from T 17, F 1.

§ 5.

$$\int_a^b \operatorname{si}(rx)\cdot\phi(x)\,dx = \int_0^1 \frac{dy}{y}\int_a^b \sin rxy\cdot\phi(x)\cdot dx, \tag{1}$$

VIII, Méth. 18, N 24, p. 461, Théorème (XXXII).

We shall prove the following theorem:

Let $\Theta(x)$, $\Theta'(x)$ *be continuous, and limited in* $\mathfrak{A}=(0,\infty)$.

Let $\frac{\Theta(x)}{q-x}$ *be absolutely integrable in* (a,∞), *where* $0<q<a$.

Let $f(x,y)=\frac{\Theta(x)}{q-x}\cdot\frac{\sin rxy}{y}$, *for* $y>0$, $x\neq q$,

$=\frac{\Theta(x)}{q-x}\cdot r$, *for* $y=0$, $x\neq q$.

Then $P_q\int_0^b dx\int_0^1 f(x,y)\,dy = \int_0^1 dy\,P_q\int_0^b f(x,y)\,dx$; (2)

b finite or infinite, $q<b$.

CASE 1°: b finite.

$P_q \int_0^b f\,dx$ is convergent for each y in $\mathfrak{B} = (0, 1)$;* Moreover it is uniformly convergent in $\mathfrak{B}$.† For (i) there are no other infinite discontinuities aside from $x = q$ and (ii) at the point $x = q$ we have:

$$\varepsilon > 0;\ \theta_0 > 0;\ \left| P_q \int_{q-\theta}^{q+\theta} \frac{\Theta(x) \,.\, \sin rxy \,.\, dx}{(q-x) \,.\, y} \right|, \quad \text{by a corollary of Hardy's,‡}$$

$$= \left[\frac{\partial}{\partial x} \frac{\Theta(x) \,.\, \sin r(x)\, y}{y} \right]_{x=q+\mu} . 2\theta; \quad -\theta \leqq \mu \leqq \theta,$$

$$= \left\{ \Theta'(q+\mu) \frac{\sin r(q+\mu)\, y}{y} + \Theta(q+\mu) \,.\, r \cos(q+\mu)\, y \right\} . 2\theta,$$

$$\leqq M \,.\, 2\theta; \quad M = \text{max. of } \{\ \} \text{ in } \mathfrak{A} = (0, b)$$

$$< \varepsilon \text{ for each } \theta < \theta_0 \text{ and all } y \text{ in } \mathfrak{B},$$

if we take $\theta_0 < \varepsilon / 2M$.

Now, by changing the value of f along $y = 0$ to $\frac{\Theta(x)}{q-x} \,.\, rx$, f becomes continuous in R except along $x = q$. Therefore,§

$$P_q \int_0^b dx \int_0^1 f\,dy = \int_0^1 dy\ P_q \int_0^b f\,dx; \quad b \text{ finite.}$$

CASE 2°: b infinite.

We must prove that $\int_a^\infty dx \int_0^1 f\,dy = \int_0^1 dy \int_a^\infty f\,dx$, where $q < a$.

$f(x, y)$ having no infinite discontinuities in $R = (a, \infty;\ 0, 1)$ and being integrable as to x and as to y in R, is regular.

$\int_a^\infty f\,dx$ converges uniformly in $\mathfrak{B}$, except at 0, since $|f| \leqq \left| \frac{\Theta(x)}{q-x} \,.\, \frac{1}{\eta} \right|$ for every y in $(\eta < 1)$; $\eta > 0$ and this is integrable in $\mathfrak{A}$.

$\int_0^1 f\,dy$ is a proper integral.

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* G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXIV, p. 24.
† G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXIV, p. 66, Definition.
‡ Vol. XXXIV, pp. 26, 27.
§ G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXV, p. 94.

Again, $\int_a^\infty dy \int_0^y f\,dy$ converges uniformly in $\mathfrak{B}$.* For, $\frac{\Theta(x)}{q-x}$ being continuous in $\mathfrak{A}$ and $\int_0^y \frac{\sin r\,xy\,dy}{y}$ being limited in R, $\left(0 \leqq \int_0^y \frac{\sin r\,xy}{y} \leqq \pi,\ r>0\right)$, $\frac{\Theta(x)}{q-x}\int_0^y \frac{\sin r\,xy\,dy}{y}$ is regular in R; $\frac{\Theta(x)}{q-x}$ is absolutely integrable in $\mathfrak{A}$; and finally $\int_0^y \frac{\sin r\,xy\,dy}{y}$ is not only limited in R but also is a continuous function of x in $(a<G)$ for each y in $\mathfrak{B}$,† and therefore integrable in any $(a<G)$.

Hence‡ we have the theorem:

$$\int_a^\infty dx \int_0^1 f\,dy = \int_0^1 dy \int_a^\infty f\,dx.$$

1° and 2° together give the theorem:

$$P_q \int_0^b \frac{\Theta(x)\,.\,\mathrm{si}\,(rx)\,dx}{q-x} = \int_0^1 \frac{dy}{y}\, P_q \int_0^b \frac{\Theta(x)\sin r\,xy\,dx}{q-x}, \qquad (2)$$

$q<b$, b finite or infinite.

$$\int_0^\infty \frac{\sin px\,\mathrm{si}\,(rx)\,dx}{q^2-x^2} = -\frac{\pi}{2q}\cos pq\,.\,\mathrm{si}\,(qr), \quad p \gtrless r, \quad \text{Nouv. Tab., T 463, F 1,}$$

or

$$= -\frac{\pi}{2q}\cos pq\,.\,\mathrm{si}\,(pq) + \frac{\pi}{2q}\sin pq\,\{\mathrm{ci}\,(pq) - \mathrm{ci}\,(qr)\}, \quad p \leqq r. \qquad (3)$$

This integral should be $P_q \int_0^\infty$ and the conditions should be

$$p \gtrless r \gtrless 0,\ q>0 \quad \text{and} \quad r \gtrless p \gtrless 0,\ q>0 \quad \text{respectively.} \qquad (4)$$

For, apply the preceding theorem (2), letting $\Theta(x) = \frac{\sin px}{q+x}$.

Then $\Theta'(x) = \frac{p\,(q+x)\cos px - \sin px}{(q+x)^2}$ and $\Theta(x)$ are both continuous and limited in $\mathfrak{A} = (0, \infty)$.

$\frac{\sin px}{q^2-x^2}$ is absolutely integrable in (a, ∞); $a>q$.

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* Lect. I, p. 467, § 660, 1.
† Lect. I, p. 390, § 563, 3.
‡ Lect. I, p. 483, § 674, 1.

The conditions of the theorem being satisfied, we have:

$$P_q \int_0^\infty \frac{\sin px \,\mathrm{si}\,(rx)\,dx}{q^2 - x^2} = \int_0^1 \frac{dy}{y} P_q \int_0^\infty \frac{\sin rxy \sin px}{q^2 - x^2}\,dx$$

$$= -\frac{\pi}{2q} \cos pq \,.\, \mathrm{si}\, qr,$$

using § 4, (17) = T 166, F 1.

In the other case where $r \gtrless p$, the interval of integration $\mathfrak{B} = (0, 1)$ must be divided into two parts; all else is similar.

Criticism.

De Haan gives Théorème (XXXII)* as an immediate application of his "Méthode 18." This method (replacing a factor by a definite integral) is due to Cauchy, being given in his "Mémoire sur diverses formules relatives à la théorie des intégrales définies, etc."† De Haan's statement‡ is as follows:

"Let $\int_p^q g(x, y)\,dy = h(x)$,

"Then $\int_a^b h(x) \,.\, \phi(x)\,dx = \int_p^q dy \int_a^b g(x, y) \,.\, \phi(x)\,dx$, provided that $g(x, y) \,.\, \phi(x)$ does not become discontinuous for any value of x in $(a < b)$ and of y in $(p < q)$."

This theorem will be useful in cases where we know the value of $\int_a^b g \,.\, \phi\,dx$.

The sufficient condition here given by de Haan is not found in Cauchy, and, indeed, is overlooked by de Haan himself in applying Théorème (XXXII) to T 463, F 1. For in that formula $\phi(x) = \frac{\sin px}{q^2 - x^2}$, which is not continuous in $R = (0, \infty\,;\ 0, 1)$.

Again, from this application, it is clear that the theorem is held to be applicable in case one of the integrals, $\int_a^b f\,dx$, becomes a principal value. From our standpoint, this must be explicitly stated. In the case of the principal value we must use such theorems as those of Hardy, employed in the proof above.

Finally, from this same context again, we see that b is meant to be infinite as well as finite. This contingency is covered by Case 2° above.

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* VIII, Méth. 18, N 24, p. 461.

† "Œuv. Comp.", Sér. 2, Tome 1, p. 467.

‡ VIII, p. 437.

§ 6.

$$\int_{-a}^{a} \frac{\sin x}{\cos 2x}\, dx = 0, \quad a < \pi/4, \qquad \text{VIII, p. 231, F (210).} \tag{1}$$

$$= 0, \quad a \gtrless \pi/4, \qquad \text{VIII, p. 231, F (211).} \tag{2}$$

De Haan gives the primitive function

$$F(x) = \int \frac{\sin x}{\cos 2x}\, dx = \frac{1}{2\sqrt{2}} \log \frac{1 + \sqrt{2}\,.\cos x}{1 - \sqrt{2}\,.\cos x}.$$

We note that the integrand, $f(x)$, is an odd function, $F(x)$ an even function.

When $a < \pi/4$, $\int_{-a}^{a} = 0$, since f is odd. (1)

When $a = \pi/4$, $P_{\pm \pi/4} \int_{-\pi/4}^{\pi/4} \frac{\sin x\, dx}{\cos 2x} = 0$, using $F(x)$. (3)

When $\pi/4 < a < 3\pi/4$, $P_{-\pi/4,\, \pi/4} \int_{-a}^{a} = 0$; also $P_{\pm \pi/4} \int_{-a}^{a} = 0$. (4)

When $a = 3\pi/4$, $P_{-\pi/4,\, \pi/4,\, \pm 3\pi/4} \int_{-3\pi/4}^{3\pi/4} = 0$. (5)

And so on, for $3\pi/4 < a \leqq \infty$, taking in each case the proper sort of principal value integral.

$$\int_{-a}^{a} \frac{\cos x\, dx}{\cos 2x} = \frac{1}{\sqrt{2}} \log \frac{1 + \sin \sqrt{2}\, a}{1 - \sin \sqrt{2}\, a}, \quad a < \pi/4, \quad \text{VIII, p. 231, F (212).} \tag{6}$$

$$\int_{-a}^{a} \frac{\cos x\, dx}{\cos 2x} = \infty, \quad a \gtrless \pi/4, \qquad \text{VIII, p. 231, F (213).} \tag{7}$$

F (213) is wrong except for the values $a = (2n + 1)\pi/4$.

De Haan gives the primitive function

$$F(x) = \int \frac{\cos x}{\cos 2x} = \frac{1}{2\sqrt{2}} \log \frac{1 + \sqrt{2} \sin x}{1 - \sqrt{2} \sin x}.$$

$f(x)$ is an even function.

When $a < \pi/4$, we get F (212) at once. (6)

When $a = \pi/4$, $P_{\pm \pi/4} \int_{-\pi/4}^{\pi/4} \frac{\cos x\, dx}{\cos 2x}$ is divergent. (7)

When $\pi/4 < a < 3\pi/4$, $P_{-\pi/4,\, \pi/4} \int_{-a}^{a} = P_{-\pi/4} \int_{-a}^{0} + P_{\pi/4} \int_{0}^{a}$

$$= \frac{1}{\sqrt{2}} \log \frac{1 + \sqrt{2} \sin a}{1 - \sqrt{2} \sin a}. \tag{8}$$

Comparing (4), we note that here $P_{\pm\pi/4}\int_{-a}^{a}$ does not converge.

When $a = 3\pi/4$, $P_{-\pi/4,\,\pi/4,\,\pm 3\pi/4}\int_{-3\pi/4}^{3\pi/4} = +\infty$. (9)

When $3\pi/4 < a < 5\pi/4$, $P_{-3\pi/4,\,-\pi/4,\,\pi/4,\,3\pi/4}\int_{-a}^{a} = \text{etc.}$ (10)

And so on as in the previous case, at (8).

Instead of F (213) we have

$$P\int_{-a}^{a}\frac{\cos x\,dx}{\cos 2x} = \frac{1}{\sqrt{2}}\log\frac{1+\sqrt{2}\sin a}{1-\sqrt{2}\sin a},\quad a > \pi/4 \neq (2n+1)\pi/4, \tag{11}$$

and $P\int_{-a}^{a}\frac{\cos x\,dx}{\cos 2x}$ is divergent, $a = (2n+1)\pi/4$. (12)

§ 7.

$$\int_0^\infty \operatorname{arc\,cot}\frac{x}{p}\cdot\frac{x\,dx}{x^2-q^2} = \frac{\pi}{4}\log\frac{p^2+q^2}{q^2},\quad \text{Nouv. Tab., T 248, F 10.} \tag{1}$$

This becomes

$$P_q\int_0^\infty \operatorname{arc\,cot}\frac{x}{p}\cdot\frac{x\,dx}{x^2-q^2} = \frac{\pi}{4}\log\frac{p^2+q^2}{q^2};\quad p>0,\ q>0. \tag{2}$$

a°. The integral is convergent at ∞, for x arc cot x/p is limited in $(0, \infty)$; and at q, as a principal value,* since b° shows that Θ'_x is continuous near $x = q$.

b°. The integrand, $f(x, p)$, is a continuous function in any $R = (0, \infty;\ p_0 < p_1)$; *i. e.*, for any $p_0 > 0$, except along $x - q = 0$.

$f'_p(x, p) = \frac{x^2}{x^2-q^2}\cdot\frac{1}{p^2+x^2}$, $(p>0)$, which is continuous in R except along $x - q = 0$.

Let $\frac{x}{x+q}\operatorname{arc\,cot}\frac{x}{p} = \Theta(x);\quad p>0.$

$$\Theta'_x = \frac{-px}{(x+q)(p^2+x^2)} + \frac{2x+q}{(x+q)^2}\cdot\operatorname{arc\,cot}\frac{x}{p};\quad p>0.$$

$$\Theta'_p = \frac{x^2 p}{(x+q)(p^2+x^2)};\quad p>0.$$

* G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXIV, p. 24.

c°. We may differentiate under the integral sign in $R_1 = (0, q/2;\ p_0 < p_1)$. For both f and f'_p are continuous in R_1.

$$\frac{d}{dp}\int_0^{q/2} f\,dx = \int_0^{q/2} \frac{\partial f}{\partial p}\,dx \quad \text{in } (p_0 < p_1).$$

d°. In $R_2 = (q/2,\ 3q/2;\ p_0, p_1)$ we may differentiate* under the sign of integration (the principal value). For q is independent of p; and by b°, Θ'_x and Θ'_p are continuous in R_2.

$$\frac{d}{dp} P_q \int_{q/2}^{3q/2} f\,dx = P_q \int_{q/2}^{3q/2} \frac{\partial f}{\partial p}\,dx \quad \text{in } (p_0 < p_1).$$

e°. In $R_3 = (3q/2,\ \infty\ ;\ p_0, p_1)$ we may do likewise.

For, $f(x, p)$ is regular in R_3 and continuous as to p for each x in $\mathfrak{A}_3$; $f'_p(x, p)$ is regular in R_3 and uniformly continuous in any $R'_3 = (3q/2 < b;\ p_0 < p_1)$, *i. e.*, for any $b > 3q/2$; $\int_{3q/2}^{\infty} f'_p\,dx$ converges uniformly in $\mathfrak{B} = (p_0 < p_1)$, since $|f'_p| \leq \frac{x^2}{x^2 - q^2} \cdot \frac{1}{p_0^2 + x^2}$ for all p in $(p_0 < p_1)$, and this is integrable in $\mathfrak{A}_3 = (3q/2,\ \infty)$. So the conditions of Lect. I, p. 493, § 683, 2 are all satisfied and we have

$$\frac{d}{dp}\int_{3q/2}^{\infty} f\,dx = \int_{3q/2}^{\infty} \frac{\partial f}{\partial p}\,dx \quad \text{in } (p_0 < p_1).$$

f°. Combining c°, d°, e°, we have

$$\frac{dI}{dp} = P_q \int_0^{\infty} \frac{\partial f}{\partial p}\,dx, \quad \text{all } p \text{ in } (p_0 < p_1),\ p_0 > 0;\ \textit{i. e.}, \text{ for any } p > 0,$$

$$= -P_q \int_0^{\infty} \frac{x^2}{q^2 - x^2} \cdot \frac{1}{p^2 + x^2}\,dx, \quad \text{breaking up into partial fractions and using T 17, F 1, § 4,}$$

$$= \frac{p\pi}{2} \cdot \frac{1}{p^2 + q^2};\quad q > 0,\ p > 0.$$

Integrating with respect to p from 0 to p (assigning to $\frac{dI}{dp}$ *any* value at $p = 0$), we have

$$I = \frac{\pi}{2}\int_0^p \frac{p\,dp}{p^2 + q^2}\,dx = \frac{\pi}{4}\log\frac{p^2 + q^2}{q^2}. \tag{2}$$

* G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXV, p. 84.

Criticism.

This proof follows the method of de Haan in VIII, p. 355, Méth. 10, N 10. De Haan's proof is formalistic, *e. g.*, the question of uniform convergence is not considered at all.

$$\int_0^1 (x \operatorname{arc\,cot} x - \frac{1}{x} \arctan x) \frac{dx}{1-x^2} = -\frac{\pi}{4} \log 2, \tag{3}$$

Nouv. Tab., T 232, F 1.

We shall establish the correctness of this evaluation.

a°. $$P_1 \int_0^\infty \frac{x}{x^2-1} \operatorname{arc\,cot} x \, . \, dx$$

$$= \lim_{\theta=0} \left\{ \int_0^{1-\theta} + \int_{1+\theta}^\infty \right\}$$

$$= \lim_{\theta=0} \left\{ \int_0^{1-\theta} dx + \int_{1/1+\theta}^{\theta} dy \frac{-1}{y^2} \cdot \frac{1/y}{\frac{1}{y^2}-1} \cdot \arctan y \right\};$$

changing variable,

$$= \lim_{\theta=0} \left\{ \int_0^{1-\theta} \frac{x \operatorname{arc\,cot} x \, dx}{x^2-1} + \int_0^{1-\theta} \frac{1}{x} \cdot \frac{\arctan x}{x^2-1} \right\}$$

$$+ \lim_{\theta=0} \int_{1-\theta}^{1/1+\theta} \frac{1}{x} \cdot \frac{\arctan x \, dx}{x^2-1},$$

since by b° this last limit exists.

b°. $$\lim_{\theta=0} \int_{1-\theta}^{1/1+\theta} \frac{1}{x} \cdot \frac{\arctan x \, dx}{x^2-1} = 0.$$

For, $$\left| \int_{1-\theta}^{1/1+\theta} \right| = \arctan \eta \int_{1-\theta}^{1/1+\theta} \frac{1}{x \, . \, (1-x^2)}$$ (by the first Theorem of the Mean), where $1-\theta \leqq \eta \leqq 1/1+\theta$,

$$\leqq \arctan \eta \left\{ \frac{1}{1+\theta} - (1-\theta) \right\} \frac{1}{(1-\theta) \cdot \left(1 - \frac{1}{1+\theta}\right)}$$

$$= \arctan \eta \cdot \frac{\theta^2}{1+\theta} \cdot \frac{1+\theta}{(1-\theta) \, . \, \theta}$$

$$= \arctan \eta \cdot \frac{\theta}{1-\theta},$$

and therefore $\lim_{\theta=0} = 0.$

c°. Accordingly,

$$P_1\int_0^\infty \frac{\operatorname{arc\,cot} x \,.\, x\,dx}{1-x^2} = \lim_{\theta=0}\int_0^{1-\theta}\left\{\frac{x \operatorname{arc\,cot} x - \frac{1}{x}\operatorname{arc\,tan}\ x}{1-x^2}\right\}dx.$$

Using (2) = T 248, F 10, with $p=q=1$ (= T 232, F 1), we have

$$-\frac{\pi}{4}\log 2 = \int_0^1 \frac{(x \operatorname{arc\,cot} x - \frac{1}{x}\operatorname{arc\,tan} x)\,dx}{1-x^2}, \qquad (3)$$

and we do not need to inquire whether this integral is proper or improper.

Criticism.

In VIII, Méth. 10, N 10, p. 355, de Haan uses $P_1\int_0^\infty \frac{x \operatorname{arc\,cot} x\,dx}{x^2-1} = \frac{\pi}{4}\log 2$ as the basis of the derivation. He immediately breaks this integral up into two divergent integrals $\int_0^1$ and $\int_1^\infty$ and then changes the variable in the latter of these. The proof here given avoids the use of divergent integrals.

Other formulas at once derived from (2) are:

$$P_q\int_0^\infty \operatorname{arc\,cot}\frac{x}{p}\cdot\frac{x\,dx}{x^4-q^4} = \frac{\pi}{8q^2}\log\frac{p^2+q^2}{(p+q)^2}; \qquad (4)$$

$q,\ p+q \neq 0$ (= Nouv. Tab., T. 248, F 13).

$$P_q\int_0^\infty \operatorname{arc\,cot}\frac{x}{p}\cdot\frac{x^3\,dx}{x^4-q^4} = \frac{\pi}{8}\log\frac{(p+q)^2(p^2+q^2)}{q^4}; \qquad (5)$$

$p, q > 0$ (= Nouv. Tab., T 248, F 14).

$$P_{\pi/4}\int_0^{\pi/2} \frac{\operatorname{arc\,tan}(p\tan x)\,dx}{\tan x\,.\,\cos 2x} = \frac{\pi}{4}\log(1+p^2); \qquad (6)$$

$p > 0$ (= Nouv. Tab., T 342, F 8).

In this last case make change of variable $x = \cot y$ and set $q=1$. Then use Hardy's Theorem on Change of Variable.*

§ 8.

$$\int_{-\infty}^\infty \frac{\operatorname{arc\,tan}(a+bx)\,dx}{1+x^2} = \frac{\pi}{2}\left\{\operatorname{arc\,tan}\frac{2a}{1-a^2-b^2} - \operatorname{arc\,tan}\frac{2ab}{1+a^2-b^2}\right\}; \qquad (1)$$

$a > 0$, Nouv. Tab., T 254, F 10.

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* G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXIV, p. 33.

We shall follow the outline of de Haan's method,* making his proof rigorous.

a°. We may differentiate with respect to b under the integral sign.

$f(x,b)=\dfrac{\arctan(a+bx)}{1+x^2}$ is regular in $R=(0,\infty;\ b_0<b_1),\ b_0>0$, and continuous in b for each x in $\mathfrak{A}$.

$f_b'(x,b)=\dfrac{x}{1+x^2}\cdot\dfrac{1}{1+(a+bx)^2}$ is regular in R; and uniformly continuous in b in $R'=(0,G;\ b_0,b_1)$ for any G, since it is continuous in R'.

$\int_0^\infty f_b'\,dx$ is uniformly convergent in $\mathfrak{B}=(b_0,b_1)$, since we have $|f_b'|\leqq\dfrac{x}{1+x^2}\cdot\dfrac{1}{1+(a+b_0x)^2}$ for all b in (b_0,b_1) and this is integrable in $\mathfrak{A}$.

Hence,† $\quad\dfrac{d}{db}\int_0^\infty f\,dx=\int_0^\infty\dfrac{\partial f}{\partial b}\,dx$, in $(b_0,b_1),\ b_0>0$.

Similarly for $\int_{-\infty}^0$. Together: $\dfrac{d}{db}\int_{-\infty}^\infty f\,dx=\int_{-\infty}^\infty\dfrac{\partial f}{\partial b}\,dx$, for $b>0$.

b°. $\dfrac{dI}{db}=\int_{-\infty}^\infty\dfrac{\partial f}{\partial b}\,dx;\ b>0,$

$$=\int_{-\infty}^\infty\frac{x}{(1+x^2)\{1+(a+bx)\}^2}\,dx$$

$$=\frac{1}{(1+a^2-b^2)^2+4a^2b^2}\Big[2ab\int_{-\infty}^\infty\frac{1}{1+x^2}\,dx$$

$$-(1+a^2)\,2a\int_{-\infty}^\infty\frac{b}{1+(a+bx)^2}\,dx$$

$$+(1+a^2-b^2)\int_{-\infty}^\infty\Big\{\frac{x}{1+x^2}-\frac{b^2x}{1+(a+bx)^2}\Big\}dx\Big],\qquad(2)$$

where because the first two integrals converge, the third must; and because $a>0,\ (1+a^2-b^2)^2+4a^2b^2\neq0$,

$$\frac{dI}{db}=\frac{1}{(1+a^2-b^2)^2+4a^2b^2}\Big[2ab\pi-(1+a^2)\,2a\pi+(1+a^2-b^2)\int_{-\infty}^\infty\{\}\Big]$$

$$=\frac{1}{(1+a^2-b^2)^2+4a^2b^2}\Big[2a\pi(b-1-a^2)+(1+a^2-b^2)\int_{-\infty}^\infty\{\}\Big].\qquad(3)$$

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* VIII, p. 355, Méth. 10, N 10.

† Lect. I, p. 493, § 683, 2.

c°. Evaluation of this last integral.

$$\int_{-\infty}^{\infty}\left\{\frac{x}{1+x^2}-\frac{b^2x}{1+(a+bx)^2}\right\}dx \text{ converges for } b>0, \text{ and}$$

$$=P_{\pm\infty}\int_{-\infty}^{\infty}\left\{\ \right\}, \text{ obviously,} \tag{4}$$

$$=P_{\pm\infty}\int_{-\infty}^{\infty}\frac{x}{1+x^2}dx-P_{\pm\infty}\int_{-\infty}^{\infty}\frac{b^2x\,dx}{1+(a+bx)^2}dx \tag{5}$$

$$=0-P_{\pm\infty}\int_{-\infty}^{\infty}\frac{y-a}{y^2+a}dy, \text{ changing variable by } x=\frac{y-a}{b},\ b>0,$$

$$=-\lim_{\Theta=\infty}\int_{-\Theta}^{\Theta}\frac{d}{dy}\left[\tfrac{1}{2}\log(y^2+1)-a\arctan y\right]dy=a\pi. \tag{6}$$

d°. Hence, $\frac{dI}{db}=\frac{-(1+a^2+b^2-2b)a}{(1+a^2-b^2)^2+4a^2b^2}$; $b>0$ [from (3) and (6)]. (7)

Give to $\frac{dI}{db}$ some value at the point $b=0$, and integrate with respect to b from 0 to b; we get finally:

$$I=\frac{\pi}{2}\left\{\arctan\frac{2a}{1-a^2+b^2}-\arctan\frac{2ab}{1+a^2-b^2}\right\}. \tag{1}$$

Criticism.

(1) De Haan differentiates under the integral sign:

$$\frac{dI}{db}=\int_{-\infty}^{\infty}\frac{x}{1+(a+bx)^2}\cdot\frac{dx}{1+x^2}$$

$$=\frac{1}{(1+a^2-b^2)^2+4a^2b^2}\left[\int_{-\infty}^{\infty}\frac{2ab+(1+a^2-b^2)x}{1+x^2}dx\right.$$

$$\left.-\int_{-\infty}^{\infty}\frac{(1+a^2)2ab+(1+a^2-b^2)b^2x}{1+(a+bx)^2}dx\right]. \tag{8}$$

Now this step requires careful justification (see a°), and this is not given. The uniform convergence of $\int_{-\infty}^{\infty}f_b'\,dx$ must be considered. We find that $b=0$ must be excluded, for $\int_{-\infty}^{\infty}f_b'(x,0)\,dx$ does not converge, let alone converge uniformly.

(2) The integrals of the right member of (8) are divergent.

(3) Such conditions must be placed upon the constants that the denominator in (8) $= (1 + a^2 - b^2)^2 + 4a^2b^2 \neq 0$. A sufficient condition is $a > 0$.

(4) We note that the method of c°, where the principal value replaces the ordinary integral at (4), is a particularly useful one. The existence of the integral $\int_{-\infty}^{\infty}$ is known. Therefore also that of $P_{\pm\infty}$. Now, in the case of the former,

$$\int_{-\infty}^{\infty} \left\{ \frac{x}{1+x^2} - \frac{b^2 x}{1+(a+bx)^2} \right\} dx \neq \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx - b^2 \int_{-\infty}^{\infty} \frac{x}{1+(a+bx)^2}\, dx,$$

whereas

$$P\int_{-\infty}^{\infty} \left\{ \frac{x}{1+x^2} - \frac{b^2 x}{1+(a+bx)^2} \right\} dx = P\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx - b^2 P\int_{-\infty}^{\infty} \frac{x}{1+(a+bx)^2}\, dx,$$

and we at once lose the first of these, $P\int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx = 0$, which means a great simplification.*

§ 9.

$$\int_a^b \arctan \frac{q}{x} \cdot \phi(x)\, dx = \int_0^{\infty} \frac{\sin qy\, dy}{y} \int_a^b e^{-xy} \phi(x)\, dx, \qquad (1)$$

VIII, p. 452, Theorem XX.

We shall prove the following theorem:

Let $\phi(x)$ be continuous and limited in $\mathfrak{A} = (0, \infty)$.

Let $\phi(x)/x$ be limited and integrable in $(0^*, \infty)$.

Then $$\int_0^{\infty} \arctan \frac{q}{x} \cdot \phi(x)\, dx = \int_0^{\infty} \frac{\sin qy\, dy}{y} \int_0^{\infty} e^{-xy} \phi(x)\, dx. \qquad (2)$$

When $y = 0$, let the integrand $f = 0$, *i. e.*, $f(x, 0) = 0$.

In $R = (0, \infty\,;\ 0, \infty)$ $f(x, y)$ is continuous, except along $y = 0$; it is integrable in $\mathfrak{A}$, for every y in $\mathfrak{B}$; in $\mathfrak{B}$ for every x in $\mathfrak{A}$.

PART 1°. We shall consider $R_1 = (0, \infty\,;\ 0, 1)$ and show that

$$\int_0^{\infty} dx \int_0^1 f\, dy = \int_0^1 dy \int_0^{\infty} f\, dx.$$

Firstly, $f(x, y)$ is regular in R_1.

* Cf. G. H. Hardy: "On an Integral Equation," *Proc. Lond. Math. Soc.*, Ser. 2, Vol. VII (1909), p. 450.

In the second place, $\int_0^\infty e^{-xy}\phi(x)\,dx$ converges uniformly in $\mathfrak{B}$, except on $y=0$, *i. e.*, in any $(\sigma<1)$; $\sigma>0$. For we have, for every y in $(\sigma<1)$, $|e^{-xy}\phi(x)|\leqq e^{-x\sigma}\,.\,M$, where M is some constant, since $\phi(x)$ is limited in $\mathfrak{A}$. And this is integrable in $\mathfrak{A}$.

Finally, we shall show $\int_0^\infty \phi(x)\,dx\int_0^y \frac{\sin qy}{y}e^{-xy}\,dy$ converges uniformly in (0, 1). This integral may be written:

$$\int_0^\infty \frac{\phi(x)\,dx}{x}\int_0^y \frac{\sin qy}{y}e^{-xy}\,x\,dy.$$

Now, by the First Theorem of the Mean,

$$\int_0^y \frac{\sin qy}{y}e^{-xy}\,x\,dy = \frac{\sin q\eta}{\eta}\int_0^y e^{-xy}\,x\,dy$$
$$= \frac{\sin q\eta}{\eta}(1-e^{-xy}), \quad \text{where } 0\leqq\eta\leqq y\leqq 1.$$

Accordingly, $\int_0^y \frac{\sin qy}{y}e^{-xy}\,x\,dy$ is limited in R_1, and for each y in $\mathfrak{B}$ is a monotone function of x, approaching 0 as x approaches ∞. Therefore, we have: $\frac{\phi(x)}{x}\int_0^y dy$ is regular in R_1, having no infinite discontinuities, being integrable in $\mathfrak{A}$ for each y in $\mathfrak{B}$ $\left(\frac{\phi(x)}{x}\text{ is integrable, }\int_0^y dy\text{ is monotone}\right)$, and integrable in $\mathfrak{B}$ for each x in $\mathfrak{A}$ $\left(\int_0^y dy\text{ is continuous}\right)$.

Now, since the integrand is regular, and breaks up into an integrable function, and one that is limited and monotone, we have* the uniform convergence of the above integral in the interval $\mathfrak{B}=(0, 1)$.

We are now able† to state the result for Part 1°:

$$\int_0^\infty dx\int_0^1 f\,dy = \int_0^1 dy\int_0^{} f\,dx. \tag{3}$$

PART 2°. We now consider $R_2=(0, \infty\,;\ 1, \infty)$ and show that

$$\int_0^\infty dx\int_1^\infty f\,dy = \int_1^\infty dy\int_0^\infty f\,dx.$$

(a) $f(x, y)$ is regular in R_2.

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* Lect. I, p. 468, § 661, 1.

† Lect. I, p. 483, § 674, 1.

(b) $\int_0^\infty e^{-xy}\,\phi(x)\,dx$ is uniformly convergent in any $(1<\beta)$, since $|e^{-xy}\,\phi(x)| \leqq e^{-x}\,.\,M$, $\phi(x)$ being limited in $\mathfrak{A}$.

(c) $\int_1^\infty \frac{\sin qy}{y}\,e^{-xy}\,dy$ is uniformly convergent in any $(0<b)$ except at $x=0$, *i. e.*, in any $(s<b)$; $s>0$. For, we have:

$$\left|\frac{\sin qy}{y}\,.\,e^{-xy}\right| \leqq q\,.\,e^{-sy} \text{ for all } x \text{ in } \mathfrak{A}.$$

(d) $\phi(x)\int_0^\infty \frac{\sin qy}{y}\,e^{-xy}\,dy$ is integrable in any $(0<b)$; for by the Second Theorem of the Mean, denoting the integrand by ψ,

$$\phi(x)\int_0^\infty \psi\,dy = \phi(x)\left\{e^{-x}\int_1^\eta \frac{\sin qy}{y}\,dy + 0\right\}$$
$$= \phi(x)\,.\,e^{-x}\,.\,(\text{a function of } y).$$

(e) $\int_0^\infty \phi(x)\,dx\int_1^y \frac{\sin qy}{y}\,e^{-xy}\,dy$ is uniformly convergent in $\mathfrak{B}=(1,\infty)$ since we have:

$$\left|\phi(x)\int_1^y \frac{\sin qy}{y}\,e^{-xy}\,dy\right| \quad \text{(by the Second Theorem of the Mean)}$$
$$= \left|\phi(x)\left\{e^{-x}\int_1 \frac{\sin qy}{y}\,dy + e^{-y}\int_\eta^y \frac{\sin qy}{y}\,dy\right\}\right|$$
$$\leqq |\phi(x)|\,.\,(e^{-x}+e^{-xy})\,.\,\max\left|\int_{y_1}^{y_2}\frac{\sin qy}{y}\,dy\right|$$
$$\leqq M\,.\,\pi\,.\,2e^{-x}, \quad \text{which is integrable in } (0,\infty).$$

Now (a), (b), (c), (d), (e) give us the conditions we require and we have:*

$$\int_0^\infty \phi(x)\,dx\int_1^\infty \frac{\sin qy}{y}\,e^{-xy}\,dy = \int_1^\infty \frac{\sin qy\,dy}{y}\int_0^\infty e^{-xy}\,\phi(x)\,dx. \qquad (4)$$

Now,

$$\int_0^\infty \frac{\sin qy}{y}\,e^{-xy}\,dy = \arctan\frac{q}{x}; \quad x>0, \quad \text{Nouv. Tab., T 365, F 1,}\dagger$$

and when $x=0$,

$$\int_0^\infty \frac{\sin qy}{y}\,dy, \text{ which } = +\pi/2,\ 0,\ -\pi/2 \text{ for } q>0,\ =0,\ <0,$$
$$= \lim_{x=0} \arctan\frac{q}{x}. \qquad (5)$$

(3), (4), (5) give (2).

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* Lect. I, p. 488, § 680, 1.

† T 365, F 1, has been verified.

This theorem is stated by Cauchy.* De Haan† gives it as an immediate application of "Méthode 18" (cf. § 5). The conditions there are not sufficient, especially if b is infinite. In the application below b is infinite.

$$\int_0^\infty \arctan\frac{q}{x} \cdot \sin px\,dx = \frac{\pi}{2p}(1 - e^{-pq}), \quad \text{Nouv. Tab., T 347, F 1.} \tag{6}$$

We shall add the conditions: $q > 0$, $p \neq 0$. (7)

An application of (2) above, $(\phi(x) = \sin px)$ and Nouv. Tab., T 261, F 1.

We define $\arctan\frac{q}{x}\sin px = \frac{\pi}{2}$, when $x = 0$.

We change $e^{-xy}\sin px$ to be 0, when $y = 0$, so that it may be integrable in $(0, \infty)$.

§ 10.

General Criticism.

1°. In the work of de Haan, no distinction is made between the principal value convergence of an integral over an infinite discontinuity and the unrestricted convergence in the case of our ordinary definition. One symbol is employed in both cases. De Haan's reason for always using the principal value definition is given in § 2, and concerns a single example, which is treated incorrectly.

The theorem, "If f is integrable in $\mathfrak{A} = (a < b)$, it is also integrable in $\mathfrak{B} = (\alpha < \beta)$, any partial field of $\mathfrak{A}$," holds good for principal value integrals *only* if α, β are not singularities across which f possesses only a principal value integral. In some instances (*e. g.*, see § 7) de Haan overlooks this point.

The notation of de Haan does not distinguish between the various kinds of principal value integrals. A list of some of these is given in § 2, 6°.

2°. Almost always the reader is without specific statements as to the range of application of the theorem; nearly all the formulas and theorems here considered are open to this criticism. This absence of conditions on functions and constants is very important.

It is tacitly assumed very often that certain letters may be infinite as well as finite. See §§ 5, 9.

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* "Œuv. Comp.", Sér. 2, Tome 1, p. 508.

† VIII, Méth. 18, N 14, p. 452, Theorem (XX).

Divergent integrals are sometimes introduced in the course of the reasoning, as *e. g.* in § 8.

3°. In double iterated limits, the order of passing to the limit is inverted without rigorous justification, practically always. The question of uniform convergence is never raised. See §§ 4, 5, 7, 8.

4°. The *methods* of de Haan, judged by modern standards, are usually incomplete or incorrect.

His *results* are remarkably free from error when one imposes proper limitations upon constants and functions. But as these limitations are not given, it is quite impossible to know, as far as the tables go, whether a formula is valid in a given case or not. Moreover, some results (see *e. g.* §§ 4, 6) are incorrect for *any* value of the constants.

NEW HAVEN, *August* 29, 1910.

The Groups of Birational Transformations of Algebraic Curves of Genus 5.

By Joseph Vance McKelvey.

1. The algebraic curves of genus 5 have been treated in brief by Wiman.* The purpose of the present paper is to show the method by which the normal curves of hyperspace of this genus and their canonical forms in the plane can be projected into each other, and also to find the groups of birational transformations under which they are invariant.

It was shown by Clebsch† that the non-hyperelliptic curve of genus p can always be birationally reduced to a curve of order $p+1$. The general curve of genus 5 is therefore a sextic with five double points, and we distinguish the following cases:

(a) General case. Sextics with five double points.

(b) Curves having a g_3^1, the canonical form being the nodal quintic.

(c) Curves having a g_2^1, the hyperelliptic forms which will not be considered in this paper.

2. (a) The adjoint curves as we shall use them are curves of order $n-3$ which pass once through each double point of C_n. They will have $n(n-3)-2\delta = 2p-2$ variable intersections with C_n and $1/2\,n(n-3)-\delta = p-1$ degrees of freedom. If $\phi_i (i=1, 2, \ldots, p)$ are p linearly independent adjoint curves of C_n, the complete system may be written in the form $\sum_{i=1}^{i=p} a_i \phi_i = 0$. This net defines a g_{2p-2}^{p-1} on C_n. Weber‡ has proved that among the ϕ-adjoints of any algebraic curve of genus p there are $1/2(p-2)(p-3)$ quadratic identities. Then, if we think of the ϕ_i as homogeneous point coordinates in linear space

* *Bihang till Svenska Vet. Akad. Handlingar*, Band XXI (1895). This article will be referred to later by W.

† *Vorlesungen über Geometrie*, Vol. I, p. 690 ff.

‡ "Ueber gewisse in der Theorie der Abel'schen Functionen auftretende Ausnahmefälle," *Math. Ann.*, Vol. XIII (1878), pp. 35–48.

of $p-1$ dimensions R_{p-1}, the whole plane may be depicted in R_{p-1} and the C_n is transformed into a hyperspace curve Γ which will be defined in R_{p-1} by the varieties whose equations are the above quadratic identities. This is called the normal curve. The ϕ_i represent spaces of $p-2$ dimensions and an arbitrary ϕ will cut Γ in as many points as the adjoint curve has variable intersections with C_n. Hence, Γ is by definition of order $2p-2$.*

Notation. $G_Q \equiv$ a group of Q points on a curve; also, in another sense, a group of transformations whose order is Q. This usage is quite well established and no ambiguity will arise.

$g^q_Q \equiv$ a set of G_Q's having q degrees of freedom.

$R_n \equiv$ space of n dimensions.

$C^{(p)}_{i,k} \equiv$ a curve of order i and genus p in R_k.

x_i and ϕ_i $(i=1, 2, \ldots, 5)$ denote point coordinates in R_4.

$P_k \equiv$ a k-fold point.

3. In the general case for $p=5$, the number of quadratic identities is three and the Γ_{2p-2} is a $C^5_{8,4}$. Now let

$$F_1(\phi_1 \phi_2 \ldots \phi_5) = 0, \quad F_2 = 0, \quad F_3 = 0 \tag{1}$$

be the three identities. Then the net

$$F \equiv \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0 \tag{2}$$

defines the system of quadrics passing through $C^5_{8,4}$. The five partial derivatives of F may be written

$$\left.\begin{aligned} &\lambda_1 F_{11} + \lambda_2 F_{21} + \lambda_3 F_{31} = 0, \\ &\lambda_2 F_{12} + \lambda_2 F_{22} + \lambda_3 F_{32} = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\lambda_1 F_{15} + \lambda_2 F_{25} + \lambda_3 F_{35} = 0, \end{aligned}\right\} \tag{3}$$

where $F_{ik} = \dfrac{\partial F_i}{\partial \phi_k}$, $\quad i = 1, 2, 3; \quad k = 1, 2, \ldots, 5.$

From (3), we obtain two important curves. Eliminating the ϕ_k, we get the discriminant of F which is a ternary quintic in λ. Call it $\Delta_5(\lambda)$. This is the condition for a double point of F and, for all values of λ_i satisfying $\Delta_5 = 0$,

* A more complete discussion of this transformation into R_{p-1} is given by Kraus: "Note über ausser-gewöhnliche Specialgruppen auf Algebraischen Curven," *Math. Ann.*, Vol. XVI (1880), p. 249. This article will be referred to later by K.

F can be written in terms of four variables instead of five. Eliminating λ, we get the three-by-five matrix in linear functions of the ϕ_k,

$$\left\| \begin{matrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} \end{matrix} \right\| = 0. \tag{4}$$

The values of ϕ_k satisfying (4) are the coordinates of the vertices of the R_4 cones that exist when the λ_i satisfy $\Delta_5 = 0$. After the transformation into terms of four variables, the vertex is

$$\phi_1' = \phi_2' = \phi_3' = \phi_4' = 0.$$

A curve D_{10} is the locus of these vertices and is the condition that (3) shall be consistent in λ, while Δ_5 is the condition that (3) shall be consistent in ϕ. Thus D_{10} and Δ_5 are in (1, 1) correspondence. Δ_5 is of further importance from the fact that unless the transformations of F leave F_i absolutely invariant our C_6 which is the plane projection of $C_{8,4}$ can have no transformations except those of this quintic, whose equation is expressible as a symmetric determinant. It must be kept in mind, therefore, that C_6, $C_{8,4}$, Δ_5 and D_{10} will be invariant under groups of transformations of the same order. The complete set of equations obtained by eliminating λ from (4) may be written in the matrix

$$\left\| \begin{matrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} \\ F_{31} & F_{32} & F_{33} & F_{34} & F_{35} \end{matrix} \right\| = 0 \text{ (all elements being linear in five variables).} \tag{5}$$

Of the ten equations shown here, only three are independent. For, if we have

$$\left.\begin{aligned} (F_{11}F_{22} - F_{21}F_{12})F_{33} + (F_{31}F_{12} - F_{11}F_{32})F_{23} + (F_{21}F_{32} - F_{31}F_{22})F_{13} = 0, \\ (F_{11}F_{22} - F_{21}F_{12})F_{34} + (F_{31}F_{12} - F_{11}F_{32})F_{24} + (F_{21}F_{32} - F_{31}F_{22})F_{14} = 0, \\ (F_{11}F_{22} - F_{21}F_{12})F_{35} + (F_{31}F_{12} - F_{11}F_{32})F_{25} + (F_{21}F_{32} - F_{31}F_{22})F_{15} = 0, \end{aligned}\right\} \tag{6}$$

the remaining seven equations follow by the proportionality of columns 3, 4 and 5, which is evident from these three. But the system of equations is restricted and a C_{27} is not defined by them, as would be the case if they were arbitrary. For example, columns 3, 4, 5 need not be proportional if 1 and 2 are, for then the bracketed expressions in (5) vanish.

Let the order be $27 - m$. Let V_{ik} denote an i-dimensional variety of order k in R_4. Then each equation in (4) defines a V_{33}. Columns (1, 2) define a V_{23}, and for all points of this V_{23}, (6) may be satisfied without all of (5) being

satisfied. (3, 4, 5) would be an exception. Hence, the order of the curve defined by these three columns and V_{23} is the value of m. Now (3, 4), (3, 5), (4, 5) define V'_{23}, V''_{23}, V'''_{23} respectively, but, excluding the vanishing of column (3), (4, 5) is a consequence of (3, 4) and (3, 5). (V'_{23}, V''_{23}) and (V'_{23}, V'''_{23}) define C'_9 and C''_9 respectively. Then, excluding (3) as mentioned above, we have

$$m = 9 + 9 - 1 = 17,$$

and the order of the curve represented by (5) is 10. The curves $D_{10,4}$ and $C_{8,4}$ can not intersect. If they do intersect, the three F_i pass through a vertex of an R_4 cone, and $C_{8,4}$ must then have a double point. $C_{8,4}$ has no double points, because its projection in R_3 would be $C^{(5)}_{6,3}$, but no such curve exists.

4. If $C_{8,4}$ be projected from a point upon it into R_3, the resulting curve will be of order 7. Similarly, if $C_{7,3}$ be projected from a point upon it into R_2, the result is a sextic. Since the genus is preserved by each projection, $C_{6,2}$ must have five double points. Both projections can be made at once from two points ξ and η on the R_4 curve. A plane π through ξ and η can, in general, cut $C_{8,4}$ in only one other point P. The image of P in the plane of projection π' will be the point common to π and π'; the double points of $C_{6,2}$ will appear when and only when π cuts $C_{8,4}$ in four points. There will evidently be five such positions of π. Let $\rho, \lambda, \mu, \sigma$ be parameters and π' be defined by $y_4 = y_5 = 0$. Then the equations of transformation are

$$\rho x_i = \lambda \xi_i + \mu \eta_i + \sigma y_i \qquad (i = 1, 2, \ldots, 5), \tag{7}$$

where y_i are the plane coordinates. Since y_4 and y_5 are zero in π', we find

$$\lambda/\rho = (\eta_5 x_4 - \eta_4 x_5)/(\xi_4 \eta_5 - \xi_5 \eta_4), \quad \mu/\rho = (\xi_4 x_5 - \xi_5 x_4)/(\xi_4 \eta_5 - \xi_5 \eta_4).$$

Then

$$\sigma y_i = (\eta_4 x_5 - \eta_5 x_4)\xi_i + (\xi_5 x_4 - \xi_4 x_5)\eta_i + (\xi_4 \eta_5 - \xi_5 \eta_4) x_i. \tag{8}$$

Making substitution (7) in $F_1 = 0$, we have

$$\lambda^2 F_1(\xi) + \mu^2 F_1(\eta) + \sigma^2 F_1(y) + 2\lambda\mu P_1(\xi, \eta) + 2\lambda\sigma P_1(\xi, y) + 2\mu\sigma P_1(\eta, y) = 0. \tag{9}$$

Since ξ and η are on $C_{8,4}$, $F_1(\xi) = 0$ and $F_1(\eta) = 0$.

Then we may write the equation in the form

$$\sigma^2 F_1(y) + 2\lambda\mu P_1 + 2\lambda\sigma P'_1 + 2\mu\sigma P''_2 = 0. \tag{10}$$

Similarly, for F_2 and F_3 we have

$$\sigma^2 F_2(y) + 2\lambda\mu P_2 + 2\lambda\sigma P_2' + 2\mu\sigma P_2'' = 0, \tag{11}$$

$$\sigma^2 F_3(y) + 2\lambda\mu P_3 + 2\lambda\sigma P_3' + 2\mu\sigma P_3'' = 0. \tag{12}$$

From the third of these equations,

$$\lambda = -\frac{\sigma^2 F_3(y) + 2\mu\sigma P_3''}{2\mu P_3 + 2\sigma P_3'}. \tag{13}$$

Make this substitution in the first two and put $\mu = 1$.

$$\sigma^3[F_1(y)P_3' - F_3(y)P_1'] + \sigma^2[P_3F_1(y) - P_1'F_3''(y) - P_1'P_3'' + P_3'P_1''] + \sigma[(P_1''P_3 + P_1P_3'')] = 0. \tag{14}$$

$$\sigma^3[F_2(y)P_3' - F_3(y)P_2'] + \sigma^2[P_3F_2(y) - P_2F_3(y) - P_2'P_3'' + P_3'P_2''] + \sigma[(P_2''P_3 + P_2P_3'')] = 0. \tag{15}$$

Eliminate σ and we have

$$\begin{vmatrix} F_1(y)P_3' - F_3(y)P_1' & P_3F_1(y) - P_1F_3(y) - P_1'P_3'' + P_3'P_1'' & P_1''P_3 + P_1P_3'' \\ & F_1(y)P_3' - F_3(y)P_1' & P_3F_1(y) - P_1F_3(y) - P_1'P_3'' + P_3'P_1'' \\ F_2(y)P_3' - F_3(y)P_2' & P_3F_2(y) - P_2F_3(y) - P_2'P_3'' + P_3'P_2'' & P_2''P_3 + P_2P_3'' \\ & F_2(y)P_3' - F_3(y)P_2' & P_3F_2(y) - P_2F_3(y) - P_2'P_3'' + P_3'P_2'' \end{vmatrix} = 0. \tag{16}$$

This is a plane C_8 in (y_1, y_2, y_3). It must, however, have a quadratic factor, because $C_{8,4}$ goes into a $C_{6,2}$ when the centers of projection are on the curve. The other factor is the $C_{6,2}$.

To project $C_{6,2}$ into $C_{8,4}$, we use the same equations of transformation as before, but the parameters must be expressed in terms of (y, ξ, η) instead of (x, ξ, η). In order to do this, solve (4), (5), (6) for $\lambda : \mu : \sigma$. This is possible by virtue of (10). We may eliminate the terms in $\lambda\mu$ and $\mu\sigma$ by multiplying

$$(4) \text{ by } \begin{vmatrix} P_2 & P_2'' \\ P_3 & P_3'' \end{vmatrix}, \ (5) \text{ by } -\begin{vmatrix} P_1 & P_1'' \\ P_3 & P_3'' \end{vmatrix}, \ (6) \text{ by } \begin{vmatrix} P_1 & P_1'' \\ P_2 & P_2'' \end{vmatrix}$$

and adding. Then

$$\frac{\lambda}{\sigma} = 1/2\,\frac{(P_3P_2'' - P_2''P_3)F_1 + (P_1P_3'' - P_3P_1'')F_2 + (P_2P_1'' - P_1P_2'')F_3}{P_1'(P_2P_3'' - P_3P_2'') + P_2'(P_3P_1'' - P_1P_3'') + P_3'(P_1P_2'' - P_2P_1'')}. \tag{17}$$

By a similar procedure we obtain

$$\frac{\mu}{\sigma} = 1/2\,\frac{(P_2P_3' - P_3P_2')F_1 + (P_3P_1' - P_1P_3')F_2 + (P_1P_2' - P_2P_1')F_3}{P_1''(P_3P_2' - P_2P_3') + P_2''(P_1P_3' - P_3P_1') + P_3''(P_2P_1' - P_1P_2')}. \quad (18)$$

The F's are quadratic in y and P_i', P_i'' are linear. Hence, letting $\rho = 1$, the equations of transformation are of the form

$$x_i = C_3(y)\,\xi_i + C_3'(y)\,\eta_i + C_2(y)\,y_i, \qquad i = 1, 2, \ldots, 5. \quad (19)$$

The form of the general transformation in the plane may be found by first projecting C_6 into R_4. By (19) this is a cubic relation. Next make a transformation in R_4 that leaves $C_{8,4}$ invariant. This transformation will always be linear. Finally, project back into R_2. By (8) this also is a linear relation. Hence, the successive projections are expressed by the equations

$$y' = f_1(x') = \phi_1(x) = \psi_3(y). \quad (20)$$

This shows that there are no transformations of higher degree than the cubic.

5. Δ_5 will have double points for all values of λ_i that make its first minors vanish. But all these minors can be made to vanish by making four properly chosen ones vanish. The net of R_4 quadrics degenerates into a net of R_4 cones K' when $\Delta_5 = 0$. The elements of any K' are planes concurrent at the vertex $P_v \equiv x_1' = x_2' = x_3' = x_4' = 0$. By means of the four conditions on the minors mentioned above, we may fix four points on K_2', viz., P_1, P_2, P_3, P_4. Then, $P_vP_1P_2$, $P_vP_2P_3$, $P_vP_3P_4$ can be made to define three elements (planes) in the same R_3. Of the general K_2 only two planes can lie in the same R_3, and if three elements are so placed, the K_2' degenerates and all of the elements will lie in two R_3's which intersect in a plane. In this plane is a line l through which all elements of the degenerate K_2' pass. An R_3 section of K_2' is an ordinary quadric surface H_2, and the section of the degenerate K_2' is the quadric cone K_2. The vertices of K_2's thus cut from K_2' will be collinear with P_v. But l contains P_v. Hence, the line of vertices of the K_2's intersects l and both these lines lie in the plane common to the two R_3's mentioned above. l is defined by $x_1'' = x_2'' = x_3'' = 0$. Now for certain values of λ_i the first minors of Δ_5 vanish, showing a double point of Δ_5, and the point P traces Δ_5 continuously while its image P' describes D_{10}. Then, since P' is replaced by a whole line when P reaches a double point, it is evident that this line corresponds to the double point of Δ_5. Whenever a double point of Δ_5 exists, then, for the values of λ_i that are the coordinates of this point, F can be expressed in terms of three variables

(x_1, x_2, x_3). The most general quadratic in three variables can be written in the form

$$2x_1x_2 - x_3^2 = 0.$$

The linear form $ax_1 + bx_2 + cx_3 = 0$ will cut this conic in (x_1', x_2', x_3') and (x_1'', x_2'', x_3''). Regarding the x's as adjoint curves, the tangent C_3's may be written

$$x_1'x_2 + x_2'x_1 - x_3'x_3 = 0, \qquad x_1''x_2 + x_2''x_1 - x_3''x_3 = 0.$$

Then F and $C_{6,2}^{(5)}$ take the form

$$2(x_1'x_2 + x_2'x_1 - x_3'x_3)(x_1''x_2 + x_2''x_1 - x_3''x_3) - (ax_1 + bx_2 + cx_3)^2 = 0.$$

An equation like this is possible for each double point of Δ_5. The two Abel forms or contact curves appear in the product term of the equation. Since there is an infinite number of curves of the form $ax_1 + bx_2 + cx_3 = 0$, there is also an infinite number of contact curves for each double point of Δ_5. When Δ_5 consists of five straight lines, the number of double points is ten. Hence, there may be ten systems of contact curves. By Kraus' (K) proof there are but three linearly independent systems. If three such systems be known, the remaining ones can be obtained linearly in terms of them. For example, if we have

$$F_1 \equiv a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0,$$
$$F_2 \equiv b_2x_2^2 + b_3x_3^2 + b_4x_4^2 = 0,$$
$$F_3 \equiv c_3x_3^2 + c_4x_4^2 + c_5x_5^2 = 0,$$

Δ_5 will consist of five straight lines. There should be therefore ten systems of contact curves. Any two of the five variables may be eliminated by linear combinations of the above F_i. Thus we get the ten systems and there are no more. By the same procedure with any three linearly independent F_i in three variables, we can get as many systems as there are double points of Δ_5. Wiman (W) states that Kraus' proof is incorrect, but he overlooks the fact that Kraus says "linearly independent" systems. Since a double point of Δ_5 calls for a linear factor of D_{10}, the D_{10} in the above illustration must consist of ten straight lines. They are of the form $x_1 = x_2 = x_3 = 0$. Obviously four such lines will go through the point $x_1 = x_2 = x_3 = x_4 = 0$. The ten lines will be concurrent by fours in five such points and will constitute the simplex of reference in R_4.

Our C_6 has a g_4^1 for each of the five double points and also for each set of four double points. Through each P_2 there is a pencil of straight lines cutting out a g_4^1 and through each set of four points can be passed a C_2 cutting C_6 in a g_4^1.

Now, by the Brill-Noether reciprocity theorem, we have the equations

$$Q + R = 2p - 2, \qquad Q - R = 2(q - r).$$

Since in this case $p = 5$ and $R = 4$, we find

$$Q = R = 4 \quad \text{and} \quad q = r = 1.$$

Now take any G_4 of the above g_4^1 as basis points. Pass a C_{n-3}, *i. e.*, a C_3, through them and the five double points. The residual is a g_4^1 from which we may in turn select a G_4 and determine the g_4^1 first used.

These g_4^1's have a simple relation to D_{10}. Project $C_{8,4}$ from a point on D_{10} into R_3. This gives a $C_{8,3}$ lying on a quadric surface. The equation of this quadric is the same as that of $F = 0$ in R_4, where it has been expressed in terms of four variables. This $C_{8,3}$ is of type (4, 4) on the quadric. The other possibilities are eliminated because the (7, 1) could be projected into a $C_{8,2}$ with a P_7, which would make its genus 0. The (6, 2) goes into a $C_{8,2}$ with a P_6 and a P_2 which is of genus 5, but on account of the P_6 a g_2^1 exists, making the curve hyperelliptic. The (5, 3) has a g_3^1 by the pencil of lines through P_5 of the $C_{8,2}$. This case has been disposed of in the nodal C_5. By examining the (4, 4), we find that it must have four double points in order to make the genus 5, for it projects into a $C_{8,2}$ with two P_4's, which with no further multiple points would make the genus 9. Project the curve $C_{8,3}$ from one of these four points P_2' into R_2 and we get a C_6 with five P_2, for the other three P_2's give P_2's in the plane and each of the two generators through P_2' cuts the $C_{8,3}$ in two points distinct from P_2. This provides two more double points, which with the three mentioned above make the genus 5. These four double points in R_3 show that from every point of $D_{10,4}$ four bisecants can be drawn to $C_{8,4}$. No two of the above double points can lie on the same generator of the R_3 quadric, for in that case the proper number of double points would not appear in C_6 if one of these two points should be used as the center of projection.

These g_4^1's are grouped in pairs, for, consider P_2' again as center of projection. Call the two generators through it A and B. All the generators of the A system cut B and project into a pencil of lines through the P_2 determined by B. Similarly for the B generators.

If F be expressed in terms of three variables, the R_3 quadric is a cone of order 2, so that the two systems of generators and therefore the two g_4^1's coincide.

The conic through four P_2's cutting out a g_4^1 may be obtained by passing a plane through two P_2's of the curve on the above quadric. Fix the plane by

a simple point P_1 and project upon the plane of C_6 as before. The plane of section must cut $C_{8,3}$ in eight points. There are two at each P_2 and four simple points including P_1. These four will in general project into four simple points. The two P_2's through which the section was made give two P_2's of $C_{6,2}$. The two generators A and B determine two P_2's as before, and the section of the quadric is thus seen to project into a conic through four P_2's of C_6. The P_2 of $C_{8,3}$ not used above projects into the fifth P_2 of C_6.

6. If the normal curve $C_{8,4}$ be linearly transformed into itself, the system

$$F \equiv \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0 \tag{21}$$

will also go into itself by the same transformation. For, $C_{8,4}$ is the complete intersection of the F_i and, since those points of the system of varieties that constitute $C_{8,4}$ remain invariant as a whole, the transformed system will pass through $C_{8,4}$. It is therefore identical with F. By this transformation, the F_i may not remain individually fixed, in which case the λ_i are linearly transformed in such a way as to put Δ_5 into itself, because it, the condition for double points of F, must remain unchanged when F remains unchanged. But in case F_i goes into F_i, the λ_i will not be altered; *i. e.*, the individual points of Δ_5 remain fixed. Since Δ_5 and D_{10} are in (1, 1) correspondence, D_{10} will also go into itself point by point with the exception of the lines corresponding to the double points of Δ_5. These lines go individually into themselves, but the points may be permuted by one or more homologies. For, if the points of D_{10} remain fixed, it can not be a proper R_4 curve. If it were, every point of R_4 would be fixed and the transformation would be an identity. D_{10} therefore consists of a point and an R_3 curve or a line and an R_2 curve. If a point and an R_3 remain fixed, call them O and R_3'. These must be the center and invariant R_3 of a homology. Any R_3 through O is invariant as a whole, for it cuts R_3' in a plane. Take R_3' and any four R_3's through O as the simplex of reference and the equation of F_i will be reducible to

$$a_i x_1^2 + f_i(x_2 x_3 x_4 x_5) = 0, \tag{22}$$

where $x_1 = 0$ is the equation of R_3'. The transformation is

$$x_1 = -x_1', \qquad x_i = x_i', \qquad i = 2, 3, 4, 5. \tag{23}$$

In this case $\Delta_5 \equiv C_1 . C_4$. The four double points thus formed require four systems of ϕ-curves. When even one such system exists, the $C_{6,2}^{(5)}$ is reducible to the form having a tacnode and three collinear double points (K). Hence, when Δ_5 is point by point invariant, the C_6 must be thus far restricted in order to be invariant under a linear G_2.

When a line and an R_2 curve remain fixed, the plane will be common to the invariant R_3's of two homologies whose centers lie on the fixed line. The F_i will be of the form

$$a_i x_1^2 + b_i x_2^2 + \psi_i (x_3 x_4 x_5) = 0, \tag{24}$$

if we take three points in the fixed plane and two points on the fixed line as the vertices of the simplex of reference. The fixed plane is defined by $x_1 = 0$, $x_2 = 0$ and the line by $x_3 = x_4 = x_5 = 0$. The centers of the two homologies are

$$x_1 = x_3 = x_4 = x_5 = 0; \quad x_2 = x_3 = x_4 = x_5 = 0.$$

The equations of transformation are

$$x_1 = -x_1', \quad x_2 = -x_2', \quad x_i = x_i', \qquad i = 3, 4, 5. \tag{25}$$

According as Δ_5 is proper or degenerate, we have the following cases:

(a) Δ_5 *is a proper* C_5. The maximum number of double points of Δ_5 and of the corresponding lines of D_{10} is six. The transformations depend upon the coefficients of F_i.

(b) $\Delta_5 \equiv C_1 . C_4$. See equations (22) and (23).

(c) $\Delta_5 \equiv C_2 . C_3$. $F_i \equiv f_i (x_1 x_2) + \psi_i (x_3 x_4 x_5) = 0$. The intersections of C_2 and C_3 call for six lines of D_{10}. There will be seven if C_3 has a double point.

(d) $\Delta_5 \equiv C_1 . C_1' . C_3$. See equations (24) and (25).

(e) $\Delta_5 \equiv C_1 . C_2 . C_2'$. Eight systems of ϕ-curves.

$$F_i \equiv a_i x_1^2 + f_i (x_2 x_3) + \psi_i (x_4 x_5) = 0.$$

(f) $\Delta_5 \equiv C_1 . C_1' . C_1'' . C_2$. Nine systems of ϕ-curves and nine corresponding lines of D_{10}. The tenth line does not correspond to a double point of Δ_5 and therefore is not the vertex of a cone through the normal curve.

(g) $\Delta_5 \equiv 5\, C_1$. D_{10} consists of ten straight lines.

$$F_i \equiv a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + a_5 x_5^2 = 0.$$

From any three linearly independent forms of F_i we can eliminate any pair of variables x_r and x_s and thus obtain the ten relations among five variables in threes.

We will illustrate two tacnodal cases, namely (a) and (d).

For (a) write the equation of the general C_3 (the ϕ-curve) through (0, 1, 0), (1, 1, 0), (1, 0, 0) and tangent to $y = 0$ at (0, 0, 1). The quadratic equation in x_i defines a C_6 with a tacnode at (0, 0, 1) and having three double points on $z = 0$. The C_3 is

$$K(x^2 y - x y^2) + L x^2 z + P y^2 z + Q y z^2 + R x y z = 0.$$

Then pass to R_4 by the transformation

$$x_1 = xy(x-y), \quad x_3 = y^2 z, \quad x_5 = xyz,$$
$$x_2 = x^2 z, \qquad\quad x_4 = yz^2.$$

$$F_1 \equiv x_2 x_3 - x_5^2 = 0, \quad F_2 \equiv x_1 x_4 + x_3 x_5 - x_2 x_3 = 0.$$

Considering these two identities, the general quadratic in x may be written with thirteen terms:

$$F_3 \equiv A x_1^2 + B x_2^2 + C x_3^2 + D x_4^2 + E x_5^2 + F x_1 x_2 + G x_1 x_3 + H x_1 x_5 + K x_2 x_4 + L x_3 x_4 + M x_2 x_5 + P x_3 x_5 + N x_4 x_5 = 0.$$

This will be invariant under $x_1 = x_4'$, $x_4 = x_1'$, if $A = D$, $F = K$, $G = L$ and $H = N$. Since this transformation can be reduced to a single change of sign, F_3 is the equation of the required sextic.

For (d), write the equation of the general C_3 through (0, 0, 1), tangent to $x = 0$ at (0, 1, 0) and to $y = 0$ at (1, 0, 0). It is

$$Kx^2 y + Mxy^2 + Nxz^2 + Qyz^2 + Rxyz = 0.$$

Then let

$$x_1 = x^2 y, \quad x_2 = xy^2, \quad x_3 = xz^2, \quad x_4 = yz^2, \quad x_5 = xyz.$$

Two identities will be

$$F_1 \equiv x_1 x_4 - x_5^2 = 0, \quad F_2 \equiv x_2 x_3 - x_4^2 = 0.$$

Similar to the preceding, we have $F_3 = 0$, but to make it invariant under $x_1 = -x_1'$, $x_4 = -x_4'$, there must be no terms linear in x_1 or x_4.

$$F_3 \equiv A x_1^2 + B x_2^2 + C x_3^2 + D x_4^2 + E x_5^2 + L x_2 x_5 + N x_3 x_5 = 0.$$

This is the equation of the sextic. Δ_5 is

$$\begin{vmatrix} 2A\lambda_3 & \lambda_1 \\ \lambda_1 & 2D\lambda_3 \end{vmatrix} \cdot \begin{vmatrix} 2B\lambda_3 & \lambda_2 & L\lambda_3 \\ \lambda_2 & 2C\lambda_3 & N\lambda_3 \\ L\lambda_3 & N\lambda_3 & -2(\lambda_1 + \lambda_2 - E\lambda_3) \end{vmatrix} = 0.$$

7. The normal curve in R_{p-1} is defined by the quadratic identities among p linearly independent adjoint curves of $C_{n,2}^{(p)}$. For $p = 5$, as already shown, this curve is of order 8. If $C_{8,4}$ be projected into R_2 from two points upon it and the image of one of these points used as a basis point in addition to the five double points, we get the system of adjoint curves,

$$\phi \equiv a_1 \phi_1 + a_2 \phi_2 + a_3 \phi_3 + a_4 \phi_4 = 0,$$

where the ϕ_i are linearly independent and may be regarded as homogeneous point coordinates in R_3. By means of these ϕ_i as functions of x, y, z the points

of the plane may be uniquely pictured as a surface in R_3 on which lies a C_7 projective with C_6. The curve is of order 7, because the adjoint curves have

$$18 - 2 \cdot 5 - 1 = 7$$

variable intersections with C_6. From these relations among the three curves, we know that $C_{6,2}^{(5)}$ and $C_{7,3}^{(5)}$ may always be considered as projections of the normal $C_{8,4}$.

If $C_{8,4}$ be projected into R_2 from O_1 and O_2, two points on the curve, the images of the two centers of projection will be the two simple intersections of C_6 and the C_2 passing through the five double points. For, the ∞^2 system of R_3's through O_1, O_2 defines a g_6^2 on C_8. Take O_1 as center and project the system into R_3'. The resulting system of planes through O_2' defines a g_6^2 on $C_{7,3}$. Then projecting from O_2', we get the ∞^2 system of straight lines in the plane defining a g_6^2 on C_6. Hence, the adjoint curves represented by the R_3's through O_1 and O_2 must consist of an arbitrary straight line and the C_2 defined by the five double points. The two remaining points in which C_2 cuts C_6 are invariant for the above system of R_3's and are therefore the images of the only two fixed points defined by the system, *i. e.*, O_1 and O_2.

If four of the points common to an R_3 and $C_{8,4}$ lie in a plane, the other four lie in a plane. Let G_4 be such a set of four points. The R_3 containing them has one degree of freedom and therefore the other four points constitute a Γ_4, where all the Γ_4's form a γ_4^1. Now, by the Brill-Noether theorem, any Γ_4 of γ_4^1 may be used as basis points of a system of R_3's which, by reason of the one degree of freedom, will define a g_4^1, and to this g_4^1 belongs the above G_4. The Γ_4's have one degree of freedom and lie therefore in a plane. Q. E. D.

There are ∞^2 planes through each point of $C_{8,4}$ cutting the curve in three additional points.

Through any two points O_1, O_2 of C_8 may be passed five planes cutting the curve in two other points. Now let O_1 be held fast while O_2 describes C_8. The set of five planes thus take ∞^1 positions, but still pass through O_1.

Therefore there are ∞^1 triads of points co-planar with each point of $C_{8,4}$. Q. E. D.

There are ∞^1 planes through each point of D_{10} cutting $C_{8,4}$ in four points. To see this, project C_8 from O of D_{10}. We get a $C_{8,3}$ of species (4, 4) on a hyperboloid H_2. Therefore a plane through O and a generator of H_2 cuts C_8 in four points. O and the ∞^1 generators define the ∞^1 planes. Q. E. D.

By projecting C_8 from P_1, a point on the curve, we get a $C_{7,3}$. With this curve is associated a ruled surface of trisecants. For, any three points co-planar with P_1 have for images three points in the line common to this plane and the R_3 of section. Therefore corresponding to each of the ∞^1 planes through P_1 containing three other points of C_8 is a trisecant of $C_{7,3}$. Since D_{10} is the locus of points from which C_8 can be projected into a (4, 4) curve on H_2, all the planes through P_1 defining a G_4 must cut D_{10}.

Now turn the plane on the line OP_1 until it contains the reciprocal G_4. It will then define the trisecant B which will intersect A at O', because P_1O is common to the two planes defining the reciprocal G_4's. O' is therefore a double point on $D_{10,3}$; the same argument holds for every position of O', since O is an arbitrary point on $D_{10,4}$. $D_{10,3}$ is therefore a double curve on S, the ruled surface of trisecants. Through P_1P_2 will pass five planes that cut $C_{8,4}$ in four points. Hence through P_2' in R_3 will pass five generators of S, which means that $C_{7,3}$ is a five-fold curve on S. In general, there will not be an R_3 tangent to $C_{8,4}$ in four points, but if F is expressed in terms of three variables, *i. e.*, if the curve lies on a $K_{2,4}$ having a line D_1 for vertex, we have ∞^1 such R_3's. The two G_4's defined by a tangent R_3 are coincident and lie in the R_2 of contact. Project $C_{8,4}$ from one of the four points of tangency. The images in R_3' of the other three points are the intersections of $C_{7,3}$ and a trisecant. These points of C_7 have a common tangent plane, namely, the intersection of the tangent R_3 with R_3'. This trisecant corresponds to two coincident G_4's and must therefore count for two. It is a double torsal generator of S (W). Project $C_{7,3}$ from one of the above points of tangency and the resulting curve is a $C_{6,2}$ with a tacnode corresponding to the other two points of tangency, while the two successive centers of projection appear as the residual intersections of C_6 and the tacnodal tangent. Again, if we project $C_{8,4}$ from a point of the line vertex D_1 of $K_{2,4}$, $C_{8,3}$ will lie on a quadric cone cutting each generator four times and not passing through the vertex. Projecting $C_{8,3}$ into R_2 from any point of the cone, we get a plane C_8 having a tacnode with four branches which is equivalent to twelve double points. Therefore, in order to have the proper genus, the $C_{8,3}$ must have four actual double points. By projecting $C_{8,3}$ from one of these double points, we get a plane C_6 with a tacnode. The images of the other three double points are collinear, because by Kraus' proof, if the adjoint curves of a C_{p+1} touch the curve at $p-1$ points, $1/2(p-4)(p+1)$ of the double points lie on a C_{p-4}. In our case $p=5$. Hence the three collinear double points.

These three being collinear, the four double points of $C_{8,3}$ must lie in a plane.

We have seen that $D_{10,3}$ is a double curve and $C_{7,3}$ is a five-fold curve on S. To find the order of the surface, let g be a trisecant of $C_{7,3}$, *i. e.*, a generator of S. Let P_1, P_2, P_3, D_1 be the points in which g cuts $C_{7,3}$ and $D_{10,3}$ respectively. Pass a plane through g. It will be tangent to S at some point T which is a double point of the plane section C_n, and g is one of the branches through T. C_n evidently consists of a C_{n-1} and the line g cutting C_{n-1} four times at each of the points P_1, P_2, P_3, and once at D_1 and T. Therefore,

$$n - 1 = 4 \cdot 3 + 1 + 1,$$

whence S is of order 15. The double points of an arbitrary plane section arise from the ten points of the double curve $D_{10,3}$ and the seven points of the five-fold curve $C_{7,3}$ that lie in the plane. This shows

$$10 + 7 \cdot \frac{5 \cdot 4}{2} = 80 \text{ double points.}$$

Now every torsal generator reduces the order of $D_{10,3}$ by 1, but adds 2 to the number of double generators, because the torsal generator itself is a double line along which the surface has a line of self-contact; any plane section has a tacnode at its point of intersection with this generator. Let δ be the number of them, then $p = 1/2 \cdot 14 \cdot 13 - 80 - \delta = 11 - \delta$.

When one of the F_i can be written in terms of three variables, then

$$D_{10} \equiv D_9 \cdot D_1,$$

in which D_1 corresponds to a double point of Δ_5. Since both curves are continuous D_1 must cut D_9. The double point so formed does not call for a linear factor of Δ_5. We have already seen that there are ∞^1 R_3's through D_1 tangent to $C_{8,4}$. The planes of tangency are uniquely determined by D_1 and the points of C_8. Hence, any plane through D_1 and a point of the normal curve defines a G_4 on the normal curve.

Forms of S_{15}. [1] When Δ_5 is a proper curve, at most six double points may exist and an equal number of double torsal generators. Thus the maximum value of δ is 6 and the minimum genus of S_{15} is 5. The torsal generators may or may not intersect.

[2] If we have $F_1(x_1, x_2, x_3) = 0$, $F_2(x_1, x_2, x_4) = 0$, $F_3(x_1, \ldots, x_5) = 0$, the normal curve lies on two quadric cones of R_4 whose vertices are $x_1 = x_2 = x_3$ and $x_1 = x_2 = x_4$. These two lines lie in the plane (x_1, x_2) and therefore will

intersect. Now, if we find Δ_5 it is evident from the forms of F_i that λ_3 may be factored out from either the fifth column or the fifth row. Hence, $\Delta_5 \equiv C_4 . C_1$. The intersections of C_4 with C_1 require that D_{10} have four linear factors each of which will be the vertex of a cone passing through $C_{8,4}$. Since a line of Δ_5 corresponds to a double point of D_{10}, the four lines must pass through that double point. It can be shown directly from the equations that the four lines are concurrent. The preceding net may be written

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 [f(x_1, x_2, x_3, x_4) + x_5 (a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) + x_5^2] = 0,$$

or putting $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 2X$,

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 [f(x_1, x_2, x_3, x_4) - X^2 + (x_5 + X)^2] = 0.$$

Then let $x_i = x_i'$, $i = 1, 2, 3, 4$. $X + x_5 = x_5'$. The net is invariant under $x_i' = x_i''$, $i = 1, 2, 3, 4$. $x_5' = -x_5''$. The center O of this perspective transformation is $x_1'' = x_2'' = x_3'' = x_4'' = 0$; the fixed space is $x_5'' = 0$. It is evident that the lines through O cutting $C_{8,4}$ in one point P_1 must cut it again in P_1'. Since D_{10} is invariant under the transformation, the four lines which are a part of it in this case must also pass through O. Let D_1, D_1', D_1'', D_1''' be the lines of D_{10}. Project $C_{8,4}$ from P_1 and find the surface of trisecants. $P_1 O$ will cut C_8 again in P_2. Join O to another point P_3, and a fourth point P_4 will be determined. There are evidently ∞^1 planes through $P_1 O$ cutting the normal curve in four points. $P_1 P_2$ are included in each G_4 of the g_4^1. Now as P_3 describes C_8 the plane of the G_4 will successively contain D_1, D_1', D_1'', D_1'''; in such cases the G_4 consists of the points of tangency of an R_3. The corresponding trisecant is a double torsal generator. These four double torsal generators intersect in the R_3 image of O, *i. e.*, in O'. The image of P_2 will be at O' in every case and $C_{7,3}$ will cut each trisecant in two points aside from O'. Since there are ∞^1 generators of S_{15} passing through O', the surface must consist in part of a cone K_k with O' as a vertex. $C_{7,3}$ evidently lies on this cone. A plane through two generators of K_k contains the five points on these generators; and since a plane through the vertex of a cone can cut the cone only in generators, the other two points in this plane must be on a third generator, *i. e.*, $K_k \equiv K_3$. If we turn the plane of our G_4 on $P_1 P_3$ or $P_1 P_4$ instead of $P_1 O$, $C_{7,3}$ will of course be the same as before, but the trisecants will not pass through O'; they will determine the part of S_{15} aside from K_3, *i. e.*, an S_{12}. $C_{7,3}$ must therefore be common to K_3 and S_{12}. The double torsal generators will appear on both

surfaces and must be identical; this means that K_3 and S_{12} are tangent along these generators. $C_{7,3}$ is always a five-fold curve on S_{15}, and in this case it is counted once on K_3 and four times on S_{12}. The complete intersection, then, of K_3 and S_{12} is C_7 as a four-fold curve and the four double torsal generators each counted twice; *i. e.*,

$$4 \cdot 7 + 2 \cdot 4 = 36.$$

The genus of S_{12} is not greater than 7 nor less than 4. For, the plane section is of order 12, $D_{6,3}$ gives six double points, $C_{7,3}$ is four-fold on S_{12} and gives $7 \cdot \frac{4 \cdot 3}{2} = 42$ double points. $p = 1/2 \cdot 11 \cdot 10 - 6 - 42 = 7$. The C_4 factor of Δ_5 may have as many as three double points, which would give rise to an equal number of double torsal generators of S_{12}, and the minimum genus is therefore 4. Nothing of this sort can happen to K_3, which is, therefore, of genus 1.

[3] In order that $\Delta_5 \equiv C_3 \cdot C_2$, the F_i must be reducible to the form

$$f_i(x_1, x_2) + \psi_i(x_3, x_4, x_5) = 0.$$

By reason of the six double points of Δ_5, $D_{10} \equiv D_4 + 6 D_1$. The six lines correspond to the six double points and there must be two factors of D_4 corresponding to the C_3 and C_2 of Δ_5. The genera must be 1 and 0; hence, $D_4 \equiv D_3 \cdot D_1$. This D_1 does not correspond to a double point of Δ_5 and is therefore not the vertex of a cone. Now project $D_{10,4}$ and $C_{8,4}$ from P_1 of the $C_{8,4}$ and build the surface of trisecants. $D_{3,3}$ and $D_{1,3}$ can not lie on the same ruled surface, for then the planes through P_1 defining the G_4's and $C_{8,4}$ must cut $D_{10,4}$ twice, which is in general not possible. Therefore S_{15} is degenerate. $D_{3,4}$ is of genus 1 and is therefore a plane curve. This plane and the line D_1 are the fixed elements in the axial perspective which the above F_i shows must exist. Since from every point of D_1 four bisecants can be drawn to $C_{8,4}$, from every point of $C_{8,4}$ can be drawn a line cutting D_1 and $C_{8,4}$. Hence, when the projection is made from P_1, $D_{1,3}$ and $C_{7,3}$ will intersect on S_{15}. To find the order of that part of S_{15} on which $D_{1,3}$ lies, pass a plane through the line. It will cut $C_{7,3}$ in one point of $D_{1,3}$ and in six others. These six will be arranged in threes on the sides of a quadrilateral, for every generator must contain three points of $C_{7,3}$. The generators will also cut $D_{1,3}$. Thus each generator is cut by four others, which makes the order of the surface 6.* Therefore,

$$S_{15} \equiv S_6 \cdot S_9.$$

* Snyder: "On the Forms of Sextic Scrolls of Genus Greater than One," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXV (1903), pp. 261–268. See type II, p. 262.

C_7 can not be more than a double curve on S_6 nor more than three-fold on S_9. Then the double points of a plane section of S_6 are seven for C_7 and one for D_1, making eight. Therefore, $p = 2$. The double points of a section of S_9 are twenty-one for C_7 and three for D_3. Therefore, $p = 4$. If C_3 has a double point, S_9 will have one double torsal generator and its genus will be 3. As in the preceding case, all the lines of D_{10} that are vertices of cones will go into lines of tangency of the parts of S_{15}. The complete intersection then of S_6 and S_9 consists of C_7 counted six times and the six lines counted twice.

[4] If the F_i are of the form

$$a_i x_1^2 + b_i x_2^2 + \psi_i (x_3, x_4, x_5) = 0,$$

Δ_5 becomes $C_3 . C_1 . C_1' = 0$ and $S_{15} \equiv S_9 . K_3 . K_3' = 0$. C_1 and C_1' each determine four collinear double points of Δ_5, but the intersection of the two lines is thus counted twice and we have but seven double points. Corresponding to these there must be two tetrads of concurrent lines, the line of centers belonging to both. Let the lines be $a_1 a_2 \ldots . a_7$ and the centers O_1 and O_2. Call the line of centers a_4. Project into R_3 from a point of the normal curve. O_1' and O_2' will be the vertices of two cones K_3 and K_3' tangent along a_4'. The remainder of S_{15} must be S_9 tangent to K_3 along $a_1' a_2' a_3' a_4'$ and to K_3' along $a_4' a_5' a_6' a_7'$. Since K_3 and K_3' can have no nodal curves, the five-fold C_7 is a simple curve on each and three-fold on S_9. We see, then, that the complete intersection of S_9 and K_3 is C_7 counted three times and $a_1' a_2' a_3'$. For S_9 and K_3', C_7 and $a_5' a_6' a_7'$. K_3 and K_3' have C_7 and a_4' in common.

[5] If F_i are of the form

$$a_i x_1^2 + f_i (x_2, x_3) + \psi_i (x_4, x_5) = 0,$$

Δ_5 becomes $C_2 . C_2' . C_1$ and $S_{15} \equiv S_6 . S_6' . K_3 = 0$. C_1 determines four collinear double points of Δ_5. The corresponding lines of D_{10} will be $a_1 a_2 a_3 a_4$ concurrent at O. C_2 and C_2' determine four more double points of Δ_5 corresponding to which there will be four skew lines $a_5 a_6 a_7 a_8$ of D_{10}. The remainder of D_{10} is a C_2 consisting of the axes of the two axial perspectives under which F_i is invariant. Project as before into R_3 and the image of O will be the vertex of the cone S_3. C_7 will be double on S_6 and S_6' and simple on K_3. The intersection of S_6 and S_6' consists of C_7 counted four times and the double torsal generators, $a_5' a_6' a_7' a_8'$, which are lines of tangency. S_6 and K_3 intersect in C_7 counted twice and the two lines of tangency $a_1' a_2'$. S_6' and S_3 have C_7 and $a_3' a_4'$ in common.

[6] If F_i are of the form

$$a_i x_1^2 + b_i x_2^2 + c_i x_3^2 + f_i(x_4, x_5) = 0,$$

Δ_5 becomes $C_2 . C_1 . C_1' . C_1'' = 0$ and $S_{15} \equiv S_6 . K_3 . K_3' . K_3'' = 0$. C_1, C_1', C_1'' each determine four collinear double points of Δ_5. We will therefore have three tetrads of lines in D_{10}. Let the centers be O_1, O_2, O_3. Let a_1, a_2, a_3 correspond to the mutual intersections of the three lines and a_4, a_5, ..., a_9 correspond to the six points on C_2. Then, for the three tetrads and their center, we may use the notation

$$O_1(a_1, a_2, a_4, a_5), \quad O_2(a_1, a_3, a_6, a_7), \quad O_3(a_2, a_3, a_8, a_9),$$

where

$$a_1 \equiv O_1 O_2, \quad a_2 \equiv O_1 O_3, \quad a_3 \equiv O_2 O_3.$$

The remaining line of D_{10} does not correspond to a double point of Δ_5; hence, it is not the vertex of an R_4 cone. Projecting as before into R_3, O_1', O_2', O_3' are the vertices of K_3, K_3', K_3'' respectively. C_7 will be a double curve on S_6 and a simple curve on the cones.

K_3 and K_3' intersect in C_7 counted once and a_1'.
K_3 and K_3'' intersect in C_7 counted once and a_2'.
K_3' and K_3'' intersect in C_7 counted once and a_3'.
S_6 and K_3 intersect in C_7 counted twice and a_4', a_5'.
S_6 and K_3' intersect in C_7 counted twice and a_6', a_7'.
S_6 and K_3'' intersect in C_7 counted twice and a_8', a_9'.

[7] When Δ_5 consists of five straight lines, the ten intersections require that D_{10} shall consist wholly of straight lines. In this case F_i is reducible to

$$a_i x_1^2 + b_i x_2^2 + c_i x_3^2 + d_i x_4^2 + e_i x_5^2 = 0,$$

in which $\Sigma \lambda_i a_i$, $\Sigma \lambda_i b_i$, ..., $\Sigma \lambda_i e_i$, $(i = 1, 2, 3)$, are the five factors of Δ_5. As already shown, the ten lines of D_{10} constitute the simplex of reference in R_4 and intersect by fours in five points. When this configuration is projected from a point of $C_{8,4}$ into R_3, the five vertices of the simplex go into the vertices of the five K_3's into which S_{15} degenerates. The ten lines go into ten lines, joining these five vertices in pairs. Thus each cone passes through the vertices of the other four and any two of the five are tangent along their common generator. C_7 in each case completes the intersection which is of order 9.

8. Since the normal curve $C_{8,4}$ and the $C_{6,2}$ are in (1, 1) correspondence, the transformations that leave $C_{6,2}$ invariant may be obtained by finding those under which $C_{8,4}$ is invariant and then getting the corresponding transformation in R_2. The method in brief is this: Find a transformation T that leaves five points of R_2 invariant. Write the equation of the most general C_3 through them. It is the equation of the complete net of adjoining curves of C_6 and has five terms. Neglecting the constant coefficients, regard these five expressions in x, y, z as $\phi_1, \phi_2, \ldots, \phi_5$, the homogeneous point coordinates in linear space of four dimensions. Find the three quadratic relations among them, two with numerical and the third with arbitrary coefficients. Apply T to the x, y, z coordinates and then find the corresponding transformation T_1 in ϕ_i. Next, determine the constants in the third quadratic relation so that it shall be invariant under T_1. This will be the equation of the C_6 that is invariant under T.

It must be kept in mind that Δ_5 will have transformations of the same order as those of C_6 if it does not remain fixed, point by point. C is defined by

$$F \equiv \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0,$$

and it is evident that the transformations that leave $C_{8,4}$ invariant must leave this net invariant. This can be done by collineations in ϕ which put F_i into F_i, leaving all points of $C_{8,4}$ fixed or permuted. Or F_i may go into linear combinations of F_i, which requires that Δ_5 have a corresponding linear transformation. In any case, Δ_5 as a whole must remain fixed because it is the condition for the double points of the system F which is invariant. Δ_5 is a quintic that can be written as a symmetric determinant. Such quintics have only the groups whose generators are G_2, G_3, G_4, G_5.* These we will now examine.

The cyclic groups G_7 and G_{13} need not be considered, as the associated curves can not be expressed in the above form.

A Linear G_2. Let $C_{6,2}$ have five distinct double points (0, 1, 0), (1, 1, 1), (1, 1, -1), (a, 0, c), (a, 0, $-c$). Write the equation of the general curve through these points. Then the five linearly independent adjoints by which we pass to R_4 may be written in the form

$$\phi_1 = c^2 x^3 + (a^2 - c^2)\, y z^2 - a^2 x z^2,$$
$$\phi_2 = c^2 (x^2 y - y z^2), \quad \phi_4 = c^2 x^2 z + (a^2 - c^2)\, x y z - a^2 z^3,$$
$$\phi_3 = c^2 (x y^2 - y z^2), \quad \phi_5 = c^2 (y^2 z - x y z).$$

* Snyder: "Plane Quintic Curves Which Possess a Group of Linear Transformations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908), pp. 1–9.

Corresponding to $T_1 \equiv \begin{pmatrix} x & y & & z \\ x & y & -z \end{pmatrix}$, we have

$$T_2 \equiv \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 \\ \phi_1 & \phi_2 & \phi_3 & -\phi_4 & -\phi_5 \end{pmatrix}.$$

To find the quadratic identities, write the quadratic equation in the five ϕ_i and substitute the values of ϕ in terms of x, y, z. To make the equation an identity some of the coefficients will be zero, leaving

$$A\phi_1\phi_3 + B\phi_1\phi_5 + D\phi_2^2 + E\phi_2\phi_4 + F\phi_2\phi_5 + G\phi_3\phi_4 + K\phi_4\phi_5 = 0.$$

This equation is identically satisfied only in case $A = -D = -K$ and $B = E = \frac{c^2}{a^2 - c^2} F = -G$.

The result may be written in the form

$$A(\phi_1\phi_3 - \phi_2^2 - \phi_4\phi_5) + B(\phi_1\phi_5 + \phi_2\phi_4 + \frac{a^2 - c^2}{c^2}\phi_2\phi_5 - \phi_3\phi_4) = 0.$$

Hence, we have

$$F_1 \equiv \phi_1\phi_3 - \phi_2^2 - \phi_4\phi_5 = 0,$$
$$F_2 \equiv c^2\phi_1\phi_5 + c^2\phi_2\phi_4 + (a^2 - c^2)\phi_2\phi_5 - c\phi_3\phi_4 = 0.$$

F_3 may now be obtained by writing the general quadratic in ϕ_i, omitting two terms by means of F_1 and F_2, and imposing the condition that it shall be invariant under T_2. This means that F_3 must either be linear in ϕ_4, ϕ_5 or contain no such linear terms at all. The former will make z a factor of C_6 and is therefore inadmissible. Hence, ϕ_4 and ϕ_5 can appear only when they form quadratic terms.

$$F_3 \equiv A\phi_1^2 + B\phi_1\phi_2 + C\phi_2^2 + D\phi_2\phi_3 + E\phi_3^2 + F\phi_4^2 + G\phi_4\phi_5 + H\phi_5^2 = 0.$$

By T_2 the net

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0$$

becomes

$$\lambda_1 F_1 + \lambda_2(-F_2) + \lambda_3 F_3 = 0;$$

hence, Δ_5 must have the transformation

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & -\lambda_2 & \lambda_3 \end{pmatrix}.$$

The forty-two inflections and one hundred twenty-four bitangents of $C_{6,2}$ are symmetrically placed with respect to $z = 0$.

A Linear G_3. Let the three vertices of the triangle of reference, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, be the double points of $C_{6,2}$. The general equation of a ϕ-curve through these points may be written

$$\begin{aligned} &K[(\omega-\omega^2)(x^2y-\omega xyz)-(\omega^2-\omega)(yz^2-\omega^2xyz)] \\ &+L[(\omega-\omega^2)(x^2z-\omega^2xyz)-(\omega-\omega^2)(yz^2-\omega^2xyz)] \\ &+M[(\omega-\omega^2)(xy^2-\omega^2xyz)-(\omega-\omega^2)(yz^2-\omega^2xyz)] \\ &+N[(\omega-\omega^2)(xz^2-\omega xyz)-(\omega^2-\omega)(yz^2-\omega^2xyz)] \\ &+P[(\omega-\omega^2)(y^2z-\omega xyz)-(\omega^2-\omega)(yz^2-\omega^2xyz)]=0. \end{aligned}$$

Transform to R_4 by

$$\begin{aligned} \phi_1 &= x^2y+yz^2+xyz, & \phi_4 &= xz^2+yz^2+xyz, \\ \phi_2 &= x^2z-yz^2, & \phi_5 &= y^2z+yz^2+xyz. \\ \phi_3 &= xy^2-yz^2, \end{aligned}$$

Corresponding to $T_1 \equiv \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$, we have

$$\left.\begin{aligned} \phi_1 &= \phi_2'+\phi_5', & \phi_3 &= -\phi_2, & \phi_5 &= \phi_2'+\phi_4'. \\ \phi_2 &= \phi_3'-\phi_2', & \phi_4 &= \phi_1'+\phi_2', \end{aligned}\right\} \equiv T_2.$$

Find the quadratic identities as before.

$$F_1 \equiv \phi_1\phi_4-\phi_2\phi_3-\phi_2\phi_5-\phi_4\phi_5=0.$$

By T_2 this becomes

$$F_2 \equiv \phi_1\phi_5+\phi_2\phi_5-\phi_3\phi_4-\phi_1\phi_4=0,$$

and F_2 becomes

$$F_2' \equiv \phi_4\phi_5+\phi_3\phi_4+\phi_2\phi_3-\phi_1\phi_5=-(F_1+F_2).$$

F_2' becomes F_1 by the same transformation; the cycle of order 3 is thus completed. F_3 may be found as before. The result is

$$\begin{aligned} F_3 \equiv\ &A\phi_1^2+B\phi_1\phi_2+C\phi_1\phi_3+D\phi_2^2+(A-D)\phi_2\phi_3+C\phi_2\phi_4 \\ &+(2A-B-C)\phi_2\phi_5+D\phi_3^2+(2A-B-C)\phi_3\phi_4+B\phi_3\phi_5 \\ &+A(\phi_4^2+\phi_5^2)=0. \end{aligned}$$

In this case, to make the net $\lambda_1F_1+\lambda_2F_2+\lambda_3F_3=0$ invariant, Δ_5 must have the group

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_2 & \lambda_1-\lambda_2 & \lambda_3 \end{pmatrix}.$$

The $C_{6,2}$ has 42 inflections and 124 bitangents. The whole number of inflections can be permuted in threes, but at least one bitangent must remain

fixed. The nodal tangents at $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ must remain fixed, because these points are invariant. To find what lines of the pencil $(1, \omega, \omega^2)$ are invariant, write the equation of the line through this point and (x, z, y). Then impose the condition on the coefficients that the line shall be invariant under T_1. The equation is

$$(\omega z_1 - \omega^2 y_1)x + (\omega^2 x_1 - z_1)y + (y_1 - \omega x_1)z = 0.$$

By T_1,

$$(\omega^2 x_1 - z_1)x + (y_1 - \omega x_1)y + (\omega z_1 - \omega^2 y)z = 0.$$

From these we find $(x_1 y_1 z_1) \equiv (1, 1, 1)$ or $(1, \omega^2, \omega)$. This means that one of the tangents at $(1, \omega, \omega^2)$ goes through $(1, 1, 1)$ and the other through $(1, \omega^2, \omega)$. By the same method we find that one of the tangents at $(1, \omega^2, \omega)$ goes through $(1, 1, 1)$ and the other through $(1, \omega, \omega^2)$; *i. e.*, the line joining these two double points is a bitangent. This leaves 123 bitangents to be permuted by the operations of G_8.

A linear G_4. Use $(1, 0, 0)$, $(1, 1, 1)$, $(1, i, -1)$, $(1, -1, 1)$, $(1, -i, -1)$ as double points. They are invariant under

$$\begin{pmatrix} x & y & z \\ x & iy & -z \end{pmatrix} \equiv T_1.$$

The equation of the general ϕ-curve through the above points is

$$B(x^2y - yz^2) + C(xy^2 - z^3) + D(y^3 - xyz) + E(x^2z - z^3) + F(xz^2 - y^2z) = 0.$$

Now pass to R_4, make T_1 on ϕ_i and find the relation between ϕ_i and ϕ_i'.

$$\left.\begin{array}{lll} \phi_1 = x^2y - yz^2 = i(x^{2\prime}y' - y'z^{2\prime}) & = i\phi_1' \\ \phi_2 = xy^2 - z^3 = -(z^{2\prime}y' - z^{3\prime}) & = -\phi_2' \\ \phi_3 = y^3 - xyz = -i(y^{3\prime} - x'y'z') & = -i\phi_3' \\ \phi_4 = x^2z - z^3 = -(x^{2\prime}z' - z^{3\prime}) & = -\phi_4' \\ \phi_5 = xz^2 - y^2z = (x'z^{2\prime} - y^{2\prime}z') & = \phi_5' \end{array}\right\} \equiv T_2.$$

By the usual method we find two identities,

$$F_1 \equiv \phi_1\phi_5 + \phi_3\phi_4 = 0,$$
$$F_2 \equiv \phi_1\phi_3 - \phi_2^2 + \phi_2\phi_4 + \phi_5^2 = 0.$$

By transforming the general equation by T_2, we find that the terms ϕ_1^2, $\phi_2\phi_5$, ϕ_3^2, $\phi_4\phi_5$ merely change sign. Hence, we may write

$$F_3 \equiv A\phi_1^2 + G\phi_2\phi_5 + H\phi_3^2 + N\phi_4\phi_5 = 0.$$

By T_2 the net

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0$$

becomes

$$\lambda_1 i F_1 + \lambda_2 F_2 - \lambda_3 F_3 = 0.$$

Δ_5 must therefore be invariant under

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ -i\lambda_1 & \lambda_2 & -\lambda_3 \end{pmatrix}.$$

Now write F_3 in the form

$$A(x^2y - yz^2)^2 + G(xy^2 - z^3)(xz^2 - y^2z) + H(y^3 - xyz)^2 + N(x^3z - z^3)(xz^2 - y^2z) = 0.$$

This is the equation of the $C_{6,2}$. At $y = 0$,

$$xz^3[Nx^2 - (G - N)z^2] = 0.$$

There are therefore three points of intersection at the vertex (1, 0, 0). Since the equation contains only even powers of y this point is a cusp and $y = 0$ is the tangent. The presence of this cusp reduces the number of inflections to forty. These and the 124 bitangents are permuted in fours. $x = 0$ is an invariant tangent and $Nx^2 - (G - N)z^2 = 0$ defines two tangents through (0, 1, 0) whose points of contact are on $y = 0$. The points on $y = 0$ interchange in pairs by T_1.

A Linear G_5. Let (1, 1, 1), $(1, \theta, \theta^2)$, $(1, \theta^2, \theta^4)$, $(1, \theta^3, \theta)$, $(1, \theta^4, \theta^3)$ be the double points wherein $\theta^5 = 1$. The general equation of a C_3 is

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ey^2z + Fxz^2 + Gx^2z + Hyz^2 + Kxyz + Lz^3 = 0.$$

If $(1, \theta, \theta^2)$ lies upon it,

$$(A + H) + (B + L)\theta + (C + G)\theta^2 + (D + K)\theta^3 + (E + F)\theta^4 = 0.$$

The remaining points give equations with the same coefficients in $(A + H)$, $(B + L)$, etc. Therefore since

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \theta & \theta^2 & \theta^3 & \theta^4 \\ 1 & \theta^2 & \theta^4 & \theta & \theta^3 \\ 1 & \theta^3 & \theta & \theta^4 & \theta^2 \\ 1 & \theta^4 & \theta^3 & \theta^2 & \theta \end{vmatrix} \neq 0,$$

$$A + H = B + L = C + G = D + K = E + F = 0.$$

The equation of the ϕ-curve then becomes

$$A(x^3 - yz^2) + B(z^3 - x^2y) + C(xy^2 - x^2z) + D(y^3 - xyz) + E(y^2z - xz^2) = 0.$$

Now pass to R_4 by the method used in the preceding cases. Corresponding to

$$T_1 \equiv \begin{pmatrix} x & y & z \\ x & \theta y & \theta^2 z \end{pmatrix},$$

we find

$$T_2 \equiv \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 \\ \phi_1 & \theta\phi_2 & \theta^2\phi_3 & \theta^3\phi_4 & \theta^4\phi_5 \end{pmatrix}.$$

The terms ϕ_1^2, $\phi_1\phi_2$, $\phi_1\phi_3$, ϕ_2^2, $\phi_3\phi_4$, $\phi_2\phi_5$, $\phi_3\phi_5$, ϕ_4^2 and $\phi_4\phi_5$ contain terms in x, y, z that are not duplicated in quadratic combinations of ϕ_i. Hence, these can not appear in the identities. We have then to consider only six terms.

$$A\phi_1\phi_4 + B\phi_1\phi_5 + C\phi_2\phi_3 + D\phi_2\phi_4 + E\phi_3^2 + F\phi_5^2 = 0.$$

By substituting the values of ϕ_i in this equation, we find

$$A = C = F \quad \text{and} \quad B = D = E.$$

Hence,

$$F_1 \equiv \phi_1\phi_4 + \phi_2\phi_3 + \phi_5^2 = 0,$$
$$F_2 \equiv \phi_1\phi_5 + \phi_2\phi_4 + \phi_3^2 = 0.$$

By means of F_1 and F_2 replace two terms $\phi_1\phi_4$ and $\phi_1\phi_5$ by their equivalents in the general quadratic equation in ϕ_i. Make the transformation T_2. Only three terms remain invariant. Hence,

$$F_3 \equiv A\phi_1^2 + B\phi_2\phi_5 + C\phi_3\phi_4 = 0.$$

By T_2 the net $\Sigma\lambda_i F_i = 0$ becomes

$$\lambda_1\theta^3 F_1 + \lambda_2\theta^4 F_2 + \lambda_3 F_3 = 0.$$

This means that Δ_5 has the group

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \theta^2\lambda_1 & \theta\lambda_2 & \lambda_3 \end{pmatrix}.$$

In the above $C_{6,2}$ we find that $x = 0$ is a bitangent with ordinary contact at $(0, 0, 1)$ and four-point contact at $(0, 1, 0)$. Thus four invariant bitangents and two invariant inflections are accounted for. This leaves 120 bitangents and 40 inflections to be permuted in fives.

The points $(1, 1, 1)$, $(1, \theta, \theta^4)$, $(1, \theta^2, \theta^3)$, $(1, \theta^3, \theta^2)$, $(1, \theta^4, \theta)$ may be used as the basis of a G_5. They are invariant under a G_2 as well. Find the adjoint curves as before.

$$\begin{aligned} \phi_1 &= x^3 - xyz, & \phi_4 &= y^3 - xz^2, \\ \phi_2 &= x^2y - y^2z, & \phi_5 &= x^2z - yz^2. \\ \phi_3 &= z^3 - xy^2, \end{aligned}$$

Two identities are

$$F_1 \equiv \phi_1\phi_4 + \phi_2\phi_3 + \phi_5^2 = 0, \qquad F_2 \equiv \phi_1\phi_3 + \phi_4\phi_5 + \phi_2^2 = 0.$$

Then the most general quadratic form that is invariant under T_2 of the preceding G_5 is

$$F_3 \equiv A\phi_1^2 + B\phi_3\phi_4 + C\phi_2\phi_5 = 0.$$

The transformation in x, y, z is

$$T_3 \equiv \begin{pmatrix} x & y & z \\ x & \theta y & \theta^4 z \end{pmatrix}.$$

The result is

$$\theta^2\lambda_1\theta^3 F_1 + \theta^3\lambda_2 . \theta^2 F_2 + \lambda_3 F_3 = 0,$$

and the net has the same equation as before.

The curve $F_3 \equiv C_6$ has $x = 0$ for inflectional tangent at the vertices $(0, 0, 1)$ and $(0, 1, 0)$. These two inflections are invariant. $x = 0$ counts for four bitangents which are invariant. The remaining bitangents and inflections are permuted in fives.

We will now show the G_2 of this C_6. By $\begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}$, ϕ_2 and ϕ_5 interchange; also ϕ_3 and ϕ_4. This interchanges F_1 and F_2 but leaves F_3 invariant. If we transform Δ_5 by $\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \lambda_1 & \lambda_3 \end{pmatrix}$, the net will be invariant.

$$\lambda_1 F_2 + \lambda_2 F_1 + \lambda_3 F_3 = 0\,; \quad i.\,e., \quad \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0.$$

A necessary condition that Δ_5 be point by point invariant when F has a G_2 is that the G_2 be reducible to a change of sign, but it is not sufficient, for the transformation

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 \\ \phi_1 & \phi_5 & \phi_4 & \phi_3 & \phi_2 \end{pmatrix},$$

mentioned above, may be put in the form

$$T \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & -x_4 & -x_5 \end{pmatrix}$$

by making the following linear transformation on ϕ_i:

$$\phi_1 = x_1, \quad \phi_2 + \phi_5 = x_2, \quad \phi_3 + \phi_4 = x_3, \quad \phi_4 - \phi_3 = x_4, \quad \phi_5 - \phi_2 = x_5.$$

Make this transformation on F_i.

$$F_1 \equiv 2x_1(x_3 + x_4) + (x_2 - x_5)(x_3 - x_4) + (x_2 + x_5)^2 = 0,$$
$$F_2 \equiv 2x_1(x_3 - x_4) + (x_2 + x_5)(x_3 + x_4) + (x_2 - x_5)^2 = 0,$$
$$F_3 \equiv A_1 x_1^2 + B_1(x_3^2 - x_4^2) + C_1(x_2^2 - x_5^2) = 0.$$

It is now evident that T interchanges F_1 and F_2. F_3 being unchanged, the G^2 of Δ_5 is the same as before. A G_{10} is the product of the above G_2 and G_5.

The following cases and curves were given by Wiman (W). When Δ_5 and $C_{8,4}$ have a G_μ, then, according to the seven cases mentioned in 6, we have groups whose orders are μ, 2μ, 2μ, 4μ, 4μ, 8μ, 16μ.

Case (a) The G_μ exists.
(b) A $G_{2\mu}$: Change of sign of one variable.
(c) A $G_{2\mu}$: Simultaneous change of signs of two variables.
(d) A $G_{4\mu}$: Two independent changes of sign.
(e) A $G_{4\mu}$: Change of sign of two pairs of variables.
(f) A $G_{8\mu}$: Three independent changes of sign.
(g) A $G_{16\mu}$: Four independent changes of sign.

The $G_{16\mu}$ exists when the F_i contain only the squares of the variables. In illustration of (f), we have a G_{192} composed of G_8 consisting of changes of sign and G_{24} the octahedral group shown by the factors $x_4 x_5 (x_4^2 + x_5^2)(x_4^2 - x_5^2)$. The curve is defined by

$$F_1 \equiv x_1^2 + x_4^2 + x_5^2 = 0,$$
$$F_2 \equiv x_2^2 + x_4^2 - x_5^2 = 0,$$
$$F_3 \equiv x_3^2 + x_4 x_5 = 0.$$

The curve with a G_{64} is defined by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$
$$x_1^2 + i x_2^2 - x_3^2 - i x_4^2 = 0,$$
$$x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0, \quad (i^2 = -1).$$

The G_{64} is composed of G_{16} and the cyclic G_4 on the variables $x_1 x_2 x_3 x_4$.

For G_{96} we have

$$x_1^2 + x_4^2 + x_5^2 = 0,$$
$$x_2^2 + j x_4^2 + j^2 x_5^2 = 0,$$
$$x_3^2 + j^2 x_4^2 + j x_5^2 = 0, \quad (j^3 = 1).$$

The components of this group are G_{16},

$$G_2 \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_3 & x_2 & x_5 & x_4 \end{pmatrix}, \qquad G_3 \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_3 & x_1 & x_2 & j x_4 & j^2 x_5 \end{pmatrix}.$$

For G_{160} we have

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$
$$x_1^2 + e x_2^2 + e^2 x_3^2 + e^3 x_4^2 + e^4 x_5^2 = 0,$$
$$e^4 x_1^2 + e^3 x_2^2 + e^2 x_3^2 + e x_4^2 + x_5^2 = 0, \quad (e^5 = 1).$$

The components of the group are G_{16},

$$G_2 \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_5 & x_4 & x_3 & x_2 & x_1 \end{pmatrix}, \qquad G_5 \equiv \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_5 & x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

9. (b) If a $C_{6,2}^{(5)}$ has a linear g_3^1, it must have a triple point.* By means of quadric inversion having the P_3 and the two P_2 for fundamental points, the curve is reduced to a nodal quintic. The adjoint curves are conics through the node (0, 0, 1). The general equation of these conics is

$$A x^2 + B x y + C y^2 + D x z + E y z = 0. \tag{26}$$

Pass to R_4 by the transformation

$$\phi_1 = x^2, \quad \phi_2 = x y, \quad \phi_3 = y^2, \quad \phi_4 = x z, \quad \phi_5 = y z. \tag{27}$$

Then three quadratic forms exist which are identities in x, y, z.

$$\phi_1 \phi_3 - \phi_2^2 = 0, \quad \phi_1 \phi_5 - \phi_2 \phi_4 = 0, \quad \phi_2 \phi_5 - \phi_3 \phi_4 = 0. \tag{28}$$

Three general quadrics in R_4 determine a $C_{8,4}$, but the set in (28) have a ruled hypersurface S in common for

$$\phi_1/\phi_2 = \phi_2/\phi_3 = \phi_4/\phi_5 = x/y = \lambda, \tag{29}$$

and $\phi_1 = \lambda \phi_2$, $\phi_2 = \lambda \phi_3$, $\phi_4 = \lambda \phi_5$ define a straight line in R_4 which is the image of the line $x = \lambda y$ in R_2. Therefore, S is the R_4 image of the plane generated

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* Snyder: "On Birational Transformations of Curves of High Genus," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908), pp. 10–18. See Art. 3.

by the pencil of lines through (0, 0, 1). The point $x=y=0$ has for its image the line $\phi_1=\phi_2=\phi_3=0$. All the lines, $x=\lambda y$, pass through the node. Hence their images, the generators of S, must cut the image of the node; *i. e.*, $\phi_1=\phi_2=\phi_3$ is the directrix. Each line of the nodal pencil cuts C_5 in three variable points. Therefore, each generator is a trisecant of the R_4 image of C_5 which lies on S. Since the node is a double point of C_5, its image must have two points on the directrix; *i. e.*, the directrix is a bisecant. Since the whole plane is uniquely pictured on S, $C_{5,2}$ and $C_{8,4}$ are invariant under the same groups of transformations. When C_5 is invariant, the system of adjoint conics must go into itself. But any of these adjoints can be expressed linearly in terms of the five and in no other way; *i. e.*, the R_4 transformations will all be linear. By these, straight lines will go into straight lines. This means that the generators of S will simply be permuted among themselves. Thus the lines of the nodal pencil will also be permuted. Moreover, adjoint conics must go into adjoint conics. When one factor of a conic is a line through the node, the other factor is some straight line which must go into a straight line by the transformation. This is possible only by collineations. Therefore, the only birational transformations under which a nodal quintic curve remains invariant are linear. Further, we may take $x=0$, $y=0$ as the nodal tangents. If C_5 is invariant, these tangents must either remain fixed or interchange. Hence, the transformations will be of the form

$$\begin{pmatrix} x & y & z \\ \lambda x & \mu y & \nu z \end{pmatrix} \text{ or } \begin{pmatrix} x & y & z \\ \lambda y & \mu x & \nu z \end{pmatrix},$$

where λ, μ, ν are roots of unity.

A G_2. A nodal C_5 will go into itself by a harmonic homology if it is symmetric about a line through the node. This line $x=0$ can not be a nodal tangent, for such a condition would call for a third tangent as the image of the second one. It would also require that $x=0$ be a tacnodal tangent. We can write the equation of the required C_5 with $x-y=0$ and $x+y=0$ as nodal tangents by omitting the odd powers of x from the general equation. Let the curve go through the vertices of the triangle; then we have

$$z^3(x^2-y^2)+z^2(Ax^2y+By^3)+z(Cx^4+Dx^2y^2+Ey^4)+Fx^4y+Gx^2y^3=0. \quad (31)$$

The center O of this homology is at the vertex (1, 0, 0). The coefficient of x^4 in (31) is the tangent to C_5 at O, $Cz+Fy=0$. This line must cut C_5 in an

even number of points apart from O. Hence, it must have either three- or five-point contact at O. In either case an odd number of inflections will be invariant. Since by Plücker's formulas the total number is thirty-nine, the remaining ones can be interchanged in pairs. The line $Cz+Fy=0$ will count for two bitangents or none according as it has five- or three-point contact. There are forty-two in all, and in either case they can be interchanged in pairs. The transformation is

$$\begin{pmatrix} x & y & z \\ -x & y & z \end{pmatrix}.$$

No perspective whose period is of odd order can exist with a nodal line as axis, for it would require an odd number of tangents at the node. None of order 4 can exist, for, retaining only those x-terms that contain x^4, the equation becomes

$$-z^3y^2+Bz^2y^3+Czx^4+Ezy^4+Fx^4y=0.$$

At the vertex (0, 0, 1) this shows $x=0$ to be a tacnodal tangent which reduces the genus of C_5.

A G_3. We may write the equation of a $C^{(5)}_{5,2}$ that is invariant under the operations of a linear G_3 by omitting the z and z^2 terms from the general equation. The equation can be put in the form

$$z^3(x^2-a^2y^2)+(ax+by)(cx+dy)(ex+fy)(gx+hy)(mx+ny)=0. \quad (32)$$

The factors of the second term in this equation show five inflectional tangents which are concurrent at (0, 0, 1) and whose points of contact are on $z=0$. If the second factor in the first term vanishes, the equation becomes $x^5=0$ or $y^5=0$. Hence, the nodal tangents $x+\alpha y=0$ and $x-\alpha y=0$ must vanish to the fifth order at the node. Since each of them cuts the branch to which the other is tangent only once, their order of contact must be 3. The five inflectional tangents count for ten tangents through (0, 0, 1). The nodal tangents count for four each. But the class of the C_5 is 18; hence, all the tangents through the node are accounted for. These will remain fixed under the G_3 given below. The curve has thirty-nine inflections. We have just seen that five of them are on the line $z=0$. The four-point contacts of the nodal tangents count for four inflections. These nine are invariant, leaving thirty to be permuted

in threes. Two of the ninety-two bitangents are accounted for by the nodal tangents. The remaining ones will be permuted. The transformation is

$$\begin{pmatrix} y & z \\ y & \omega z \end{pmatrix} \text{ or } \begin{pmatrix} y & z \\ y & \omega^2 z \end{pmatrix}.$$

If the constants are such that (32) is symmetric in x and y, we have

$$z^2(x^2+y^2) + A(x^4y + xy^4) + B(x^2y + xy^2) + C(x^5+y^5) = 0. \quad (33)$$

This curve admits of a G_2 as well as a G_3. Hence, it is invariant under the following G_6,

$$\begin{pmatrix} x & y & z \\ y & x & \omega z \end{pmatrix} \text{ or } \begin{pmatrix} x & y & z \\ y & x & \omega^2 z \end{pmatrix}.$$

Nine inflections and two bitangents will be invariant, as before. The others will be permuted in sixes.

A group G_5 is of the form

$$\begin{pmatrix} x & y & z \\ \theta x & \theta^4 y & z \end{pmatrix}.$$

The equation of the invariant C_5 is

$$z^3xy + a z x^2 y^2 + b x^5 + c y^5 = 0. \quad (34)$$

This curve has the nodal tangents $x=0$ and $y=0$, each having four-point contact. Thus these two lines count for two bitangents, eight simple tangents through $(0, 0, 1)$ and four inflectional tangents. The remaining ninety bitangents, ten simple tangents through the node and thirty-five inflectional tangents will be grouped in fives. If (l, m, n) is a point of inflection, four others are associated with it, namely $(\theta l, \theta^4 m, n)$, $(\theta^2 l, \theta^3 m, n)$, $(\theta^3 l, \theta^2 m, n)$, $(\theta^4 l, \theta m, n)$. These five points lie on the conic $n^2xy - lmz^2 = 0$. Therefore, the thirty-five points of inflection are arranged by fives on seven conics tangent to each other at their points of intersection with $z=0$. If $n=0$, the conic becomes $z^2=0$, which cuts each of the conics twice at their common points of tangency. The point $(1, -(b/c)^{1/5}, 0)$ is a point of inflection. The inflectional tangent is

$$x + \left(\frac{c}{b}\right)^{1/5} y + \frac{a}{5\, b^{3/5}\, c^{2/5}} z = 0. \quad (35)$$

Hence, $z=0$ cuts the curve in five points of inflection.

If in (34) $b=c$, the equation becomes

$$z^3xy + a z x^2 y^2 + b(x^5 + y^5) = 0. \quad (36)$$

The C_5 is now invariant under

$$\begin{pmatrix} x & y & z \\ y & x & z \end{pmatrix},$$

and therefore has a G_2 in addition to the G_5 just given. We must now account for one invariant inflection out of the thirty-five. It will also be necessary to fix four more so that the number will be divisible by both five and two. Let $(c/b)^{1/5} = \theta$, where $\theta^5 = 1$. Then the point $(1, -(b/c)^{1/5}, 0)$ becomes $(\theta, -1, 0)$. (35) becomes

$$x + \theta y + \frac{a}{5b}\theta^3 z = 0. \tag{37}$$

This tangent and its point of contact are invariant under the transformation

$$\begin{pmatrix} x & y & z \\ \theta y & \theta^4 x & z \end{pmatrix},$$

obtained by combining G_2 and G_5. Though it is formally a G_{10}, the transformations of this group interchange the points in pairs. Hence, the other four inflections on z are interchanged. This leaves thirty inflections. The ten simple tangents through $(0, 0, 1)$ and the ninety bitangents remain as in G_5. (36) is therefore invariant under the above G_{10}. Since (36) is only a special case of (34), we may apply G_5 to any three distinct points of inflection (a, b, c), (a', b', c'), (a'', b'', c'') and obtain twelve other inflections. The fifteen points will be distinct, provided no one of the first three goes into either of the others by the successive transformations. Next apply G_2 to each of these. We thus get fifteen others, which completes the thirty.

By G_5:			By G_2:		
a	b	c	b	a	c
θa	$\theta^4 b$	c	$\theta^4 b$	θa	c
$\theta^2 a$	$\theta^3 b$	c	$\theta^3 b$	$\theta^2 a$	c
$\theta^3 a$	$\theta^2 b$	c	$\theta^2 b$	$\theta^3 a$	c
$\theta^4 a$	θb	c	θb	$\theta^4 a$	c

These ten points lie on the conic $c^2 xy - abz^2 = 0$. Hence, they constitute the complete intersection of our C_5 with the above conic. A similar set of points may be found from (a', b', c') and (a'', b'', c''). If we denote the three conics by C_2, C_2' and C_2'', the thirty inflections are the complete intersections of C_5 with the family of sextics $C_2 . C_2' . C_2'' + Kz C_5 = 0$.

We will now combine G_3 and G_5 for a G_{15}. To make the curve invariant under

$$\begin{pmatrix} x & y & z \\ \theta x & \theta^4 y & z \end{pmatrix},$$

let $a=0$ in (34). The equation is

$$z^3 x y + b x^5 + c y^5 = 0. \tag{38}$$

As in G_5, the nodal tangents count for two bitangents, eight simple tangents through the node O and four inflectional tangents. $x + (c/b)^{1/5} y = 0$ is an inflectional tangent. It is independent of ω in the G_{15} and therefore has but four images. These five inflectional tangents account for the remaining ten tangents through O. There are then thirty inflections and ninety bitangents to be permuted by G_{15}. When $b=c$, we have a G_{30} defined by

$$\begin{pmatrix} x & y & z \\ \theta y & \theta^4 x & \omega z \end{pmatrix}.$$

The tangent $x + (c/b)^{1/5} y = 0$ becomes $x + \theta y = 0$. Its point of contact is $(\theta, -1, 0)$, which is invariant. By G_5 the four images are $(\theta^2, -\theta^4, 0)$, $(\theta^3, -\theta^3, 0)$, $(\theta^4, -\theta^2, 0)$, $(1, -\theta, 0)$. G_{30} interchanges these four in pairs as in G_{10}.

CORNELL UNIVERSITY, *June*, 1909.

*The General Theory of Linear q-Difference Equations.**

By R. D. Carmichael.

Introduction.

By means of a transformation of the form $z = (m_1 x + m_2)/(\mu_1 x + \mu_2)$, the system of functional equations

$$H_i\left(\frac{az+b}{cz+d}\right) = \sum_{j=1}^{n} \lambda_{ij}(z) H_j(z) \qquad (i = 1, \ldots, n),$$

in the n unknown functions $H_1(z), \ldots, H_n(z)$, may be transformed into the system of difference equations

$$G_i(x+1) = \sum_{j=1}^{n} \bar{\lambda}_{ij}(x) G_j(x) \qquad (i = 1, \ldots, n),$$

or into the system of q-difference equations

$$G_i(qx) = \sum_{j=1}^{n} \bar{\lambda}_{ij}(x) G_j(x) \qquad (i = 1, \ldots, n),$$

according as the substitution $z' = (az+b)/(cz+d)$ has one or two double points. To do this it is necessary to choose the transforming substitution so that in the first case the single double point is carried to infinity, and in the second case the two double points are carried to zero and to infinity respectively.

The essential general properties of the solutions of linear difference equations are known.†

The present paper is devoted to an investigation of the existence and properties of solutions of linear q-difference equations.

In § 1, for the case when $|q| \neq 1$, I prove the existence of two fundamental systems of solutions, one of simple character at infinity and the other of simple

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* Read before the American Mathematical Society (Chicago), April 28, 1911.

† See the papers by Carmichael and by Birkhoff in *Transactions of the American Mathematical Society*, Vol. XII. References to the previous literature of difference equations will be found in these papers.

character at zero. These two systems of solutions are analogous to the two systems of solutions of difference equations whose existence I pointed out at the close of my difference-equation paper (*loc. cit.*, pp. 133–134).

In § 2 an investigation of the relations between these two fundamental systems of solutions leads to a theory analogous to the interesting Birkhoff characterization of the solutions of a system of difference equations (see §§ 5 and 7 of the paper already referred to).

In § 3 I consider the exceptional case when $|q| = 1$. A method is given for obtaining fundamental systems of solutions in explicit finite form.

Rev. F. H. Jackson* has given a treatment of some questions connected with q-difference equations. Reference should also be made to some of the general theorems on functional equations due to A. Grévy† and L. Leau,‡ from which some of the results of the present paper can be deduced.

§ 1. *Existence of two Fundamental Systems of Solutions when* $|q| \neq 1$.

Let
$$G_i(qx) = q^\alpha x^\alpha \sum_{j=1}^{n} A_{ij}(x)\, G_j(x) = q^\beta x^\beta \sum_{j=1}^{n} B_{ij}(x)\, G_j(x) \qquad (i = 1, \ldots, n) \quad (1)$$
be a system of n first-order linear homogeneous q-difference equations involving the n unknown functions $G_1(x), \ldots, G_n(x)$ of the complex variable x, the known quantities entering into the equation being defined as follows:

1. α, β, q are constants and $|q| \neq 1$.
2. The functions $A_{ij}(x)$ and $B_{ij}(x)$ are single-valued and
$$A_{ij}(x) = A_{ij} + A'_{ij}x^{-1} + A''_{ij}x^{-2} + \ldots \qquad (i, j = 1, \ldots, n) \quad |x| \geqq R,$$
$$B_{ij}(x) = B_{ij} + B'_{ij}x + B''_{ij}x^2 + \ldots \qquad (i, j = 1, \ldots, n) \quad |x| \leqq r.$$

3. The constants A_{ij} and the constants B_{ij} are such that the roots $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ respectively of the characteristic equations at infinity and at zero,
$$|A_{ij} - \delta_{ij}\rho| = 0, \quad |B_{ij} - \delta_{ij}\rho| = 0, \qquad \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$
are finite and different from zero and verify the relations
$$q^\gamma A_i - A_j \neq 0, \quad q^\gamma B_i - B_j \neq 0 \qquad (i, j = 1, \ldots, n;\ i \neq j),$$
γ being any positive or negative integer or zero.

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, pp. 305–314.

† Paris thesis, 1894.

‡ Paris thesis, 1897.

When the roots of a characteristic equation satisfy the given relations, we shall say that they are of simple character.

By means of the transformations

$$G_i(x) = e^{\frac{\alpha}{2}\log q(\eta^2+\eta)}\overline{G}_i(x) = e^{\frac{\beta}{2}\log q(\eta^2+\eta)}\overline{G}_i'(x) \qquad (i=1, \ldots, n), \quad (2)$$

where $\eta = \log x/\log q$, system (1) may be replaced by two simpler systems, one of which has $\alpha = 0$ and the other of which has $\beta = 0$. Making these substitutions and reducing, we have

$$\overline{G}_i(qx) = \sum_{j=1}^{n} A_{ij}(x)\overline{G}_j(x) \qquad (i=1, \ldots, n), \quad (3)$$

$$\overline{G}_i'(qx) = \sum_{j=1}^{n} B_{ij}(x)\overline{G}_j'(x) \qquad (i=1, \ldots, n). \quad (4)$$

For the sake of simplicity in formulæ it is convenient to reduce (3) and (4) to normal forms valid in the neighborhood of infinity and of zero respectively. We begin with system (3).

Let us put

$$\overline{G}_i(x) = \sum_{j=1}^{n} \alpha_{ij}F_j(x) \qquad (i=1, \ldots, n), \quad (5)$$

where α_{ij}, $i, j = 1, \ldots, n$, are constants whose determinant $|\alpha_{ij}|$ is different from zero. Making this substitution in (3) and solving for $F_i(qx)$, $i=1, \ldots, n$, one may write the result in the form

$$F_i(qx) = \sum_{j=1}^{n} \overline{A}_{ij}(x)F_j(x) \qquad (i=1, \ldots, n),$$

where the functions $\overline{A}_{ij}(x)$, being linear combinations of the functions $A_{ij}(x)$ with constant coefficients, may be expanded in powers of x^{-1} for $|x| \geq R$. From the well-known theory of linear substitutions* and the fact that the roots of the characteristic equation at infinity are of simple character, it follows that a proper determination of the constants α_{ij} will reduce the constant term in $\overline{A}_{ij}(x)$ to zero when $i \neq j$. The preceding system of equations may then be written in the form

$$F_i(qx) = A_iF_i(x) + \sum_{j=1}^{n}(a_{ij}x^{-1} + a_{ij}'x^{-2} + \ldots)F_j(x) \qquad (6)$$
$$(i=1, \ldots, n) \quad |x| \geq R.$$

*Compare my difference-equation paper, *loc. cit.*, pp. 126–127, where for a similar transformation the reduction is carried out in detail.

In order to remove the term in x^{-1} from each of the non-diagonal coefficient functions of the second members, we may employ the transformation

$$F_i(x) = \bar{F}_i(x) + \sum_{j=1}^{n} \frac{\beta_{ij}}{x} \bar{F}_j(x), \quad \beta_{ii} = 0, \quad (i = 1, \ldots, n), \quad (7)$$

where the constants β_{ij}, $i \neq j$, are to be determined. Clearly the determinant of the transformation is not identically zero. If we substitute the above values of $F_i(x)$ in the preceding system of equations, and in the result write only those terms which do not involve x^{-2}, x^{-3}, ..., we have, when $|x| \geq R$,

$$\begin{aligned}
&\bar{F}_i(qx) + \sum_{j=1}^{n} \frac{\beta_{ij}}{qx} \bar{F}_j(qx) \\
&\quad = A_i \bar{F}_i(x) + A_i \sum_{j=1}^{n} \frac{\beta_{ij}}{x} \bar{F}_j(x) + \sum_{i=1}^{n} \left\{ \left(\frac{a_{ij}}{x} + \ldots \right) \left[\bar{F}_j(x) + \sum_{k=1}^{n} \frac{\beta_{jk}}{x} \bar{F}_k(x) \right] \right\} \\
&\quad = \left(A_i + \frac{a_{ii}}{x} + \ldots \right) \bar{F}_i(x) + \sum_{j=1}^{n}{}' \left(\frac{A_i \beta_{ij} + a_{ij}}{x} + \ldots \right) \bar{F}_j(x) \quad (i = 1, \ldots, n),
\end{aligned}$$

where the accent in Σ' denotes that in the summation the term for which $j = i$ is to be omitted.

In order to solve these equations for $\bar{F}_i(qx)$, $i = 1, \ldots, n$, in terms of $\bar{F}_i(x)$, $i = 1, \ldots, n$, we shall first find the cofactors of the determinant of the system,

$$\Delta \equiv \left| \delta_{ij} + \frac{\beta_{ij}}{qx} \right| = 1 + \text{ terms in } x^{-2}, x^{-3}, \ldots \qquad |x| \geq R.$$

The cofactor Δ_{ij} of the element in the i-th row and the j-th column has evidently no constant term, except when i equals j, when there is a constant term 1. The term in x^{-1} for $i \neq j$ is clearly $-\beta_{ji}/qx$. For, if we cross out the i-th row and the j-th column of Δ, there remain $n - 2$ elements with a constant term 1; and therefore the expansion of this minor can contain only one term in x^{-1}, namely, that term which contains all the $n - 2$ factors 1 and whose remaining factor is not in the same rows or columns as these. Therefore this term is $\pm \beta_{ji}/qx$; and, since an even number of interchanges of rows and columns brings the elements β_{ij}/qx and β_{ji}/qx to the place in the first row second column, and second row first column respectively, the negative sign must be chosen. In the cofactor Δ_{ii} there is no term in x^{-1}; for every term in the expansion containing

one factor x^{-1} necessarily contains another. Hence the cofactors may be written

$$\Delta_{ij} = -\frac{\beta_{ji}}{qx} + \text{terms in } x^{-2}, x^{-3}, \ldots . \qquad (i \neq j);$$

$$\Delta_{ii} = 1 + \text{terms in } x^{-2}, x^{-3}, \ldots .$$

Solving the last system of equations by the aid of these cofactors, we readily obtain

$$\Delta\bar{F}_i(qx) = \left(A_i + \frac{a_{ii}}{x} + \ldots .\right)\bar{F}_i(x) + \sum_{j=1}^{n}{}' \left\{\frac{(qA_i - A_j)\beta_{ij} + qa_{ij}}{x} + \ldots .\right\}\bar{F}_j(x)$$
$$(i = 1, \ldots, n).$$

Since the roots of the characteristic equation at infinity are of simple character, it follows that it is always possible to determine β_{ij}, $i \neq j$, so that $(qA_i - A_j)\beta_{ij} + qa_{ij} = 0$. If the values of β_{ij} so determined are substituted in the preceding system and each of the equations is then divided by Δ, the result takes the form

$$\bar{F}_i(qx) = \left(A_i + \frac{a_{ii}}{x}\right) + \sum_{j=1}^{n}\left(\frac{b_{ij}}{x^2} + \frac{b'_{ij}}{x^3} + \ldots .\right)\bar{F}_j(x) \qquad (8)$$
$$(i = 1, \ldots, n) \qquad |x| \geq R.$$

It is now easy to see that by means of the transformation

$$\bar{F}_i(x) = \bar{f}_i(x) + \sum_{j=1}^{n}\frac{\gamma_{ij}}{x^2}\bar{f}_j(x), \quad \gamma_{ii} = 0, \quad (i = 1, \ldots, n), \quad (9)$$

system (8) goes over into the form

$$\bar{f}_i(qx) = (A_i + a_{ii}x^{-1} + b_{ii}x^{-2})\bar{f}_i(x) + \sum_{j=1}^{n}(c_{ij}x^{-3} + \ldots .)\bar{f}_j(x)$$
$$(i = 1, \ldots, n) \qquad |x| \geq R.$$

By repeated use of transformations similar to (7) and (9), the exponent of x in the coefficients of the transformation being increased by unity at each step, it is clear that our system of equations may be reduced to

$$f_i(qx) = (A_i + A'_i x^{-1} + \ldots . + A_i^{(s)}x^{-s})f_i(x) + \sum_{j=1}^{n}(d_{ij}x^{-s-1} + \ldots .)f_j(x) \qquad (10)$$
$$(i = 1, \ldots, n) \qquad |x| \geq R,$$

where s is any positive integer.

By combining into one all the transformations thus far employed one readily proves that *the change from* (1) *to* (10) *may be effected at once by a transformation of the type*

$$G_i(x) = e^{\frac{\alpha}{2}\log q(\eta^2+\eta)} \sum_{j=1}^{n} \pi_{ij}\left(\frac{1}{x}\right) f_j(x) \quad (i = 1, \ldots, n), \quad (11)$$

where the $\pi_{ij}\left(\frac{1}{x}\right)$ *are polynomials in* $1/x$, the determinant of the transformation not being identically zero.

In a similar way *one may find polynomials* $P_{ij}(x)$ *in* x *such that the transformation*

$$G_i(x) = e^{\frac{\beta}{2}\log q(\eta^2+\eta)} \sum_{j=1}^{n} P_{ij}(x) g_j(x) \qquad (i = 1, \ldots, n) \quad (12)$$

is non-singular and carries (1) *over into*

$$g_i(qx) = (B_i + B_i' x + \ldots + B_i^{(\sigma)} x^{\sigma}) g_i(x) + \sum_{j=1}^{n} (e_{ij} x^{\sigma+1} + \ldots) g_j(x) \quad (13)$$
$$(i = 1, \ldots, n) \qquad |x| \leq r,$$

where σ *is any positive integer.* It is hardly necessary to remark that the forms of the various preliminary transformations here are what those of the preceding case become when $1/x$ is replaced by x.

If $A = q^{\mu}$, it is easy to determine constants c', c'', $\ldots$, $c^{(2s)}$ such that

$$f(x) = x^{\mu}(1 + c'x^{-1} + \ldots + c^{(s)}x^{-s})$$

verifies the equation

$$f(qx) = (A + A'x^{-1} + \ldots + A^{(s)}x^{-s}) f(x) + x^{\mu}(c^{(s+1)}x^{-s-1} + \ldots + c^{(2s)}x^{-2s}).$$

Evidently this may be written

$$f(qx) = \left(A + \ldots + A^{(s)}x^{-s} + \frac{c^{(s+1)}x^{-s-1} + \ldots + c^{(2s)}x^{-2s}}{1 + c'x^{-1} + \ldots + c^{(s)}x^{-s}}\right) f(x) = q^{\mu}\lambda(x) f(x),$$

where $\lambda(x)$ has the constant term 1 and is analytic when $|x|$ is greater than the greatest absolute value of a zero of $1 + c'x^{-1} + \ldots + c^{(s)}x^{-s}$. Consequently, if we write $A_i = q^{\mu_i}$, there exist functions $\lambda_i(x)$ and a constant $\bar{R}$ greater than or equal to R such that system (10) may be written in the form

$$f_i(qx) - q^{\mu_i}\lambda_i(x) f_i(x) = \sum_{j=1}^{n} \psi_{ij}(x) f_j(x) \quad (i = 1, \ldots, n), \quad (14)$$

where each of the $n^2 + n$ functions $\psi_{ij}(x)$, $\lambda_i(x)$ is analytic for $|x| \geq \bar{R}$, the

expansion for $\psi_{ij}(x)$ beginning with a term in x^{-s-1}, and where the functions $\lambda_i(x)$ have each the constant term 1 and are such that for every i the equation

$$f_i^{(1)}(qx) - q^{\mu_i}\lambda_i(x)f_i^{(1)}(x) = 0$$

has a solution in the form

$$f_i^{(1)}(x) = U_i(x) \equiv x^{\mu_i}(1 + c_i'x^{-1} + \ldots + c_i^{(s)}x^{-s}).$$

In a similar way it may be shown that if $B_i = q^{m_i}$ there exist functions $l_i(x)$ and a constant r different from zero and equal to or less than r such that system (13) may be written in the form

$$g_i(qx) - q^{m_i}l_i(x)g_i(x) = \sum_{j=1}^{n} \phi_{ij}(x)g_j(x) \quad (i = 1, \ldots, n), \quad (15)$$

where each of the $n^2 + n$ functions $\phi_{ij}(x)$, $l_i(x)$ is analytic for $|x| \leqq \bar{r}$, the expansion for $\phi_{ij}(x)$ beginning with a term in $x^{\sigma+1}$, and where the functions $l_i(x)$ have each the constant term 1 and are such that for every i the equation

$$g_i^{(1)}(qx) - q^{m_i}l_i(x)g_i^{(1)}(x) = 0$$

has a solution in the form

$$g_i^{(1)}(x) = V_i(x) \equiv x^{m_i}(1 + \bar{c}_i'x + \ldots + \bar{c}_i^{(\sigma)}x^{\sigma}).$$

Systems (14) *and* (15) *are the normal forms of* (1), *the first being valid in the neighborhood of infinity and the second in the neighborhood of zero.* To each of these normal forms the method of successive approximation is conveniently applicable. By means of this method we shall now prove the existence of a fundamental system of solutions of the equations in the normal form (14).

Consider the set of systems each of n linear equations, all except the first being non-homogeneous:

$$\left.\begin{aligned} &f_i^{(1)}(qx) - q^{\mu_i}\lambda_i(x)f_i^{(1)}(x) = 0, \\ &f_i^{(2)}(qx) - q^{\mu_i}\lambda_i(x)f_i^{(2)}(x) = \sum_{j=1}^{n} \psi_{ij}(x)f_j^{(1)}(x), \\ &f_i^{(3)}(qx) - q^{\mu_i}\lambda_i(x)f_i^{(3)}(x) = \sum_{j=1}^{n} \psi_{ij}(x)f_j^{(2)}(x), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (16)$$

We have already found a solution $U_i(x)$ of the first of these systems of equations; we shall consider here the more general solution

$$u_i(x) = C_i(x)U_i(x) \equiv C_i(x)x^{\mu_i}(1 + c_i'x^{-1} + \ldots + c_i^{(s)}x^{-s}) \quad (i = 1, \ldots, n), \quad (17)$$

where $C_i(x)$ is any function satisfying the functional equation $C_i(qx) = C_i(x)$.

All the other systems have the general non-homogeneous form

$$h_i(qx) - q^{\mu_i}\lambda_i(x)h_i(x) = \eta_i(x) \qquad (i = 1, \ldots, n), \tag{18}$$

only a single unknown function $h_i(x)$ entering into the i-th equation of the system $i = 1, \ldots, n$. Two formal solutions are readily obtained as follows: Let

$$h_i(x) = U_i(x)\,\bar{h}_i(x)$$

and substitute in the preceding equation. Dividing the result by

$$U_i(qx) = q^{\mu_i}\lambda_i(x)\,U_i(x),$$

we have

$$\bar{h}_i(qx) - \bar{h}_i(x) = \frac{\eta_i(x)}{U_i(qx)},$$

a system with the two formal solutions

$$\bar{h}_i(x) = -\sum_{\nu=0}^{\infty} \frac{\eta_i(q^\nu x)}{U_i(q^{\nu+1}x)}, \qquad \bar{h}_i(x) = \sum_{\nu=0}^{\infty} \frac{\eta_i(q^{-\nu-1}x)}{U_i(q^{-\nu}x)}.$$

Hence two formal solutions of (18) are

$$h_i(x) = -U_i(x)\sum_{\nu=0}^{\infty} \frac{\eta_i(q^\nu x)}{U_i(q^{\nu+1}x)}, \qquad h_i(x) = U_i(x)\sum_{\nu=0}^{\infty} \frac{\eta_i(q^{-\nu-1}x)}{U_i(q^{-\nu}x)}. \tag{19}$$

Either of these may be used to obtain a sequence of formal solutions of (16). It will turn out that one of these sequences will lead to a (convergent) solution and the other to a divergent formal solution of (14). The first leads to a (convergent) solution if $|q| > 1$, the second if $|q| < 1$. As the reasoning is entirely similar in the two cases, it is sufficient to carry it out for one. We shall suppose that $|q| > 1$ and shall therefore use the first formal solution (19).

Employing the notation

$$S_{xi}(\eta) \equiv -\sum_{\nu=0}^{\infty} \eta(q^\nu x)\,\frac{U_i(x)}{U_i(q^{\nu+1}x)},$$

and using for each system that particular formal solution which is obtained by adding the solution $u_i(x)$ of the homogeneous equation to the solution derived from the first equation (19), we have

$$f_i^{(1)}(x) = u_i(x),$$

$$f_i^{(2)}(x) = u_i(x) + S_{xi}\Big(\sum_{j=1}^{n} \psi_{ij}u_j\Big),$$

$$f_i^{(3)}(x) = u_i(x) + S_{xi}\Big\{\sum_{j=1}^{n} \psi_{ij}u_j + \sum_{j=1}^{n} \psi_{ij}S_{xj}\Big(\sum_{k=1}^{n} \psi_{jk}u_k\Big)\Big\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

By actual substitution one may readily verify that a solution of system (14) is obtained in the limit functions (if they exist) of the sequences $f_i^{(1)}, f_i^{(2)}, \ldots,$ $i = 1, \ldots, n$; that is, in

$$f_i(x) = u_i(x) + S_{xi}(\sum_{j=1}^{n} \psi_{ij} u_j) + S_{xi}\{\sum_{j=1}^{n} \psi_{ij} S_{xj}(\sum_{k=1}^{n} \psi_{jk} u_k)\}$$
$$+ S_{xi}[\sum_{j=1}^{n} \psi_{ij} S_{xj}\{\sum_{k=1}^{n} \psi_{jk} S_{xk}(\sum_{l=1}^{n} \psi_{kl} u_l)\}] + \ldots . \quad (i = 1, \ldots, n). \quad (20)$$

For a fixed j let us take $C_i(x) = \delta_{ij}$. We thus obtain a particular formal solution of (14), which we will denote by $f_{ij}(x)$, $i = 1, \ldots, n$. Making this substitution for every j, we obtain the n particular formal solutions

$$f_{1j}(x), f_{2j}(x), \ldots, f_{nj}(x) \qquad (j = 1, \ldots, n),$$

where

$$f_{ij}(x) = \delta_{ij} U_j(x) + S_{xi}(\psi_{ij} U_j) + S_{xi}\{\sum_{k=1}^{n} \psi_{ik} S_{xk}(\psi_{kj} U_j)\}$$
$$+ S_{xi}[\sum_{k=1}^{n} \psi_{ik} S_{xk}\{\sum_{l=1}^{n} \psi_{kl} S_{xl}(\psi_{lj} U_j)\}] + \ldots . \qquad (i, j = 1, \ldots, n). \qquad (21)$$

There exist constants R', M, c_1 and c_2 such that, for every i and j,

$$|\psi_{ij}(x)| < M|x|^{-s-1}, \quad c_1 < |U_j(x)x^{-\mu_j}| < c_2, \quad |x| \geqq R'.$$

Then the convergence of all the series in (21) and of the series (21) itself for $|x| \geqq R'$ will follow readily from similar convergence in the series

$$\bar{S}_{xi}\left(\frac{Mc_2}{x^{s+1-\mu_j}}\right) + \bar{S}_{xi}\left\{\sum_{k=1}^{n} \frac{M}{x^{s+1}} \bar{S}_{xk}\left(\frac{Mc_2}{x^{s+1-\mu_j}}\right)\right\} + \ldots , \qquad (22)$$

where

$$\bar{S}_{xi}(\eta) \equiv \sum_{\nu=0}^{\infty} |\eta(q^\nu x)| \frac{|U_i(x)|}{|U_i(q^{\nu+1}x)|}.$$

We will suppose now that the arbitrary integer s has been chosen greater than twice the absolute value of any μ. Considering the first term of (22), we have

$$\bar{S}_{xi}\left(\frac{Mc_2}{x^{s+1-\mu_j}}\right) < \sum_{\nu=0}^{\infty} \left| \frac{Mc_2}{(q^\nu x)^{s+1-\mu_j}} \frac{c_2}{c_1} \right| = |x^{\mu_j - s - 1}| \sum_{\nu=0}^{\infty} \frac{Mc_2^2}{|c_1 q^{\nu(s+1-\mu_j)}|} = q_1 |x^{\mu_j - s - 1}|,$$

where q_1 denotes the sum of the last written series—a series which converges, since $s > |\mu_j|$. From this result it follows that the series $x^{-\mu_j} S_{xi}(\psi_{ij} U_j)$ is uniformly convergent for $|x| \geqq R'$ and that its sum is less in absolute value than $q_1|x^{-s-1}|$. Moreover, each term of the series is analytic for $|x| \geqq R'$; and, since

the series converges uniformly throughout the region, the sum is analytic in the same region.

Taking the second term of (22) and using the result of the last paragraph, we have

$$\bar{S}_{xi}\left\{\sum_{k=1}^{n}\frac{M}{x^{s+1}}\bar{S}_{xk}\left(\frac{Mc_2}{x^{s+1-\mu_j}}\right)\right\} < \bar{S}_{xi}\left\{\frac{Mn}{x^{s+1}}\,q_1 x^{\mu_j-s-1}\right\} = \sum_{\nu=0}^{\infty}\frac{Mnq_1}{|(q^\nu x)^{2s+2-\mu_j}|}\,\frac{c_2}{c_1}$$

$$= q_1|x^{\mu_j-2s-2}|\sum_{\nu=0}^{\infty}\frac{Mnc_2}{c_1|q^{\nu(2s+2-\mu_j)}|} = q_1q_2|x^{\mu_j-2s-2}|,$$

where q_2 denotes the sum of the last preceding series. From this we conclude that the third term in the series in (21) is in absolute value less than $q_1q_2|x^{\mu_j-2s-2}|$, and that it is the product of x^{μ_j} by a function which is analytic for $|x| \geqq R'$.

Continuing this process we reach the result that every term in (21) is the product of x^{μ_j} by a function which is analytic for $|x| \geqq R'$, and that the series is term by term less in absolute value than the series

$$\delta_{ij}c_2|x^{\mu_j}| + q_1|x^{\mu_j-s-1}| + q_1q_2|x^{\mu_j-2s-2}| + q_1q_2q_3|x^{\mu_j-3s-3}| + \cdots, \qquad (23)$$

where

$$q_1 = \sum_{\nu=0}^{\infty}\frac{Mc_2^2}{c_1|q^{\nu(s+1-\mu_j)}|}, \quad q_\alpha = \sum_{\nu=0}^{\infty}\frac{Mnc_2}{c_1|q^{\nu(\alpha s+\alpha-\mu_j)}|}, \quad \alpha = 2, 3, \ldots.$$

Since $q_2, q_3, \ldots$ is a decreasing sequence with the limit zero, it follows that series (23) converges when $|x| > 1$. Hence there exists an R'', $R'' > 1$, $R'' \geqq R'$, such that the series obtained by multiplying (21) term by term by $x^{-\mu_j}$ is uniformly convergent and its terms are analytic for $|x| \geqq R''$. Hence all the functions $x^{-\mu_j}f_{ij}(x)$, $i, j = 1, \ldots, n$, defined by (21) are analytic for $|x| \geqq R''$.

Moreover, since the series in (21) is term by term less than the series (23), and since the limit as x approaches infinity of $U_j(x)x^{-\mu_j}$ is 1, it follows that

$$\lim_{x=\infty} x^{-\mu_j}f_{ij}(x) = \delta_{ij} \qquad (i, j = 1, \ldots, n).$$

From this we see that the determinant of which the elements are $f_{ij}(x)$ is not identically zero, and this is clearly a necessary and sufficient condition that the solutions $f_{ij}(x)$ are linearly independent in the sense that no solution is expressible linearly in terms of the others, the multipliers being functions satisfying the equation $C(qx) = C(x)$.

If now we write

$$G_{ij}^{(\infty)}(x) = e^{\frac{a}{2}\log q(\eta^2+\eta)} \sum_{k=1}^{n} \pi_{ik}\left(\frac{1}{x}\right) f_{kj}(x), \quad \eta = \log x/\log q, \quad (i, j = 1, \ldots, n),$$

it is clear from (11) that we have n solutions

$$G_{1j}^{(\infty)}(x),\ G_{2j}^{(\infty)}(x),\ \ldots,\ G_{nj}^{(\infty)}(x) \qquad (j = 1, \ldots, n)$$

of equation (1). Furthermore, since the transformation (11) is non-singular, it is evident that these solutions are linearly independent and constitute therefore a fundamental system of solutions of (1). On account of the properties of $f_{ij}(x)$ it is clear that we may write

$$G_{ij}^{(\infty)}(x) = e^{\frac{a}{2}\log q(\eta^2+\eta)} x^{\mu_j} V_{ij}^{(\infty)}(x) \qquad (i, j = 1, \ldots, n),$$

where $V_{ij}^{(\infty)}(x)$ is analytic in the neighborhood of infinity.

If an expression of the above form for $G_{ij}^{(\infty)}(x)$ is substituted in (1), $V_{ij}^{(\infty)}(x)$ being expanded as a power-series in $1/x$, and if the constants are determined by direct reckoning, it will turn out that each of the n formal solutions obtained in this way is unique up to a constant multiplier. Consequently, if these constant multipliers are properly chosen, these formal solutions are the actual solutions whose existence has just been determined.

This solution of (1) has been obtained by starting from the normal form (14) and employing the method of successive approximation. Starting from (15) and working in a similar way, it may be shown that we have also a fundamental system of solutions

$$G_{1j}^{(0)}(x),\ G_{2j}^{(0)}(x),\ \ldots,\ G_{nj}^{(0)}(x) \qquad (j = 1, \ldots, n)$$

which may be written in the form

$$G_{ij}^{(0)}(x) = e^{\frac{\beta}{2}\log q(\eta^2+\eta)} x^{m_j} V_{ij}^{(0)}(x) \qquad (i, j = 1, \ldots, n),$$

where $V_{ij}^{(0)}(x)$ is analytic in the neighborhood of infinity, and that this solution may be obtained by substituting in the equation as in the foregoing case and determining the constants in a direct way. The principal modification in the argument consists in using the second instead of the first formal solution (19) of the non-homogeneous equation (18). The discussion follows so closely along the lines of the preceding that it is unnecessary to repeat it.

The argument above has been carried out on the assumption that $|q| > 1$. If $|q| < 1$, the only essential modification of the argument consists in interchanging the rôles of the two formal solutions (19) of (18). In this case the

second should be used for solutions about infinity and the first for solutions about zero. The results as to the nature of the solutions at these points still remain valid.

The nature of each of the functions of the two fundamental systems of solutions obtained above is so far determined only in the neighborhood of infinity and of zero respectively. In order to determine their nature outside of these regions, we work as follows: If we consider the first two members of (1) and solve for $G_i(x)$ in terms of $G_i(qx)$, $i = 1, \ldots, n$, we may write the result in the form

$$G_i(x) = q^{-\alpha} x^{-\alpha} \sum_{j=1}^{n} \bar{A}_{ij}(qx)\, G_j(qx) \quad (i = 1, \ldots, n). \tag{24}$$

Then from (1) itself and the preceding system, we have the relations

$$G_{ij}(qx) = q^{\alpha} x^{\alpha} \sum_{k=1}^{n} A_{ik}(x)\, G_{kj}(x) \qquad (i, j = 1, \ldots, n), \tag{25}$$

$$G_{ij}(x) = q^{-\alpha} x^{-\alpha} \sum_{k=1}^{n} \bar{A}_{ik}(qx)\, G_{kj}(qx) \quad (i, j = 1, \ldots, n), \tag{26}$$

where $G_{ij}(x)$ is to be identified with either $G_{ij}^{(\infty)}(x)$ or $G_{ij}^{(0)}(x)$.

Let $r_1, r_2, r_3, \ldots$ be an infinite sequence of increasing numbers such that $r_{i+1} = |q| r_i$ or $r_{i+1} = |q|^{-1} r_i$, $i = 1, 2, \ldots$, according as $|q| > 1$ or $|q| < 1$. Consider the infinite system of circles of radii $r_1, r_2, r_3, \ldots$ and center at the origin, and assume that r_1 has been so chosen that the first two of these circles lie within the region in which $V_{ij}^{(0)}(x)$ is analytic. These circles divide the part of the plane external to the first one into an infinite system of circular rings, which we shall call *fundamental regions*. If we know the value of $G_{ij}^{(0)}(x)$ at every point in one of these rings, we can compute its value at any point in the next larger by means of (25) if $|q| > 1$, and by means of (26) if $|q| < 1$. From this fact it is easy to determine the position of the singularities of $G_{ij}^{(0)}(x)$. If $\mu = q^t \nu$, where t is any positive or negative integer (not zero), we shall say that μ is *externally* or *internally congruent* to ν according as $|\mu| > |\nu|$ or $|\mu| < |\nu|$. Excluding the points zero and infinity, we see that the remaining singularities of each function $G_{ij}^{(0)}(x)$ are all included in the set of points externally congruent to the singularities of $A_{ik}(x)$ or of $\bar{A}_{ik}(x)$, $i, k = 1, \ldots, n$, according as $|q| > 1$ or $|q| < 1$.

In a similar way it is readily shown that the singularities of each function $G_{ij}^{(\infty)}(x)$, other than zero and infinity, are all included in the set of points inter-

nally congruent to the singularities of $\overline{A}_{ik}(x)$ or of $A_{ik}(x)$, $i, k = 1, \ldots, n$, according as $|q| > 1$ or $|q| < 1$.

Furthermore, if $A_{ik}(x)$, $i, k = 1, \ldots, n$, are rational functions, it is clear that the singularities (other than zero and infinity) of the functions in either solution consist entirely of poles.

The principal results which we have obtained may be stated in the following theorem:

THEOREM I. *The system of q-difference equations* (1) *or* (24) *has two fundamental systems of solutions of the forms*

$$G_{ij}^{(\infty)}(x) = e^{\frac{\alpha}{2}\log q\,(\eta^2+\eta)}\, x^{\mu}\, V_{ij}^{(\infty)}(x), \quad \eta = \frac{\log x}{\log q}, \qquad (i, j = 1, \ldots, n),$$

$$G_{ij}^{(0)}(x) = e^{\frac{\beta}{2}\log q\,(\eta^2+\eta)}\, x^{m_j}\, V_{ij}^{(0)}(x) \qquad (i, j = 1, \ldots, n),$$

where $V_{ij}^{(\infty)}(x)$ *and* $V_{ij}^{(0)}(x)$ *are analytic in the neighborhood of infinity and of zero respectively. These solutions may be determined by substitution (a power-series being assumed for each of the functions* V*) and direct reckoning out of constants. Besides the points zero and infinity, all the singularities of* $G_{ij}^{(\infty)}(x)$ *are included in the set of points internally congruent to the singularities of* $\overline{A}_{ik}(x)$ *or of* $A_{ik}(x)$, $i, k = 1, \ldots, n$, *according as* $|q| > 1$ *or* $|q| < 1$; *and all the singularities of* $G_{ij}^{(0)}(x)$ *are included in the set of points externally congruent to the singularities of* $A_{ik}(x)$ *or of* $\overline{A}_{ik}(x)$, $i, k = 1, \ldots, n$, *according as* $|q| > 1$ *or* $|q| < 1$. *The general solution* $G_i(x)$, $i = 1, \ldots, n$, *may be written in either of the forms*

$$G_i(x) = \sum_{j=1}^{n} C_j^{(\infty)}(x)\, G_{ij}^{(\infty)}(x) = \sum_{j=1}^{n} C_j^{(0)}(x)\, G_{ij}^{(0)}(x) \quad (i = 1, \ldots, n),$$

where the functions $C_j^{(\infty)}(x)$ *and* $C_j^{(0)}(x)$, $j = 1, \ldots, n$, *satisfy the equation* $C(qx) = C(x)$ *but are otherwise arbitrary.*

Further, if $A_{ik}(x)$, $i, k = 1, \ldots, n$, *are rational functions of* x, *the singularities (other than zero and infinity) of the functions in each of the two fundamental solutions consist entirely of poles.*

Let us now consider a single q-difference equation of the n-th order in the two forms

$$\begin{aligned} &H(q^n x) + (q^n x)^{\alpha} A_1(x)\, H(q^{n-1}x) + (q^n x)^{\alpha}(q^{n-1}x)^{\alpha} A_2(x)\, H(q^{n-2}x) \\ &\quad + \ldots + (q^n x)^{\alpha} \ldots (q^2 x)^{\alpha}(qx)^{\alpha} A_n(x)\, H(x) = 0, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &H(q^n x) + (q^n x)^{\beta} B_1(x)\, H(q^{n-1}x) + (q^n x)^{\beta}(q^{n-1}x)^{\beta} B_2(x)\, H(q^{n-2}x) \\ &\quad + \ldots + (q^n x)^{\beta} \ldots (q^2 x)^{\beta}(qx)^{\beta} B_n(x)\, H(x) = 0, \end{aligned} \tag{28}$$

where the unknown quantities entering into the equation are defined as follows:

1. α, β, q are constants and $|q| \neq 1$.
2. The functions $A_i(x)$ and $B_i(x)$ are single-valued and

$$A_i(x) = A_i + A_i' x^{-1} + A_i'' x^{-2} + \cdots \qquad (i = 1, \ldots, n) \quad |x| \geqq R,$$
$$B_i(x) = B_i + B_i' x + B_i'' x^2 + \cdots \qquad (i = 1, \ldots, n) \quad |x| \leqq r.$$

3. The constants A_i and the constants B_i are such that the characteristic equations at infinity and at zero,

$$\rho^n + A_1 \rho^{n-1} + A_2 \rho^{n-2} + \cdots + A_{n-1}\rho + A_n = 0, \tag{29}$$
$$\rho^n + B_1 \rho^{n-1} + B_2 \rho^{n-2} + \cdots + B_{n-1}\rho + B_n = 0, \tag{30}$$

have their respective roots $a_1, \ldots, a_n$ and $b_1, \ldots b_n$ of simple character.

The transformations

$$H(x) = e^{\frac{\alpha}{2}\log q\,(\eta^2+\eta)}\,\overline{H}(x), \qquad \eta = \log x/\log q,$$
$$H(x) = e^{\frac{\beta}{2}\log q\,(\eta^2+\eta)}\,\overline{H}'(x),$$

carry equations (27) and (28) respectively over into

$$\overline{H}(q^n x) + A_1(x)\,\overline{H}(q^{n-1}x) + \cdots + A_n(x)\,\overline{H}(x) = 0, \tag{31}$$
$$\overline{H}'(q^n x) + B_1(x)\,\overline{H}'(q^{n-1}x) + \cdots + B_n(x)\,\overline{H}'(x) = 0. \tag{32}$$

It is evident that these two equations are equivalent respectively to the two systems

$$\overline{H}_1(qx) = \sum_{j=1}^{n}\{-A_j(x)\}\,\overline{H}_j(x), \quad \overline{H}_i(qx) = \overline{H}_{i-1}(x) \quad (i = 2, \ldots, n), \tag{33}$$

$$\overline{H}_1'(qx) = \sum_{j=1}^{n}\{-B_j(x)\}\,\overline{H}_j'(x), \quad \overline{H}_i'(qx) = \overline{H}_{i-1}'(x) \quad (i = 2, \ldots, n), \tag{34}$$

where $\overline{H}_n(x) = \overline{H}(x)$ and $\overline{H}_n'(x) = \overline{H}'(x)$. The characteristic equation of (33) at infinity takes the form (29), and the characteristic equation of (34) at zero takes the form (30).

Equations (33) and (34) are of the form (3) and (4) respectively and consequently the preceding discussion is applicable to the former systems. Corresponding to the solved form (24) of (1) are analogous solved forms of (31) and (33) respectively; namely,

$$\overline{H}(x) + \overline{A}_1(x)\,\overline{H}(qx) + \overline{A}_2(x)\,\overline{H}(q^2x) + \cdots + \overline{A}_n(x)\,\overline{H}(q^n x) = 0, \tag{35}$$

$$\overline{H}_i(x) = \overline{H}_{i+1}(qx), \ (i = 1, \ldots, n-1); \ \overline{H}_n(x) = \sum_{j=1}^{n}\{-\overline{A}_{n-j+1}(x)\}\,\overline{H}_j(qx), \tag{36}$$

where the functions $\overline{A}_i(x)$, $i = 1, \ldots, n$, are readily expressed in terms of the functions $A_i(x)$, $i = 1, \ldots, n$.

If now we apply our previous results to the systems of equations and interpret them with reference to the single equation of the n-th order, we obtain the following theorem:

THEOREM II. *The n-th order q-difference equation* (27), (28) *or* (35) *has two fundamental systems of solutions of the forms*

$$H_i^{(\infty)}(x) = e^{\frac{\alpha}{2}\log q\,(\eta^2+\eta)}\, x^{\mu_i}\, V_i^{(\infty)}(x), \quad \eta = \frac{\log x}{\log q}, \qquad (i = 1, \ldots, n),$$

$$H_i^{(0)}(x) = e^{\frac{\beta}{2}\log q\,(\eta^2+\eta)}\, x^{m_i}\, V_i^{(0)}(x) \qquad (i = 1, \ldots, n),$$

where $V_i^{(\infty)}(x)$ *and* $V_i^{(0)}(x)$ *are analytic in the neighborhood of infinity and of zero respectively. These solutions may be determined by substitution (a power-series being assumed for each of the functions* V*) and direct reckoning out of constants. Besides the points zero and infinity, all the singularities of* $H^{(\infty)}(x)$ *are included in the set of points internally congruent to the singularities of* $\overline{A}_i(x)$ *or of* $A_i(x)$, $i = 1, \ldots, n$, *according as* $|q| > 1$ *or* $|q| < 1$; *and all the singularities of* $H_i^{(0)}(x)$ *are included in the set of points externally congruent to the singularities of* $A_i(x)$ *or of* $\overline{A}_i(x)$, $i = 1, \ldots, n$, *according as* $|q| > 1$ *or* $|q| < 1$. *The general solution* $H(x)$ *may be written in either of the forms*

$$H(x) = \sum_{j=1}^{n} C_j^{(\infty)}(x)\, H_j^{(\infty)}(x) = \sum_{j=1}^{n} C_j^{(0)}(x)\, H_j^{(0)}(x),$$

where the functions $C_j^{(\infty)}(x)$ *and* $C_j^{(0)}(x)$, $j = 1, \ldots, n$, *satisfy the equation* $C(qx) = C(x)$ *but are otherwise arbitrary.*

Further, if $A_i(x)$, $i = 1, \ldots, n$, *are rational functions of* x, *the singularities (other than zero and infinity) of the functions in each of the two fundamental solutions consist entirely of poles.*

REMARK. In view of the preceding work it is obvious that the results of this section may be obtained by direct reckoning out of the solutions, by the power-series method suggested above, together with appropriate convergence proofs. Such a method, however, is not available in the case of difference equations. Consequently it has seemed best to employ the method of successive approximation (already used by me for difference equations), so as to develop the two parts of this general theory along the same lines and by the same methods. The matrix method used by Birkhoff for difference equations is also applicable to q-difference equations; it furnishes therefore a second single method by which both cases may be treated.

§ 2. *Relations between the two Fundamental Solutions.*

The question of the relations existing between the two fundamental solutions of system (1) furnishes an interesting problem. It is clear that either of these solutions may be expressed in terms of the other. Thus we have

$$G_{ij}^{(0)}(x) = \sum_{k=1}^{n} C_{ik}(x)\, G_{kj}^{(\infty)}(x) \quad (i, j = 1, \ldots, n), \tag{37}$$

$$G_{ij}^{(\infty)}(x) = \sum_{k=1}^{n} \overline{C}_{ik}(x)\, G_{kj}^{(0)}(x) \quad (i, j = 1, \ldots, n), \tag{38}$$

where $C_{ik}(x)$ and $\overline{C}_{ik}(x)$ are functions satisfying the equation $C(qx) = C(x)$. Our problem reduces, then, to that of the determination of the nature of these functions $C_{ik}(x)$ and $\overline{C}_{ik}(x)$. In the general case they are of a complicated character. But in the most interesting and important case, namely, that in which the coefficients of the original equation are polynomials (or even when they are merely rational functions), the result takes a simple and elegant form.

We suppose that system (1), subject to the previously assigned conditions, is also such that it may be written in the form

$$G_i(qx) = \sum_{k=1}^{n} P_{ik}(x)\, G_k(x) \qquad (i = 1, \ldots, n), \tag{39}$$

where the coefficients $P_{ik}(x)$ are polynomials in x of degree α or less. The constant β which enters into system (1) is in this case zero.

According as $|q| > 1$ or $|q| < 1$, the functions $G_{ij}^{(0)}(x)$ or $G_{ij}^{(\infty)}(x)$ are analytic except at zero and infinity. We will consider first the case in which $|q| > 1$. If (37) is solved for $C_{ik}(x)$, $i, k = 1, \ldots, n$, in terms of $G_{ij}^{(0)}(x)$ and $G_{ij}(x)$, it is readily seen that every function $C_{ik}(x)$ is analytic except perhaps at zero, innnity, the singularities of $G_{ij}^{(\infty)}(x)$, and the zeros of the function defined by the determinant $|G_{ij}^{(\infty)}(x)|$. But on account of the properties of the functions $G_{ij}^{(\infty)}(x)$ and of the form of the determinant (see Theorem I), it is clear that there exists a fundamental region (sufficiently remote from the origin) in which the functions $G_{ij}^{(\infty)}(x)$, $i, j = 1, \ldots, n$, have no singularity and the function $|G_{ij}^{(\infty)}(x)|$ has no zero. Hence the functions $C_{ik}(x)$, $i, k = 1, \ldots, n$, are analytic in this fundamental region; and therefore they are analytic throughout the plane, except at zero and infinity.

In case $|q| < 1$, we may argue in a similar way to show that the functions $\overline{C}_{ik}(x)$, $i, k = 1, \ldots, n$, are analytic throughout the plane, except at zero and infinity.

It is clear that if the coefficients $P_{ik}(x)$ in (39) are restricted to be rational (but not necessarily polynomial), a discussion analogous to that in the foregoing case can be readily carried out. If $|q|>1$, the functions $C_{ik}(x)$, away from zero and infinity, will be analytic except for poles; and if $|q|<1$, the functions $\overline{C}_{ik}(x)$ will have this property.

These results may be stated in the following theorem:

THEOREM III. *Let a system of q-difference equations* (1) *be written in the form*

$$G_i(qx)=\sum_{k=1}^{n} P_{ik}(x)G_k(x) \qquad (i=1, \ldots, n);$$

and let the two simple fundamental solutions be connected by the relations

$$G_{ij}^{(r)}(x)=\sum_{k=1}^{n} C_{ik}(x)G_{kj}^{(s)}(x), \quad C_{ik}(qx)=C_{ik}(x) \quad (i, j, k=1, \ldots, n),$$

where $r, s = 0, \infty$ *or* $\infty, 0$ *according as* $|q|>1$ *or* $|q|<1$. *Then:*

If the coefficients $P_{ik}(x)$ *are polynomials in* x, *the functions* $C_{ik}(x)$ *are analytic except at zero and infinity;*

If the coefficients $P_{ik}(x)$ *are rational in* x, *the functions* $C_{ik}(x)$, *away from zero and infinity, are analytic except for poles.*

An inverse to some of the preceding results may be stated in the form of the following theorem:

THEOREM IV. *Let* $G_{ij}^{(0)}(x)$ *and* $G_{ij}^{(\infty)}(x)$, $i, j=1, \ldots, n$, *be two sets of* n^2 *single-valued functions which, away from zero and infinity, are analytic except for poles, and which have the property*

$$\lim_{x=0} G_{ij}^{(0)}(x)e^{\frac{-\beta}{2}\log q(\eta^2+\eta)}x^{-m_j}=V_{ij}^{(0)}, \quad |V_{ij}^{(0)}|\neq 0; \quad m_i-m_j\neq \text{integer}, i\neq j,$$

$$\lim_{x=\infty} G_{ij}^{(\infty)}(x)e^{\frac{-\alpha}{2}\log q(\eta^2+\eta)}x^{-\mu_j}=V_{ij}^{(\infty)}, \quad |V_{ij}^{(\infty)}|\neq 0; \quad \mu_i-\mu_j\neq \text{integer}, i\neq j,$$

where α *and* β *are integers (positive, negative or zero),* $\eta=\log x/\log q$, *the two sets of functions being connected by the relations*

$$G_{ij}^{(0)}(x)=\sum_{k=1}^{n} C_{ik}(x)G_{kj}^{(\infty)}(x), \quad C_{ik}(qx)=C_{ik}(x), \quad (i, j, k=1, \ldots, n).$$

Then the two sets of functions $G_{ij}^{(0)}(x)$ *and* $G_{ij}^{(\infty)}(x)$ *are the two simple fundamental solutions of a system of q-difference equations*

$$G_i(qx)=\sum_{k=1}^{n} P_{ik}(x)G_k(x) \qquad (i=1, \ldots, n),$$

in which the coefficients $P_{ik}(x)$ *are rational in* x.

The proof of the theorem requires the determination of the existence and the properties of functions $P_{ik}(x)$ such that

$$G_{ij}^{(0)}(qx) = \sum_{k=1}^{n} P_{ik}(x) G_{kj}^{(0)}(x) \qquad (i, j = 1, \ldots, n), \tag{40}$$

$$G_{ij}^{(\infty)}(qx) = \sum_{k=1}^{n} P_{ik}(x) G_{kj}^{(\infty)}(x) \qquad (i, j = 1, \ldots, n). \tag{41}$$

Each of the two systems of equations may be solved for $P_{ik}(x)$. The two representations of $P_{ik}(x)$ must be consistent, in view of the relations connecting $G_{ij}^{(0)}(x)$ and $G_{ij}^{(\infty)}(x)$. From either of them it is clear that the functions $P_{ik}(x)$, away from zero and infinity, are single-valued and analytic except for poles.

If (40) is solved for $P_{ik}(x)$ in terms of $G_{ij}^{(0)}(x)$ and $G_{ij}^{(0)}(qx)$, $i, j, k = 1, \ldots, n$, and the limit for $x = 0$ is taken, then, in view of the first limit in the hypothesis of the theorem, we have

$$\lim_{x=0} P_{ik}(x) x^{-\beta} = \frac{|V_{rs}^{(ik)}|}{|V_{rs}^{(0)}|},$$

where $|V_{rs}^{(0)}|$ is the determinant with $V_{rs}^{(0)}$ in the r-th row and s-th column and $|V_{rs}^{(ik)}|$ is what $|V_{rs}^{(0)}|$ becomes when the k-th row is replaced by $q^{m_j} V_{ij}^{(0)}$, $j = 1, \ldots, n$. From this it follows that $P_{ik}(x)$ has at $x = 0$ a zero of order β (or a pole of order $-\beta$ if β is negative). Furthermore, the q-difference equation has at zero the characteristic equation

$$|P_{ik} - \delta_{ik}\rho| = 0, \quad P_{ik} = \frac{|V_{rs}^{(ik)}|}{|V_{rs}^{(0)}|}.$$

A direct reckoning will show that the roots of this equation are $\rho = q^{m_j}$, $j = 1, \ldots, n$.

In like manner we may employ (41) and the second limit in the hypothesis of the theorem to show that $P_{ik}(x)$ has at infinity a pole of order α (or a zero of order $-\alpha$ if α is negative) and that the roots of the characteristic equation at infinity are $\rho = q^{n_j}$, $j = 1, \ldots, n$.

From the fact that $P_{ik}(x)$ is analytic except for poles we conclude that it is rational in x. From the characteristic equations at zero and infinity respectively, we conclude that the q-difference equation of the theorem has solutions of the requisite property at each of these points. This completes the demonstration of the theorem as stated.

§ 3. *The Case when* $|q| = 1$.

When $|q| = 1$, the theory of the q-difference equation is essentially different from that when $|q| \neq 1$, previously considered. In treating the former we distinguish two cases, according as there is or is not a positive integer n such that $q^n = 1$.

CASE I. We will suppose first that there exists no integer n such that $q^n = 1$. That there is in this case in general no analytic solution (not identically zero) is easily seen from the consideration of a first-order equation with constant coefficients:

$$f(qx) = af(x), \qquad a > 1.$$

We have evidently

$$f(q^a x) = af(q^{a-1}x) = a^2 f(q^{a-2}x) = \ldots = a^a f(x).$$

For any x it is clear that there exists an α greater than any preassigned M and such that $q^\alpha x$ is less than any preassigned distance e from x. When $f(x)$ is not equal to zero at the given point x the quotient of the functional values at $q^\alpha x$ and x, that is $f(q^\alpha x)/f(x)$, can therefore be made as large as one pleases, although $q^\alpha x$ and x are separated by a less distance than e. Hence the equation $f(qx) = af(x)$ has no continuous solution except $f(x) \equiv 0$.

CASE II. We will suppose now that q is a root of unity and that n is the least integer such that $q^n = 1$. In the present case it is more convenient to consider from the outset a single equation than a system of equations, for evidently any single equation of any order can be written in the form

$$\psi_1(x) f(qx) + \psi_2(x) f(q^2 x) + \ldots + \psi_n(x) f(q^n x) = 0. \tag{42}$$

That solutions (not identically zero) do not exist when the ψ's are unrestricted is readily shown; for we have the following system of equations:

$$\left.\begin{aligned}
\psi_1(x) f(qx) + \psi_2(x) f(q^2 x) + \ldots + \psi_n(x) f(q^n x) &= 0, \\
\psi_n(qx) f(qx) + \psi_1(qx) f(q^2 x) + \ldots + \psi_{n-1}(qx) f(q^n x) &= 0, \\
\psi_{n-1}(q^2 x) f(qx) + \psi_n(q^2 x) f(q^2 x) + \ldots + \psi_{n-2}(q^2 x) f(q^n x) &= 0, \\
\ldots\ldots\ldots

This system is consistent only if its determinant is zero; that is, if

$$\begin{vmatrix} \psi_1(x) & \psi_2(x) & \cdots & \psi_n(x) \\ \psi_n(qx) & \psi_1(qx) & \cdots & \psi_{n-1}(qx) \\ \psi_{n-1}(q^2x) & \psi_n(q^2x) & \cdots & \psi_{n-2}(q^2x) \\ \cdots & \cdots & \cdots & \cdots \\ \psi_2(q^{n-1}x) & \psi^3(q^{n-1}x) & \cdots & \psi_1(q^{n-1}x) \end{vmatrix} = 0. \tag{44}$$

We therefore suppose that the coefficients $\psi_i(x)$ are connected by the relation (44).

So far the only restriction on q is that it is a primitive n-th root of unity. The numbers $q, q^2, q^3, \ldots, q^{n-1}, q^n$ are all the n-th roots of unity, and hence this set of powers contains the primitive n-th root of unity of smallest argument. Without loss of generality we may suppose that this root is q itself; and this we shall do. Then the points $q, q^2, \ldots, q^n$ are situated on the unit circle at equal intervals, in such wise that in making a positive circuit along the circumference the points are encountered in the order written. If rays are drawn from the origin through these n points, the plane is divided into n equal sectors; and these we shall call a set of fundamental regions of the plane. If these rays are all rotated about the origin through the same angle, then in their new position they will define another set of fundamental regions.

We have seen that a necessary condition for the existence of a solution (not identically zero) is that the determinant in (44) shall be of rank $n-1$ at most. We suppose in general that this determinant is of rank $n-\alpha$, where α is a number of the sequence $1, 2, \ldots, n-1$. Then, among the quantities

$$f(qx),\ f(q^2x),\ \ldots,\ f(q^nx),$$

there is a set α in number in terms of which every other can be expressed linearly. Hence there are α fundamental regions in which $f(x)$ may be assigned at will; it is then determined throughout the entire plane. On account of this fact we shall say that the order of the equation is α.

When the equation is of order 1, in accordance with this definition, there is a single relation among the coefficients; otherwise there is more than one. The general case is therefore that in which the order is 1. Since in this case a solution is determined when its value is assigned in a single fundamental region, it is clear that the most general solution is of the form $C(x)f_1(x)$, where $f_1(x)$ is any particular solution (not identically zero) and $C(x)$ is an arbitrary function satisfying the equation $C(qx) = C(x)$.

In the case of a first-order equation it is easy to obtain a particular solution in explicit finite form. The ordinary theory of linear algebraic equations applied to the system (43) yields us the values of the ratios

$$f(qx) : f(q^2x) : \ldots\ldots : f(q^nx);$$

and hence we have an equation of the form

$$f(qx) = \psi(x) f(x), \tag{45}$$

where $f(x)$ is the same as the $f(x)$ in (42). To find a solution of this equation, we proceed thus:

The product

$$f(q^nx) f(q^{n-1}x) \ldots\ldots f(qx)$$

is unchanged by the substitution of qx for x; and hence its value is a function satisfying the relation $C(qx) = C(x)$. Consequently there exists a particular solution $f_1(x)$ for which

$$f_1(q^nx) f_1(q^{n-1}x) \ldots\ldots f_1(qx) = 1.$$

But

$$f_1(q^{i+1}x) = \psi(q^ix) f_1(q^ix).$$

Making repeated substitutions from this formula into the preceding equation, we have

$$\psi(q^{n-1}x)\,[\psi(q^{n-2}x)]^2 \ldots\ldots [\psi(qx)]^{n-1}\,[\psi(x)]^n\,[f_1(x)]^n = 1.$$

Hence

$$f_1(x) = \{\psi(q^{n-1}x)\,[\psi(q^{n-2}x)]^2 \ldots\ldots [\psi(qx)]^{n-1}\,[\psi(x)]^n\}^{-1/n}$$

is a particular solution of (45) and hence of (43); the general solution is $C(x) f_1(x)$, where $C(x)$ satisfies the equation $C(qx) = C(x)$ but is otherwise arbitrary.

The above discussion suggests a general method for solving (42) when the rank of its determinant in (44) is $n - \alpha$. In a form most convenient for this case the method may be described as follows: To any $n - \alpha$ independent equations of the set (43) adjoin other equations subject to the following conditions: 1) The n equations of the resulting set are independent and consistent. 2) The left members of the α new equations are invariant under the substitution of qx for x. They are linear in $f(q^nx), \ldots\ldots, f(x)$. 3) The second members of these α equations are arbitrary functions of x satisfying the relation $C(qx) = C(x)$.

One of the possible forms for these equations is

$$\lambda(q^{n-1}x)f(q^{n-1}x) + \ldots\ldots + \lambda(qx)f(qx) + \lambda(x)f(x) = C(x),$$

where $\lambda(x)$ is any conveniently chosen function of x.

If the system is solved algebraically for $f(q^n x), \ldots\ldots, f(x)$, the resulting expression for $f(x)$ will contain linearly α arbitrary functions which satisfy the equation $C(qx) = C(x)$. And hence this gives the general solution of (42), since it is of order α.

In carrying out this process it is important to secure that the arbitrary functions shall enter into the solution independently; that is, in such way that they can not be combined in the solution into fewer functions of the same type.

We shall illustrate the above method by a treatment of the special case in which $q = \omega =$ cube root of unity of least argument and $\alpha = 2$. It is easy to show that the general equation for this case may be written in the form

$$f(\omega^2 x) + \psi(x)\psi(\omega x)f(\omega x) + \psi(x)f(x) = 0, \quad \psi(x)\psi(\omega x)\psi(\omega^2 x) = 1.$$

In accordance with our general method, if $\psi(x) \not\equiv 1$,* we may adjoin the equations

$$f(\omega^2 x) + f(\omega x) + f(x) = C_1(x),$$
$$\omega^2 x f(\omega^2 x) + \omega x f(\omega x) + f(x) = C_2(x).$$

The value of $f(x)$ is obtained algebraically from the last three equations.

INDIANA UNIVERSITY, *July*, 1911.

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* If $\psi(x) \equiv 1$, a fundamental system of solutions is 1 and x.

Concerning Linear Projective Order.

By Arthur Richard Schweitzer.

Introduction.

Hölder, in the *Mathematische Annalen*, Vol. LXV (1908), p. 161, considers the geometry of the projective line. In particular, in §§ 1–5 he has given several treatments of the linear projective order theory. In our investigations in the logic of geometry we have been impressed with the desirability * of connecting the foundations of linear projective order as intimately as possible with the descriptive † line; and it would seem that the previous treatments ‡ of the subject, including Hölder's, have not exhibited the close relation between the descriptive and projective linear orders as coherently as a suitable analysis of the theories in question really permits.§

In the present paper we aim to give an axiomatic analysis of linear projective order which has a very simple relation with a linear descriptive system.‖ Aside from this advantage, a significant property of our analysis is the relation which it bears to our planar descriptive system ${}^2R_2$ ¶ and from which springs a new application of a fundamental logical principle. The latter may be stated ** as follows :

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* See concluding paragraphs of § 2 of the present paper and our article, "A Theory of Geometrical Relations," American Journal of Mathematics, Vol. XXXI (1909), pp. 365–410. In the sequel reference is made by pages to this paper by the notation "T. G. R."

† On the meaning of the term "descriptive" see B. Russell, "The Principles of Mathematics," Cambridge (1903), § 374; in T. G. R. we have used this term in a generalized sense.

‡ For the literature reference may be made to Russell, *loc. cit.*, §§ 204–205, and to Hölder's article.

§ Hölder, *loc. cit.*, p. 179, note 1, and p. 181, notes 2, 3.

‖ The system which is here relevant is our system ${}^1R_1$, T. G. R., p. 378.

¶ T. G. R., p. 382 and § 3 of this article.

** This statement does not differ essentially from that given by E. H. Moore, *Bulletin of the American Mathematical Society*, Vol. XVI (1909) pp. 111–112. The principle is essentially heuristic and accordingly may be compared with the empirical position of J. S. Mill, "A System of Logic," 4th edition, London (1856). In our specific application, by *concept* we understand *theory*.

The existence of resemblances between given concepts implies the existence of a general concept which underlies the particular concepts and unifies them with respect to those resemblances.

Like the system ${}^2R_2$, our axioms are phrased in terms of an undefined relation $\alpha R\beta\gamma$ which may be interpreted concretely by the statement: "If a person swims from the point β to the point γ, then the point α is at his right."

§1. *A New Analysis of Linear Projective Order: The System* ${}^2R_2^{(0)}$.

I. AXIOMS.

1. There exists at least one point.*

2. The existence of α implies the existence of α_0, β_0, γ_0 such that $\alpha_0 R\beta_0\gamma_0$ or $\alpha_0 R\gamma_0\beta_0$.

3. $\alpha R\beta\gamma$ implies $\alpha\bar{R}\gamma\beta$.†

4. $\alpha R\beta\gamma$ implies $\beta R\gamma\alpha$.

5. $\alpha R\beta\gamma$, $\xi \neq \alpha, \beta, \gamma$ imply $\xi R\beta\gamma$ or $\alpha R\xi\gamma$.

6. $\alpha R\beta\gamma$, $\xi R\beta\gamma$, $\xi \neq \alpha$ imply $\alpha R\xi\gamma$ or $\xi R\alpha\gamma$.

7. $\alpha R\beta\gamma$, $\alpha \neq \beta$, $\beta \neq \gamma$, $\alpha \neq \gamma$ imply the existence of ξ such that $\beta R\xi\gamma$ and $\alpha R\xi\gamma$.

II. DEFINITIONS.

1. $\gamma\delta$ is *collinear* with $\alpha\beta$ $(\alpha, \beta \neq \gamma, \delta)$ means, $\alpha R\beta\gamma$ or $\beta R\alpha\gamma$; $\alpha R\beta\gamma$ implies $\delta R\beta\gamma$ or $\alpha R\delta\gamma$; and $\beta R\alpha\gamma$ implies $\delta R\alpha\gamma$ or $\beta R\delta\gamma$ (compare Fig. 1; we have $\alpha R\beta\gamma$ and $\delta = \delta_1'$ or δ_2' or δ'').

2. From definition 1 is obtained the definition of "$\gamma\delta$ *separates* $\alpha\beta$" by the proper substitution of "and" for "or"; compare Fig. 2.

3. ξ is in the *interior* of $\alpha\beta$ means, $\alpha \neq \beta$ and the existence of γ such that $\alpha R\beta\gamma$ implies $\xi R\beta\gamma$ and $\alpha R\xi\gamma$; compare Fig. 3.‡

Instead of phrasing the preceding system of axioms (absolutely) in terms of an alternating relation R, one may construct an equivalent system

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* Greek letters are used to denote points.

† The rule over the R is a symbol of negation.

‡ Compare the definitions, T. G. R., pp. 382–383.

(relatively) in terms of a transitive and symmetrical* relation K between triads† as follows:

System ${}^2K_2^{(0)}$.

1. There exists at least one point.

2. The existence of α implies the existence of α_0, β_0, γ_0 such that $\alpha_0\beta_0\gamma_0 K$.

3. $\alpha\beta\gamma K$ implies $\alpha\beta\gamma\overline{K}\alpha\gamma\beta$.

3′. $\alpha\beta\gamma K$ implies $\alpha\gamma\beta K$.

4. $\alpha\beta\gamma K$ implies $\alpha\beta\gamma K\beta\gamma\alpha$.

5. $\alpha\beta\gamma K$, $\xi \neq \alpha, \beta, \gamma$ imply $\xi\beta\gamma K$ or $\alpha\xi\gamma K$.

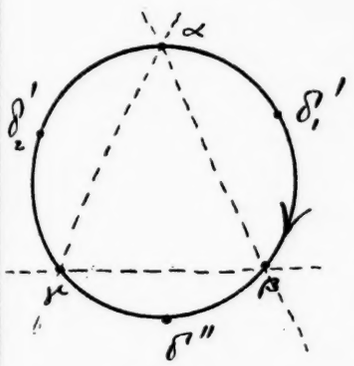

FIG. 1.

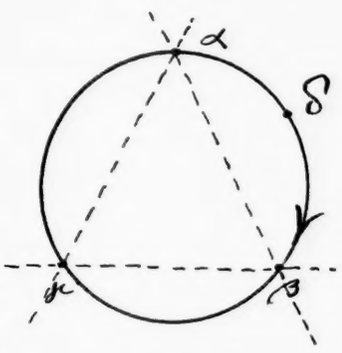

FIG. 2.

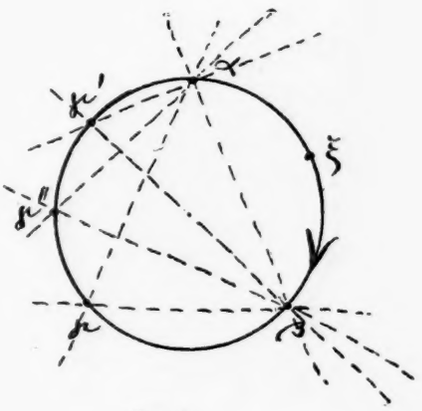

FIG. 3.

6. $\alpha\beta\gamma K$, $\xi\beta\gamma K$, $\xi \neq \alpha$ imply $\alpha\xi\gamma K$.

6′. $\xi\beta\gamma K\alpha\xi\gamma$ implies $\xi\beta\gamma K\alpha\beta\gamma$.

7. $\alpha\beta\gamma K$, $\alpha \neq \beta$, $\beta \neq \gamma$, $\alpha \neq \gamma$ imply the existence of ξ such that $\alpha\beta\gamma K\beta\xi\gamma$ and $\alpha\beta\gamma K\alpha\xi\gamma$.

8. $\alpha\beta\gamma K\alpha'\beta'\gamma'$ implies $\alpha\beta\gamma K$.

9. $\alpha\beta\gamma K\alpha'\beta'\gamma'$ implies $\alpha'\beta'\gamma' K$.

10. $\alpha\beta\gamma K\xi\eta\zeta$, $\xi\eta\zeta K\alpha'\beta'\gamma'$ imply $\alpha\beta\gamma K\alpha'\beta'\gamma'$.

11. $\alpha\beta\gamma K$, $\alpha'\beta'\gamma' K$ imply $\alpha\beta\gamma K\alpha'\beta'\gamma'$ or $\alpha\beta\gamma K\beta'\alpha'\gamma'$.

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* For definitions see Russell, *loc. cit.*, § 208.

† Compare the abstract, *Bulletin of the American Mathematical Society*, Vol. XIII (1906) p. 58. On the notation of the system ${}^2K_2^{(0)}$ see T. G. R., pp. 375, 394. In connection with this system one may define "Gleichartigkeit" of triads as we have done in the case of tetrads, *Bull. Amer. Math. Soc.*, Vol. XV (1908), p. 80; compare Grassmann, *Gesammelte Werke*, Vol. I, Part I, p. 163; Hölder, *loc. cit.*, § 4, uses the same term in a different sense.

On the relation between the preceding systems ${}^2R_2^{(0)}$ and ${}^2K_2^{(0)}$ we may refer to the systems ${}^2R_2$ and ${}^2K_2$.*

III. THEOREMS.

On the basis of the system ${}^2R_2^{(0)}$ one can easily prove:

1. $\alpha R\xi\gamma$ and $\xi R\beta\gamma$ imply $\alpha R\beta\gamma$.
2. $\alpha R\beta\gamma$ and $\xi R\beta\gamma$, $\xi \neq \alpha$ imply $\alpha R\xi\beta$ or $\xi R\alpha\beta$.
3. $\alpha R\beta\gamma$ and $\alpha R\xi\gamma$, $\xi \neq \beta$ imply $\xi R\beta\gamma$ or $\beta R\xi\gamma$.

Then with the aid of the preceding theorems it is easily shown that the axioms of Hölder, *loc. cit*, § 5, follow from our axioms.

In contrast with the system ${}^2R_2$ it may also be shown that if $\alpha R\beta\gamma$ and $\xi \neq \alpha, \beta, \gamma$, then ξ is in the interior of precisely three compartments† associated with $\alpha\beta\gamma$; that is, one of the following three cases must hold:

I.	II.	III.
$\alpha R\beta\gamma$	$\alpha R\beta\gamma$	$\alpha R\beta\gamma$
$\alpha R\xi\gamma$	$\xi R\alpha\gamma$	$\alpha R\xi\gamma$
$\beta R\xi\gamma$	$\xi R\beta\gamma$	$\xi R\beta\gamma$
$\alpha R\beta\xi$	$\alpha R\beta\xi$	$\beta R\alpha\xi$

Thus the seven compartments into which a triangle separates a descriptive plane are here reduced, as it were, to three. Moreover, if $\alpha R\beta\gamma$, then $\xi R\beta\gamma$, $\alpha R\xi\gamma$ and $\alpha R\beta\xi$ together are impossible; likewise there is no point in the interior of both $\alpha\beta$ and $\beta\alpha$‡. There exists, however, a point ξ in the interior of $\alpha\beta$; and if $\alpha R\xi\gamma$ and $\xi R\beta\gamma$, then ξ is in the interior of $\alpha\beta$.

§ 2. *Relation to a Linear Descriptive System.*

The system ${}^2R_2^{(0)}$ is very simply related to the linear descriptive system ${}^1R_1$.§ Namely, from the system ${}^1R_1$ can be derived an associated system by a process

*Cf. T. G. R., pp. 382, 393–394 and §2 of the present paper. The latter is related exclusively to the system ${}^2R_2^{(0)}$ unless otherwise specified.

†The term *compartment* seems to be in closer harmony with the planar descriptive connotations of our paper than the term "Strecke" (segment) which Hölder uses.

‡See T. G. R., p. 382, definitions 3, 4.

§Cf. T. G. R., p. 378.

which may perhaps be called *tactical adjunction;* one writes throughout in ${}^1R_1$, $\theta_1 R \theta_2 \gamma$ for $\theta_1 R \theta_2$. The following axioms are then obtained;*

1. There exists α.
2. The existence of α implies the existence of α_0, β_0 $[\gamma]$ such that $\alpha_0 R \beta_0 \gamma$ or $\beta_0 R \alpha_0 \gamma$.
3. $\alpha R \beta \gamma$ implies $\beta \bar{R} \alpha \gamma$.
4. $\alpha R \beta \gamma$ and $\xi \neq \alpha, \beta$ imply $\alpha R \xi \gamma$ or $\xi R \beta \gamma$.
5. $\alpha R \beta \gamma$ and $\alpha R \xi \gamma$, $\xi \neq \beta$ imply $\xi R \beta \gamma$ or $\beta R \xi \gamma$.
6. $\alpha R \beta \gamma$ implies the existence of ξ_1 such that $\beta R \xi_1 \gamma$ and $\alpha R \xi_1 \gamma$.
7. $\alpha R \beta \gamma$ implies the existence of ξ_2 such that $\xi_2 R \alpha \gamma$ and $\xi_2 R \beta \gamma$.
8. $\alpha R \beta \gamma$ implies the existence of ξ such that $\alpha R \xi \gamma$ and $\xi R \beta \gamma$.

If now one assumes:

3′. $\alpha R \beta \gamma$ implies $\beta R \gamma \alpha$

then axioms 7, 8 follow from 1–6, and the latter axioms become essentially the system ${}^2R_2^{(0)}$. This genesis of the system ${}^2R_2^{(0)}$ suggests at once that with reference to a fixed point † of the system ${}^2R_2^{(0)}$ the set of axioms ${}^1R_1$ is definable. For we may define:‡

$\delta R_{\alpha\beta\gamma\varepsilon}$ means: 1°. There exist α, β, γ such that $\alpha R \beta \gamma$. 2°. $\delta R \varepsilon \alpha$.

Thus $\delta R_{\alpha\beta\gamma\varepsilon}$ is true if $\delta = \beta$ and $\varepsilon = \gamma$, but $\delta \bar{R}_{\alpha\beta\gamma\varepsilon}$ if either $\delta = \alpha$ or $\varepsilon = \alpha$.

The system ${}^2R_2^{(0)}$ we term the *projective analogue* of the system ${}^1R_1$. It is true that the projective analogues ${}^3R_3^{(0)}$, ${}^4R_4^{(0)}$, etc., of our systems § ${}^2R_2$, ${}^3R_3$, etc., respectively possess a genesis similar ‖ to that of ${}^2R_2^{(0)}$; but these analogues we shall discuss elsewhere¶.

It may be observed that the relation of the system ${}^1K_1$ ** to its corresponding projective analogue ${}^2K_2^{(0)}$ seems less simple than the relation of ${}^1R_1$ to ${}^2R_2^{(0)}$.

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* In deriving these axioms we have augmented, for the purposes of the present paper, the conclusion of axioms 6 and 7 of ${}^1R_1$ by the statements "$\alpha R \xi_1$" and "$\xi_2 R \beta$" respectively. It may be noted that this change in the axioms does not invalidate the independence system given for axiom 4 of ${}^1R_1$ (T. G. R., p. 379).

† The system ${}^2R_2^{(0)}$ does not, of course, justify the selection of a particular point; strictly, the linear descriptive system defined is a function of point variables.

‡ The definition of a linear descriptive system on the basis of linear projective order is, in itself, well known; cf. Russell, *loc. cit.*, § 204.

§ T. G. R., pp. 382, 387, etc.

‖ Thus the projective analogue ${}^3R_3^{(0)}$ of ${}^2R_2$ arises by adjoining to the latter a point δ and assuming that $\alpha R \beta \gamma \delta$ implies $\gamma R \delta \alpha \beta$.

¶ Compare T. G. R., p. 366.

** T. G. R., p. 394.

§ 3. *Relation to a Planar Descriptive System.*

Tactically, the linear projective axioms ${}^2R_2^{(0)}$ and the planar descriptive system ${}^2R_2$* are of the same dimensions: they are both stated in terms of (alternating) triads. A further comparison of the two theories discloses a remarkable resemblance.† For this purpose it is convenient to phrase the system ${}^2R_2$ as follows:‡

System ${}^2R_2$.

1. There exists at least one point.

2. The existence of α implies the existence of α_0, β_0, γ_0 such that $\alpha_0 R \beta_0 \gamma_0$ or $\alpha_0 R \gamma_0 \beta_0$.

3. $\alpha R \beta \gamma$ implies $\alpha \bar{R} \gamma \beta$.

4. $\alpha R \beta \gamma$ implies $\beta R \gamma \alpha$.

5. $\alpha R \beta \gamma$, $\xi \neq \alpha, \beta, \gamma$ imply $\xi R \beta \gamma$ or $\alpha R \xi \gamma$ or $\alpha R \beta \xi$.

6. $\alpha R \beta \gamma$, $\xi R \beta \gamma$, $\xi \neq \alpha$ imply $\alpha R \xi \gamma$ or $\xi R \alpha \gamma$ or $\alpha R \beta \xi$ or $\beta R \alpha \xi$.

7. $\alpha R \beta \gamma$, $\alpha \neq \beta$, $\beta \neq \gamma$, $\alpha \neq \gamma$ imply the existence of ξ such that $\beta R \xi \gamma$ and $\alpha R \gamma \xi$.

8. $\alpha R \beta \gamma$, $\varepsilon R \beta \gamma$, $\alpha R \varepsilon \gamma$, $\alpha R \beta \varepsilon$, $\varepsilon \neq \alpha, \beta, \gamma$ imply: the existence of ξ such that $\xi \bar{R} \alpha \varepsilon$, $\xi \bar{R} \varepsilon \alpha$, and the existence of δ' such that $\delta' R \beta \gamma$ implies $\delta' R \xi \gamma$ and $\delta' R \beta \xi$, and the existence of δ'' such that $\delta'' R \gamma \beta$ implies $\delta'' R \xi \beta$ and $\delta'' R \gamma \xi$.

Thus our linear axioms 1–4 are identical with axioms 1–4 of ${}^2R_2$; axioms 5 and 6 of ${}^2R_2^{(0)}$ affirm the truth of axioms 5 and 6 of ${}^2R_2$. The resemblance between linear axiom 7 and the corresponding axiom of ${}^2R_2$ is evident. There is no linear projective axiom corresponding to axiom 8 of ${}^2R_2$; yet the latter axiom is consistent with our linear projective axioms, since the hypothesis of axiom 8 of ${}^2R_2$ is not fulfilled; compare theorems, § 1 of the present paper.

The preceding considerations show that the axioms of § 1 and the system ${}^2R_2$ can be logically combined into a single system merely by replacing, in axiom 7 of ${}^2R_2$, the term $\alpha R \gamma \xi$ by ($\alpha R \xi \gamma$ or $\alpha R \gamma \xi$). We have thus arrived

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* T. G. R., p. 382.

† The essential basis for the resemblance in question is assumed to be identity (compare J. S. Mill, *loc. cit.*, Book I, Chapter IV, § 6, paragraphs 2, 3; also Bradley, "Appearance and Reality," London (1902), p. 592, III). This resemblance suggests that the geometry of the projective line may be considered as a quasi-planar descriptive geometry; in fact it seems not inappropriate to call the former a *singular* geometry of the descriptive plane.

‡ The present statement of our system ${}^2R_2$ does not differ essentially from that given in T. G. R., p. 382.

at a result which we believe to be a new application of the heuristic principle mentioned in our introduction.* With reference to this principle our result may be stated as follows:

The same general† theory underlies the theories of linear projective order and planar descriptive order.‡

Further details of this general theory seem desirable.

§ 4. *Consistence and Independence of the Axioms of the General Theory.*

There exists at least one set of points which satisfies the axioms of the general theory of § 3, but which contravenes the systems ${}^2R_2^{(0)}$ and ${}^2R_2$. As this set we may take the points in the interior and on the boundary of the unit circle in a cartesian plane and, say, continuous values of the coordinates; the relation $\alpha R\beta\gamma$ is valid if, and only if,

$$\begin{vmatrix} \alpha_1 & \alpha_2 & 1 \\ \beta_1 & \beta_2 & 1 \\ \gamma_1 & \gamma_2 & 1 \end{vmatrix} > 0 \qquad \begin{cases} \alpha_1^2 + \alpha_2^2 \gtreqless 1. \\ \beta_1^2 + \beta_2^2 \gtreqless 1. \\ \gamma_1^2 + \gamma_2^2 \gtreqless 1. \end{cases}$$

This region has the property that on its boundary the system ${}^2R_2^{(0)}$ is valid, while the system ${}^2R_2$ is satisfied in its interior.

We consider now the independence of the axioms of the general theory. As usual, we denote by C_i a class of points such that in this class axiom i ($i = 1, 2, \ldots, 8$) is contradicted and the remaining axioms are satisfied or are non-effective.

C_8. Construct a complete set of 35 ternary alternating products in seven elements, 1, 2, 3, 4, 5, 6, 7, such that every triad is reducible to the form $[\xi\eta\zeta]$, where $\xi < \eta < \zeta$. If in this set the triads

$$[125]\ [236]\ [347]\ [451]\ [562]\ [673]\ [714]$$

are replaced by their conjugates,

$$[215]\ [326]\ [437]\ [541]\ [652]\ [763]\ [174],$$

then the resulting set is a C_8. Another set of the latter kind is obtained by considering a complete set of 84 ternary alternating products in nine elements,

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*Cf. E. H. Moore, "Atti del IV Congresso Internazionale dei Matematici," Rome (1909), p. 98; O. Bolza, *Bulletin of the American Mathematical Society*, Vol. XVI (1910), p. 402.

†The generalization which is relevant to our application of the above principle is a *denotative* generalization, or generalization in extension, as distinguished from the *connotative* or intensive generalization.

‡With this result compare Russell, *loc. cit.*, § 382.

1, 2, 3, ..., 9, such that every triad is reducible to the form $[\xi\eta\zeta]$, where $\xi < \eta < \zeta$, and replacing the triads

$$[126]\ [237]\ [348]\ [459]\ [561]\ [672]\ [783]\ [894]\ [915]$$

by their conjugates. It would seem of interest to examine the availability of the infinite sequence of analogous sets that here suggests itself.

C_7. Take any one of the complete sets of ${}^nC_3$ ternary alternating products in n elements, 1, 2, 3, 4, ..., n ($n = 3, 4, 5, \ldots$), such that each product is reducible to the form $[\xi\eta\zeta]$, where $\xi < \eta < \zeta$.

C_8–C_1. For the remaining independence sets reference may be made to AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), p. 386.

*On Certain Expansions of Elliptic, Hyperelliptic and Related Periodic Functions.**

By F. R. Moulton.

§ 1. Introduction.

The functions treated in this paper are defined by differential equations of the types which arise in many physical problems. For this reason special interest attaches to the real solutions. The problem of finding the properties of the solutions and their expansion is, however, one of analytic functions, and therefore it is not convenient to limit the discussion to real values of the variables. Consequently the analytic character of the solutions is determined and, because of the possible practical applications, especial attention is given to developing convenient methods for constructing them.

When the differential equations treated here are suitably specialized, they define certain elliptic or hyperelliptic functions. One of the most perfect theories in analysis is unquestionably that of elliptic functions. Nothing could be more systematic and elegant than the development of the properties of these functions when they are defined by the conditions that they shall be uniform and without essential singularities for all finite values of the independent variable, and that they shall have an algebraic addition theorem or that they shall be doubly periodic. These strong conditions make possible the detailed development of the theory, but they prevent its ready extension to more general cases. In this paper the point of view is altogether different. The functions are defined by differential equations, and the loss suffered in not explicitly prescribing their properties is partly balanced by the greater elasticity of the methods.

In physical problems the questions of periodicity of motion and the asymptotic approach of the dependent variables to limiting values are of primary interest. In the case of elliptic functions there can not be more than two periods,

* Read before the American Mathematical Society under a different title, April 28, 1911.

only one of which can be real; but in the case of hyperelliptic and more general abelian functions the number of real periods is not limited. In § 2 the question of the existence of the real and purely imaginary periods is treated. In this connection the qualitative character of the real curves defined by the differential equation is considered. In § 3 a number of special cases are treated, the simplest of which gives rise to the ordinary elliptic functions. In § 4 a method is given for expressing the periods by converging series in certain parameters of the differential equations. In § 5 the explicit construction of the periodic solutions from the differential equation is treated, and some of the important properties of the solutions are derived. In § 6 a method is given for constructing the solution from the first integral of the differential equations. In § 7 a method is given for constructing the solution, its properties being known from the discussion of § 5, from the values of the dependent variable and its derivatives for special values of the independent variable. The whole problem of constructing the solutions is there reduced simply to solving sets of ordinary linear algebraic equations. In § 8 application is made to the Legendre elliptic functions, and new expansions are given for them which, in § 9, are proved to converge for all finite real values of the independent variable if the absolute value of the modulus x^2 is less than unity. The series are very convenient for practical use and are developed to the twelfth power of x inclusive.

§ 2. THE DIFFERENTIAL EQUATIONS AND PERIODS OF THE SOLUTIONS.

Consider the differential equation

$$\frac{d^2x}{dt^2} = x'' = \sum_{-\infty}^{+\infty} A_j x^j, \tag{1}$$

where the A_j are constants. Suppose the series converges if $r < |x| < R$. This problem includes that of the motion of a particle moving subject to a central force varying as any positive or negative power of the distance. The determination of the motion of a particle in the equatorial plane of an oblate or prolate spheroid, or in fact in the equatorial plane of any body whose surface can be expressed by zonal harmonics, depends upon an equation of the type of (1). Equation (1) also includes all those problems whose solutions are obtained from the inversion of elliptic and hyperelliptic integrals. Reference to any treatise on elliptic functions will show the abundance of these problems.

Equation (1) admits the integral

$$x' = \sqrt{f(x)} = \sqrt{\sum_{j=-\infty}^{+\infty} B_j x^j + B \log x}, \tag{2}$$

where B_0 is the constant of integration and where

$$B = 2A_{-1}, \qquad B_j = \frac{2A_{j-1}}{j}, \quad j = -\infty, \ldots, +\infty, \text{ except } 0.$$

The series $f(x)$ also converges if $r < |x| < R$.

It follows from Cauchy's fundamental theorem on analytic differential equations that if, for any finite t_1, the dependent variable x takes such a value x_1 that the right member of (1) is regular, the solution of (1) is a regular function of t in the vicinity of t_1. Therefore the solution of (1) is regular for at least all finite values of t for which x stays in the ring $r < |x| < R$. It does not, of course, follow that for given initial conditions finite values of t exist such that x takes all values in the ring.

Suppose the roots of $f(x) = 0$ in the ring $r < r_1 \leqq |x| \leqq R_1 < R$ are $\alpha_1, \ldots, \alpha_p$, where p is finite except in the trivial case $f(x) \equiv 0$. Suppose the α_j are all distinct. Now join the circumferences r and R by a cut C which does not pass through an α_j and which has no double point. From (2) we have

$$t - t_1 = \int_{x_1}^{x} \frac{dx}{\sqrt{f(x)}}. \tag{3}$$

It is well known from the general theory* that the values of t obtained by integrating from $x = x_1$ back to $x = x_1$, over any path not passing through an α_j or crossing r_1, R_1 or C, are linear functions with integral coefficients of $k-1$ primitive periods $\omega_2, \ldots, \omega_p$ and I_1, where $\omega_2, \ldots, \omega_p$ are the values of the integrals along closed paths around α_1 and α_2, α_1 and α_3, $\ldots$, α_1 and α_p respectively, and where I_1 is the integral from 0 to α_1. These periods may not all be distinct and they are in general complex.

We shall confine our attention now to the real and purely imaginary periods. Suppose the coefficients of (1) and (2) are all real. Suppose $\alpha_1, \ldots, \alpha_m$ are the real positive roots of $f(x) = 0$, and that $\beta_1, \ldots, \beta_n$ are the real negative roots of $f(x) = 0$ satisfying the inequalities $r_1 \leqq |x| \leqq R_1$. It follows from the hypotheses above that $f(x)$ is finite, continuous, and single-valued in the real intervals $-R_1 < x < -r_1$, $r_1 < x < R_1$, and it changes sign only at the α_j and β_j.

* Puiseux, *Journal de Mathématiques*, 1850, and Picard, *Traité d'Analyse*, Vol. II, Chap. VIII.

Suppose $f(x_1) > 0$, where $\alpha_i < x_1 < \alpha_{i+1}$, and that $f(x)$ does not vanish for $\alpha_i < x < \alpha_{i+1}$. Therefore x_1' is real and we may suppose it is positive. The solution x will be a real function of t and x will increase with increasing real values of t until it reaches α_{i+1}. It follows from (3) that this value will be attained for a finite value of t if α_{i+1} is a simple root of $f(x) = 0$. If $x = \alpha_{i+1}$ is a multiple root, x will approach α_{i+1} asymptotically as t becomes infinite. Suppose α_{i+1} is a simple root and that x has attained this value at t_{i+1}. It can not pass beyond this value, for then $f(x)$ would become negative and x' imaginary, whereas x is a regular function of t having real coefficients in the vicinity of this point. For the same reason it can not become complex as t takes increasing real values. It can not become a constant at this point, for x'' is

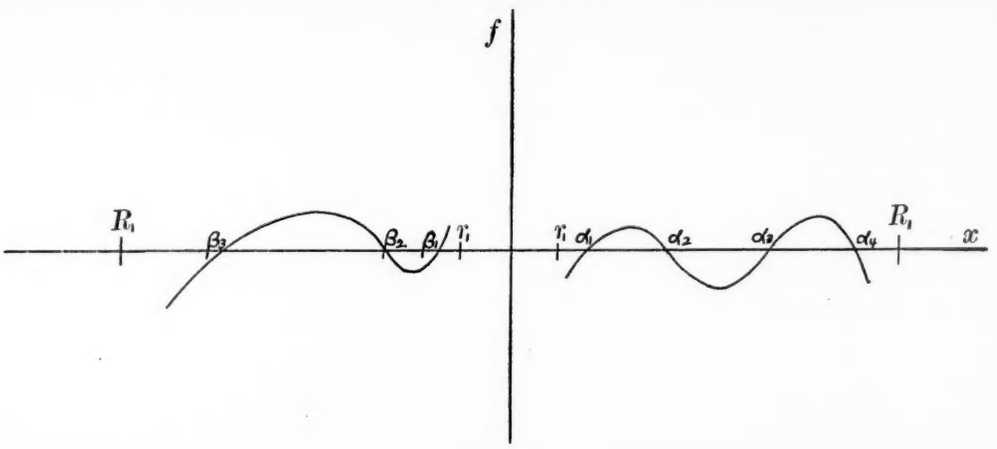

FIG. 1.

distinct from zero, since α_{i+1} is a simple root of $f(x) = 0$. Therefore as t increases beyond t_{i+1}, x' becomes negative and x decreases continuously to α_i, which it reaches in a finite time if α_i is a simple root of $f(x) = 0$. Then the sign of x' changes and x increases to α_{i+1} again. That is, under the conditions set down x oscillates between α_i and α_{i+1} in the period

$$T(\alpha_i, \alpha_{i+1}) = 2\int_{\alpha_i}^{\alpha_{i+1}} \frac{dx}{\sqrt{f(x)}}. \tag{4}$$

Similar results are true for all other closed intervals in the ring whose end-points are simple real roots of $f(x) = 0$ for which $f(x)$ is positive. The periods are not necessarily distinct.

In an interval in which $f(x)$ is negative we make the transformation

$$t = \sqrt{-1}\,\tau,$$

after which the problem in τ has the same properties it previously had in t.

That is, there are purely imaginary periods corresponding to the closed intervals in which $f(x)$ is negative, and for them x is real.*

The solution involves two arbitrary constants, one of which is B_0; the other is additive to t, since (1) is unchanged by the transformation $t = t_0 + \tau$. The constant t_0 has no influence upon the analytic character of the solutions, but it is quite otherwise with B_0. Since this constant is additive in $f(x)$, changing it simply produces a vertical shift in the curves of Fig. 1. Suppose B_0 is increased. As a minimum approaches the x-axis, the two real periods associated with the adjacent intervals for which $f(x)$ is positive approach infinity. On the other hand, the associated imaginary period approaches a finite determinate limit, as can be shown by forming the expression for the period and passing to the limit. If α_{i+1} and α_{i+2} are the two roots which approach equality, we have

$$f(x) = -(x - \alpha_{i+1})(\alpha_{i+2} - x)\psi(x),$$

where $\psi(x) > 0$ for $\alpha_{i+1} \leqq x \leqq \alpha_{i+2}$. Let the value at which α_{i+1} and α_{i+2} unite be α and $\psi(\alpha) = A > 0$. Let $x = \alpha + \xi$. When $\alpha_{i+2} - \alpha_{i+1}$ is sufficiently small, $\psi(\alpha + \xi)$ can be expanded in a power-series in ξ which converges for all x in $\alpha_{i+1} \leqq x \leqq \alpha_{i+2}$. Therefore the imaginary period is

$$\sqrt{-1}\, T = \frac{2}{\sqrt{A}} \int_{\alpha_{i+1}-\alpha}^{\alpha_{i+2}-\alpha} \frac{[1 + \phi'(\alpha)\xi + \ldots.]\, d\xi}{\sqrt{(\alpha - \alpha_{i+1})(\alpha_{i+2} - \alpha) + (\alpha_{i+1} + \alpha_{i+2} - 2\alpha)\xi - \xi^2}}, \qquad (5)$$

where $\phi = \dfrac{\sqrt{A}}{\sqrt{\psi}}$. The limit of this integral as $\alpha_{i+1} \doteq \alpha$, $\alpha_{i+2} \doteq \alpha$ is $\dfrac{2\pi}{\sqrt{A}}$. As B_0 increases beyond the value for which $\alpha_{i+1} = \alpha_{i+2}$, the purely imaginary period and one of the real periods giving real values of x disappear, and there remains in place of them a single period, which is real, giving real values of x. If the value of B_0 decreases until a maximum of $f(x)$ crosses the x-axis, the rôles of the real and imaginary periods are interchanged.

Suppose the positive roots of $f(x) = 0$ in the interval $r_1 < |x| < R_1$ are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Suppose $f(x)$ is positive in the intervals $\alpha_1\alpha_2$, $\alpha_3\alpha_4$ and negative in $\alpha_2\alpha_3$. In the positive intervals x is real for real t, and in the negative intervals it is real for purely imaginary values of $t = \sqrt{-1}\,\tau$. In Fig. 2 these curves are given. Those which belong to real values of t are given in full lines and

* While elliptic functions can have only two periods whose ratio can not be real, the functions defined by these more general equations can have any number of periods. This point has been insisted on by Casorati in two memoirs in *Acta Mathematica*, Vol. VIII, 1886.

that which belongs to imaginary values of t is given in chain dots. In its case the t-axis should be read the τ-axis.

Since the solutions carry an arbitrary constant added to t, they may be displaced in the horizontal direction without changing their character. The figure has been drawn so that x starts at α_1 at $t=0$. It arrives at α_2 at $t=\frac{1}{2}T_{12}$ and returns to α_1 at $t=T_{12}$. But if t takes purely imaginary increments when it arrives at $\frac{1}{2}T_{12}$, x will describe the curve belonging to the interval $\alpha_2\alpha_3$. At $t=\frac{1}{2}T_{12}+\sqrt{-1}\,\frac{1}{2}T_{23}$, x will have the value α_3, and will return to α_2 at

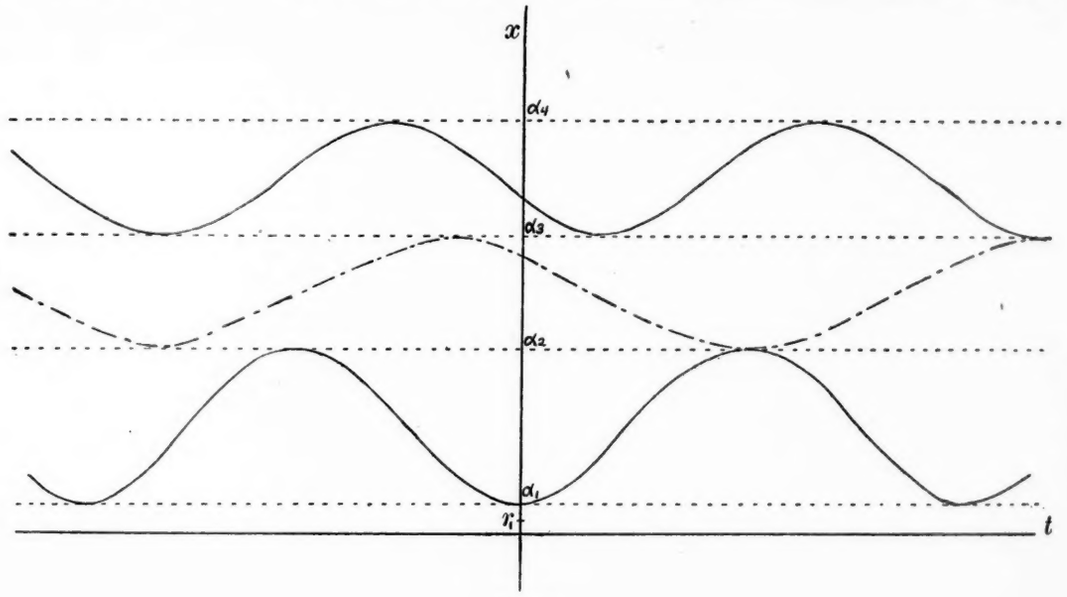

FIG. 2.

$t=\frac{1}{2}T_{12}+\sqrt{-1}\,T_{23}$. But if t takes real increments after $t=\frac{1}{2}T_{12}+\sqrt{-1}\,\frac{1}{2}T_{23}$, x will describe the curve belonging to the interval $\alpha_3\alpha_4$, and will arrive at α_4 at $t=\frac{1}{2}T_{12}+\sqrt{-1}\,\frac{1}{2}T_{23}+\frac{1}{2}T_{34}$.

The discussion of the branches for $|x|>R$ can not be made unless more properties of $f(x)$ are given.

§ 3. SOME SPECIAL CASES.

CASE 1. *$B=0$ and only a finite number of the B_{-j} distinct from zero.*

If only a finite number of the B_{-j} are distinct from zero, $r=0$. If B is distinct from zero, the constant B_0 is not defined for $x=0$. If x has initially some value distinct from zero, and if for $t=t_1$ it becomes zero, t_1 is an essential

singular point. If $B_{-j}=0$ for $j=1$ and $j>m$, and if $B_{-m}\neq 0$, the nature of the function in the vicinity of $t=t_1$, where $x=0$ for $t=t_1$, can easily be determined from (2). We have at once

$$\left.\begin{aligned} t-t_1 &= \int_0^x \frac{x^{\frac{m}{2}}\,dx}{\sqrt{B_{-m}+B_{-m+1}x+\cdots}} \\ &= \frac{2}{2+m}\,\frac{x^{\frac{m+2}{2}}}{\sqrt{B_{-m}}}\,[1+C_1x+\cdots]. \end{aligned}\right\} \tag{6}$$

Inverting this series, we have

$$x=\left[\frac{(2+m)^2}{4}B_{-m}(t-t_1)^2\right]^{\frac{1}{m+2}}\left\{1+D_1(t-t_1)^{\frac{1}{m+2}}+\cdots\right\}. \tag{7}$$

If the constants $A_{\pm j}$ are real, the coefficients D_i in (7) are real. The variable x considered as a function of $t-t_1$ has $m+2$ determinations which vanish at $t-t_1=0$. If m is even, they are all complex, or two are real and m of them are complex for both positive and negative real values of $t-t_1$ according as B_{-m} is negative or positive. If B_{-m} is negative, two branches are real for both positive and negative purely imaginary values of $t-t_1$. If m is odd, there are $m+1$ complex values of x and one real value for both positive and negative real values of t. In this case there is one real value of x for both positive and negative purely imaginary values of t.

Suppose m is even, B_{-m} is positive, and $f(x)$ has simple zeros at $x=\alpha_1, \alpha_2, \beta_1, \beta_2$, where $\alpha_1<\alpha_2<R$, $-\beta_1<-\beta_2<R$. Suppose $x=0$ at $t=t_1$. There are two real branches in the neighborhood of $t=t_1$ for both positive and negative real values of $t-t_1$. Let us take $t-t_1$ positive and first consider the branch for which x' becomes positive. It follows from (2) that on this branch x' stays positive and x increases until it reaches the value α_1. For real increasing values of $t-t_1$, x' changes sign and x decreases to zero, which it reaches at a finite value of $t-t_1=T(0,\alpha_1)$. At this value of $t-t_1$, x' changes sign by passing through infinity and x remains positive after passing through zero. Then the cycle to α_1 and return is repeated. There is a similar cycle to β_1 and return with the period $T(0,\beta_1)$ obtained by starting with the branch for which x' became negative for real $t-t_1$ near zero.

If $t-t_1$ takes purely imaginary increments after arriving at any odd multiple of $\frac{1}{2}T(0,\alpha_1)$ on the branch which lies between 0 and α_1, x will describe a real curve between α_1 and α_2; and starting from the other branch

there is similarly a real curve between β_1 and β_2 for purely imaginary increments to $t-t_1$ after it arrives at any multiple of $T(0, \beta_1)$. Fig. 3 shows the curves for real $t-t_1$ in full lines, and for imaginary $t-t_1$ in chain dots. The curves are drawn satisfying the initial condition $x=0$ at $t=0$, and by continuous variation of t from this value all the branches shown can be obtained.

Now suppose that m is again even, that $f(x)$ has simple zeros at $x=\alpha_1, \alpha_2, \beta_1, \beta_2$, but that B_{-m} is negative. The only qualitative difference between this case and that just treated is that the full curves belonging to real $t-t_1$,

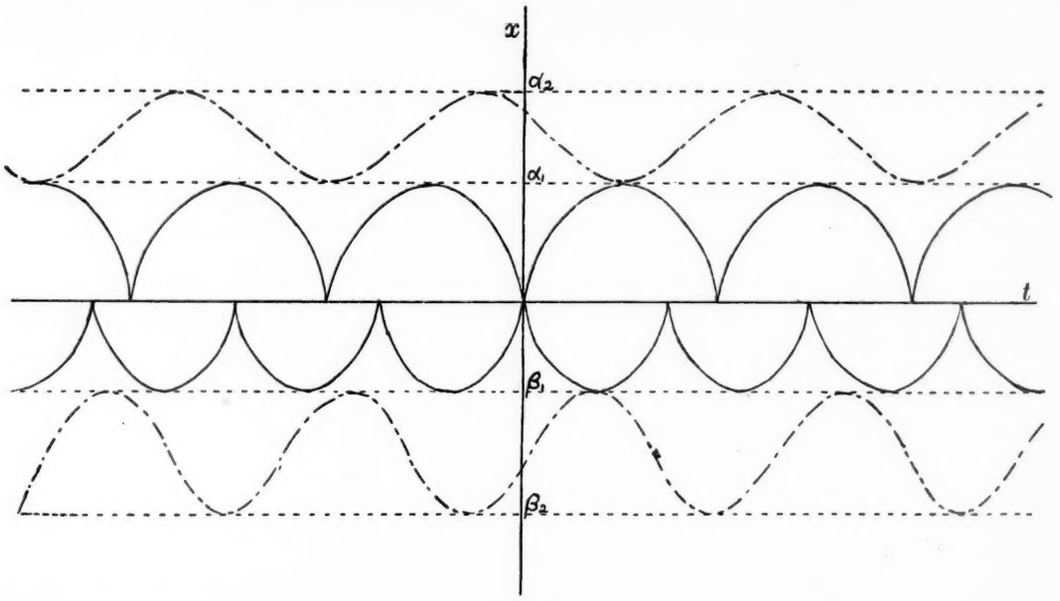

FIG. 3.

and the chain-dot curves belonging to a purely imaginary $t-t_1$, are interchanged in position.

Suppose m is odd, B_{-m} is again positive, and $f(x)$ has simple zeros at $x=\alpha_1, \alpha_2, \beta_1, \beta_2$, where $\alpha_1<\alpha_2<R$, $-\beta_1<-\beta_2<+R$. In this case there is only one real value of x both when $t-t_1$ is positive and when it is negative, and in both cases x is positive. If $t-t_1$ is a positive or negative pure imaginary, there is one real branch on which x is negative. For real $t-t_1$, whether increasing or decreasing from $t-t_1=0$, x increases to α_1 and then returns to zero after a finite time $T(0, \alpha_1)$. For purely imaginary $t-t_1$, x decreases to β_1 and returns to zero after the finite time $\sqrt{-1}\,T(0, \beta_1)$. If $t-t_1$ increases by purely imaginary increments from $\frac{1}{2}T(0, \alpha_1)$, x remains

real and increases to α_2, and then, decreasing, returns to α_1 at the finite time $\sqrt{-1}\,T(\alpha_1, \alpha_2)$. If $t-t_1$ increases by real increments from $\sqrt{-1}\,\frac{1}{2}\,T(0, \beta_1)$, x remains real and decreases to β_2 and then returns to β_1 after the finite interval $T(\beta_1, \beta_2)$. Fig. 4 gives the real curves with the arbitrary determined so that $x=0$ at $t=0$. By continuous variation of t through real and purely imaginary values over the proper ranges, all the curves will be described.

The case where B_{-m} is negative differs qualitatively only in that the curves for real and purely imaginary values of $t-t_1$ are interchanged in position.

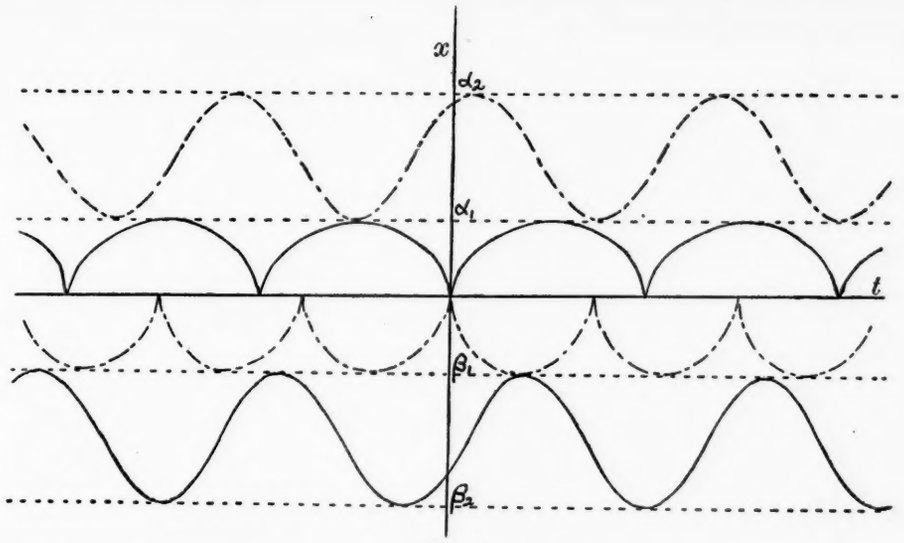

FIG. 4.

CASE 2. *$B=0$ and only a finite number of the B_j distinct from zero.*

We now consider the case where $B=0$, $B_n \neq 0$ and the $B_j=0$, $j>n$. In this case $R=\infty$. The question of chief interest is the nature of x as a function of t for those values of t for which $x=\infty$. Let $x=1/\xi$ and (2) becomes

$$\xi' = -\frac{\sqrt{\ldots + B_{-m}\xi^{m+n} + \ldots + B_{-1}\xi^{n+1} + B_0\xi^n + \ldots + B_n}}{\xi^{\frac{n-4}{2}}}. \tag{8}$$

If $n \leq 4$, the case of elliptic or exponential functions, ξ is a regular function of t at t_1, where $\xi=0$, $x=\infty$ at $t=t_1$. In other cases we find from

$$-(t-t_1) = \int_0 \frac{\xi^{\frac{n-4}{2}}\,d\xi}{\sqrt{B_n + B_{n-1}\xi + \ldots}} \tag{9}$$

that

$$\xi = \left[\frac{(n-2)^2}{4} B_n (t-t_1)^2\right]^{\frac{1}{n-2}} \left\{1 + \ldots\right\}. \tag{10}$$

The discussion of the real branches of the curves defined by this equation is the same as that of (7).

Suppose α is the largest positive root of $f(x)=0$ and that β is the negative root having the largest modulus. Suppose further that these roots are simple. Then it follows from (9) that ξ will vary from 0 to $1/\alpha$ or to $1/\beta$ in a finite interval of time $T(\alpha, \infty)$ or $T(\beta, \infty)$. Under the hypotheses which have been adopted, if $t-t_1$ increases continuously beyond $T(\alpha, \infty)$ or $T(\beta, \infty)$, by real increments

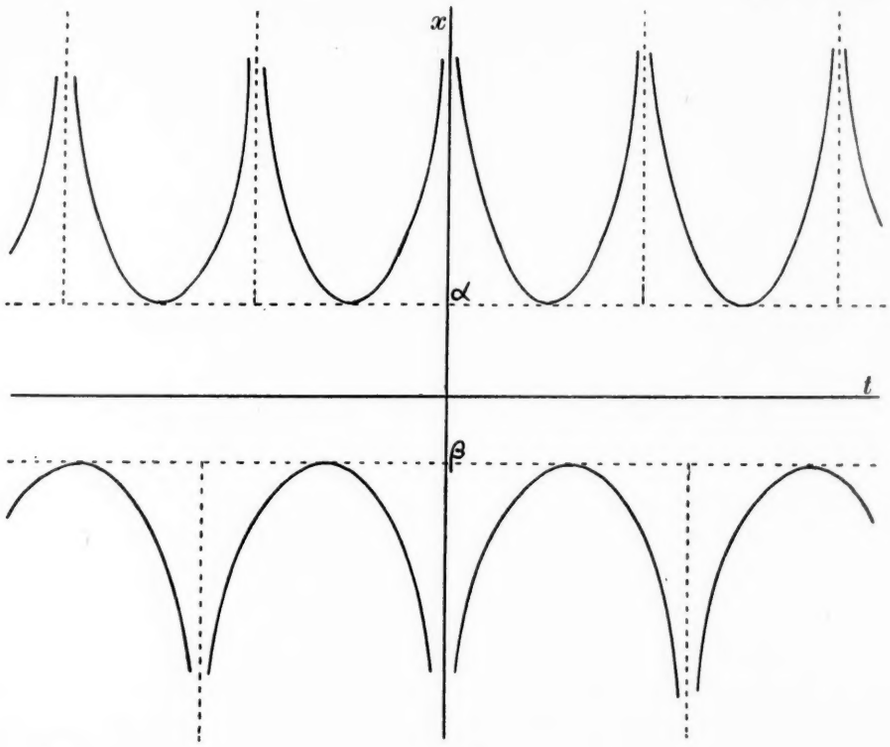

FIG. 5.

if B_n is positive and by purely imaginary increments if B_n is negative, ξ' will change sign and ξ will return to zero. For B_n positive and n even the infinite branches of the real curves are given in Fig. 5.

If B_n were negative, the curves would be similar for purely imaginary values of t.

If n is odd and B_n positive, there are positive real branches for real values of $t-t_1$, and negative real branches for purely imaginary values of $t-t_1$. Qualitatively the figure for the real branches would be the same as Fig. 5,

except that the negative branches would belong to purely imaginary values of $t - t_1$. If B_n were negative, the results would differ qualitatively from those where B_n is positive only in that the real and purely imaginary values of $t - t_1$ would be interchanged in position.

CASE 3. *$f(x)$ a polynomial of the fourth or third degree.*

Suppose the roots of $f(x) = 0$ are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and that the coefficient of x^4 is unity. Suppose $\alpha_1, \ldots, \alpha_4$ are real and distinct and that the notation has been so chosen that $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$. We now make the linear fractional transformation

$$\frac{x-\alpha_3}{x-\alpha_2} \cdot \frac{\alpha_4-\alpha_2}{\alpha_4-\alpha_3} = \frac{\xi-\xi_3}{\xi-\xi_2} \cdot \frac{\xi_4-\xi_2}{\xi_4-\xi_3}, \tag{11}$$

which transforms $\alpha_2, \alpha_3, \alpha_4$ into ξ_2, ξ_3, ξ_4 respectively, and any other point x forming a certain anharmonic ratio with $\alpha_2, \alpha_3, \alpha_4$ into a ξ forming the same anharmonic ratio with ξ_2, ξ_3, ξ_4. We assign the values $\xi_2 = -1$, $\xi_3 = +1$, $\xi_4 = +1/\varkappa$, where $\varkappa$ is determined by the condition that the point α_1 shall be transformed into $-1/\varkappa$. This is, of course, the Legendre normal form. Then (11) becomes

$$\frac{(1+\varkappa)^2}{(1-\varkappa)^2} = \frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2} \cdot \frac{\alpha_4-\alpha_2}{\alpha_4-\alpha_3} = C > 1.$$

Solving, we get

$$\varkappa = \frac{\sqrt{C}-1}{\sqrt{C}+1} \text{ and } \varkappa = \frac{\sqrt{C}+1}{\sqrt{C}-1}. \tag{12}$$

The former of these two values of $\varkappa$ is less than unity and will be adopted. It leaves the transformed points in the same order on the real axis as the original ones. Then equation (2) becomes

$$\left.\begin{aligned} \xi' &= \sqrt{A}\sqrt{(1-\xi^2)(1-\varkappa^2\xi^2)}, \\ A &= \frac{(\alpha_2-\alpha_1)(\alpha_4-\alpha_3)}{(1-\varkappa)^2}. \end{aligned}\right\} \tag{13}$$

If $f(x)$ were of the third degree, that is, if α_4 were infinite, the transformation would lead to the same result.

In the case under consideration ξ is a regular function of t for all finite values of t. It has four periods,

$$T(-1/\varkappa, -1),\ T(-1, +1),\ T(+1, +1/\varkappa),\ T(1/\varkappa, +\infty, -\infty, -1/\varkappa).$$

The first and third are obviously equal. Since the transformation $\xi = -1/\varkappa\eta$ leaves the form of the first of (13) invariant, the second and fourth periods are

also equal. One period is real and the other purely imaginary. The real curves are shown in Fig. 6.

CASE 4. *The differential equation* $x' = \sqrt{(1-x^2)(1-\varkappa^2 x^2)}$.

The case $n > 2$ will be discussed here only to show in what general respects the functions defined by this differential equation differ from those heretofore

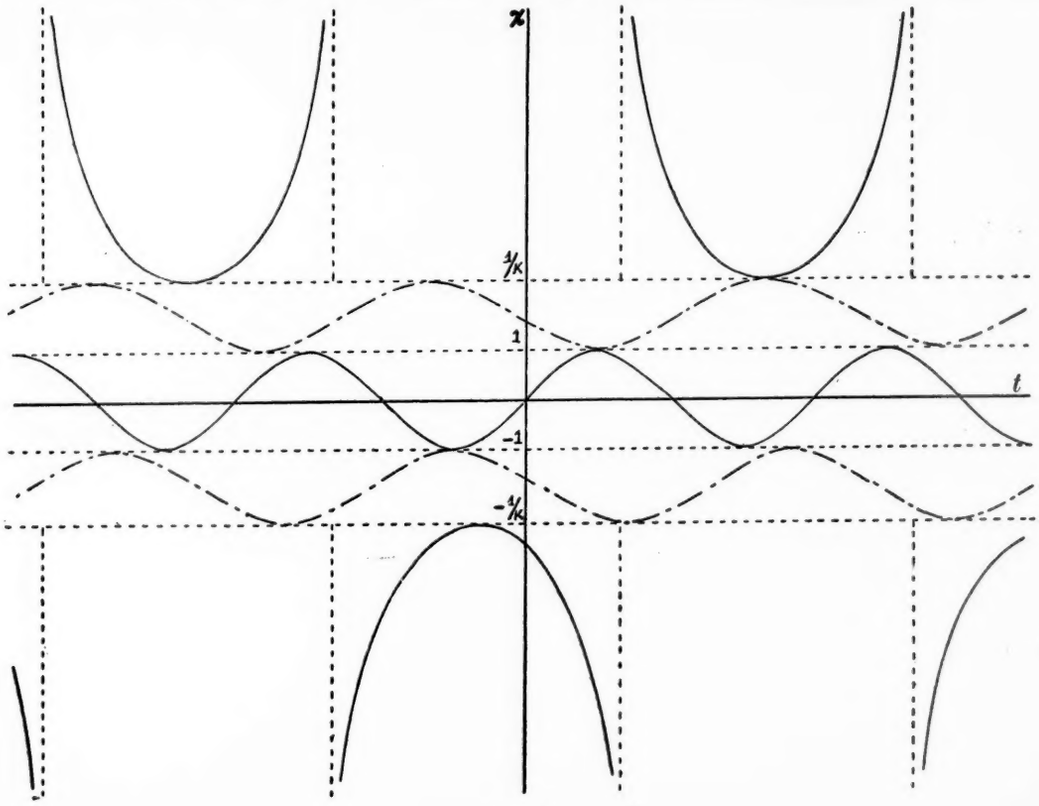

FIG. 6.

treated. For simplicity $f(x)$ is taken in the Legendre normal form. But it is noted that if we should start with $f(x)$ having the zeros $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, the transformation (11) would not in this case reduce the differential equation precisely to the normal form.

The solutions of

$$x' = \sqrt{(1-x^2)(1-\varkappa^2 x^2)} \tag{14}$$

are regular functions of t, certainly for all finite values, except possibly those for which $x = \pm 1$, $\varkappa x = \pm 1$, $x = \infty$. We shall examine the function at these points in turn.

Suppose $x=0$ at $t=0$. One of the determinations of the radical is real and positive. If we choose this one and take t real and positive, x will increase to unity, which it will reach at the finite time

$$t_1 = T(0,1) = \int_0^1 \frac{dx}{\sqrt[n]{(1-x^2)(1-x^2x^2)}}. \tag{15}$$

If n were even and the negative determination of the radical had been taken, x would have arrived at -1 at the time t_1.

We must investigate the nature of x as a function of t in the vicinity of $t=t_1$. Let $x=1-\xi$; then (14) gives

$$t-t_1 = -\int_0^\xi \frac{d\xi}{\sqrt[n]{\xi(2-\xi)(1-x^2+2x^2\xi-x^2\xi^2)}}. \tag{16}$$

Expanding the integrand as a series and integrating, we get

$$-(t-t_1) = \xi^{\frac{n-1}{n}}[a_0 + a_1\xi + \ldots.],$$

the a_i being real and a_0 positive. Inverting this series, we have

$$\left.\begin{aligned} &\xi = b_1\tau + b_2\tau^2 + \ldots., \quad b_1 > 0, \\ &\tau = \{[-(t-t_1)]^n\}^{\frac{1}{n-1}}. \end{aligned}\right\} \tag{17}$$

It follows from the second of these equations that τ is not changed by multiplying $t-t_1$ by any n-th root of unity. Therefore, there are n values of $t-t_1$ having the same modulus which give the same value of τ. As $t-t_1$ passes through zero, τ passes through zero; and its value after changing sign is, for a given modulus, equal to that before multiplied by $e^{\frac{n\pi\sqrt{-1}}{n-1}}$, which is a complex number except when $n=2$. Therefore, if $t-t_1$ is real any real curve belonging to it becomes complex when $t-t_1$ passes through zero. It is obvious therefore that these functions can not represent the motions of physical bodies.

If n is odd, ξ has two real values, one positive and one negative, for real negative values of $t-t_1$. In this case ξ is not real for any other real or purely imaginary values of $t-t_1$. If n is even, ξ has only one real value for real values of $t-t_1$, and this value is positive and the same whether $t-t_1$ is positive or negative with a given modulus. But in this case ξ is also real, though it is negative, for $t-t_1$ equal to a real positive or negative number multiplied by any n-th root of -1. If n is 2, these are the purely imaginary values of $t-t_1$ encountered in the problems heretofore considered. In general, if n is even

and not congruent to zero mod 4, two of these values are conjugate pure imaginaries. But if n is a multiple of 4, all these values of $t-t_1$ are complex.

It follows from this discussion that if we start with $x=0$, x' real and positive at $t=0$, x will take the value $+1$ at a finite time $t=t_1$. At this point there are $n-1$ branches of the curves satisfying the differential equation, and there are values of $t-t_1$, real or complex, for which x remains real and increases, and others for which it remains real and decreases. On the branch on which it increases it arrives at $+1/\varkappa$ at a finite time $t=t_2$. At this point the description of the branching differs from that at $x=+1$ only in that positive and negative as applied to ξ, where $x=1-\xi$ as before, must be interchanged. Consequently there is a real increasing value of x beyond $x=1/\varkappa$ for real positive values of $t-t_2$. If we follow this curve, x will tend toward infinity, which it will reach at the time

$$T(1/\varkappa, \infty) = \int_{1/\varkappa}^{\infty} \frac{dx}{\sqrt[n]{(1-x^2)(1-\varkappa^2 x^2)}}, \tag{18}$$

which is infinite unless $n<4$.

Suppose $n=3$ and let $x=-1/\xi$. Then, if $x=\infty$ at $t=t_3$, we have

$$t-t_3 = \int_0^{\xi} \frac{d\xi}{\xi^{2/3}\sqrt[3]{(1-\xi^2)(\varkappa^2-\xi^2)}},$$

$$t-t_3 = \xi^{1/3}[a_0 + a_1\xi + \cdots],$$

the a_i being real and a_0 positive. From this we get

$$\left.\begin{aligned} \xi &= b_0(t-t_3)^3 + \cdots, \\ x &= \frac{-1}{)^3}\{c_0 + c_1(t-t_3) + \cdots\}, \quad c_0>0. \end{aligned}\right\} \tag{19}$$

The discussions for -1, $-1/\varkappa$ and ∞ are essentially the same. We see that the details are quite different when n is even from those when it is odd, and that $x=\infty$ for finite values of t only if n is 2 or 3. In the former they are regular points, and in the latter poles of order 3. For $n=3$ the real curves are shown in Fig. 7.

§ 4. Explicit Computation of the Periods.

Suppose it is desired to find the period corresponding to the interval α_i, α_{i+1}. One method is obviously to compute the numerical value of (4) by mechanical quadratures, as Legendre did in the case of elliptic integrals. But this is not sufficient in the present discussion, because in constructing the solutions in §§ 5–7 it will be necessary to have the periods developed in converging series.

Since the zeros of $f(x)$ in $r_1 \leqq |x| \leqq R_1$ are $\alpha_1, \ldots, \alpha_p$, we may write

$$f(x) = (x - \alpha_1) \ldots (x - \alpha_p) \psi(x), \tag{20}$$

where $\psi(x)$ is finite and distinct from zero in the ring $r_1 \leqq |x| \leqq R_1$. Suppose α_i and α_{i+1} are real and that $f(x_0) > 0$ for $\alpha_i < x_0 < \alpha_{i+1}$, so that the period is real.

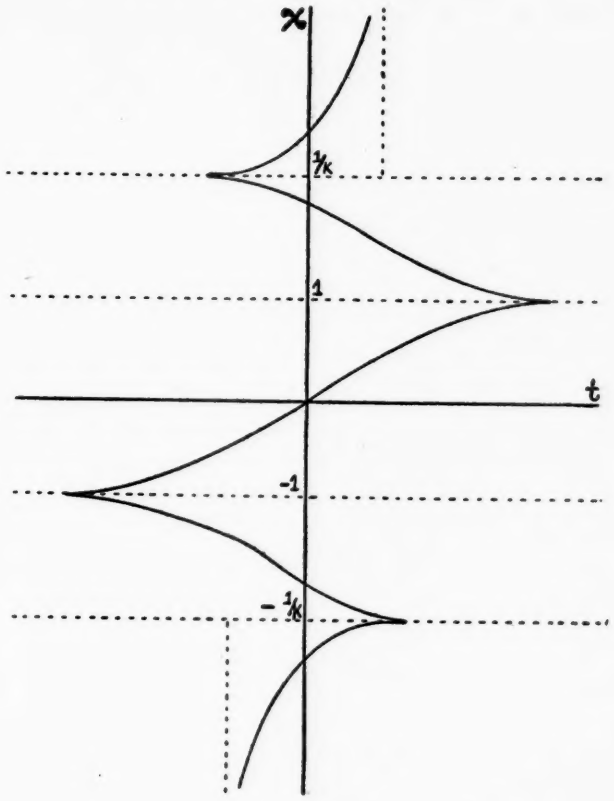

FIG. 7.

Let

$$x = \frac{\alpha_i + \alpha_{i+1}}{2} + \frac{\alpha_{i+1} - \alpha_i}{2} \xi. \tag{21}$$

Then (20) becomes

$$f(x) = (1 - \xi^2) \prod_{j=1}^{p} \left[\frac{\alpha_i + \alpha_{i+1} - 2\alpha_j}{2} + \frac{\alpha_{i+1} - \alpha_i}{2} \xi \right] \chi(\xi), \tag{22}$$

where $\chi(\xi)$ is finite and distinct from zero for all ξ corresponding to $r_1 \leqq |x| \leqq R_1$. Those α_j which are complex occur in conjugate pairs. In these cases we take

their products and obtain factors of the form $\beta_j\xi^2+\gamma_j\xi+\delta_j$, where β_j, γ_j and δ_j are real constants. Then we may write (22) in the form

$$f(x)=\frac{A^2(\alpha_{i+1}-\alpha_i)^2}{4}(1-\xi^2)(1+\rho_1\xi)$$
$$\ldots\ldots(1+\rho_k\xi)[1+\lambda_1\xi+\mu_1\xi^2]\ldots\ldots[1+\lambda_l\xi+\mu_l\xi^2]F(\xi), \quad (23)$$

where $\rho_1, \ldots, \rho_k, \lambda_1, \mu_1, \ldots, \lambda_l, \mu_l$ and the coefficients of $F(\xi)$ are real constants, and where $F(0)=1$. Moreover $\rho_1, \ldots, \rho_k$ are less than unity because α_i and α_{i+1} are consecutive real roots of $f(x)=0$.

In order to compute (4) we must take the square root of the reciprocal of $f(x)$. The square roots of the reciprocals of the factors $(1+\rho_1\xi)\ldots\ldots(1+\rho_k\xi)$ are expansible as power-series in ξ which converge for all $|\xi|\leqq 1$. The expansion of the square root of the reciprocal of $1+\lambda_j\xi+\mu_j\xi^2$ as a power-series in ξ converges if $|\xi|<1/\sqrt{|\mu_j|}$. We shall suppose at present that the $|\mu_j|$ are all less than unity. Then (4) becomes

$$T(\alpha_i, \alpha_{i+1})=\frac{2}{A}\int_{-1}^{+1}\frac{[1+P_1\xi+P_2\xi^2+\ldots\ldots]d\xi}{\sqrt{1-\xi^2}}, \quad (24)$$

where $P_1, P_2, \ldots\ldots$ are polynomials in $\rho_1, \ldots, \rho_k, \lambda_j, \mu_j$ and the coefficients of $F(\xi)$. The series under the integral sign converges for $|\xi|\leqq 1$. Since

$$\int_{-1}^{+1}\frac{\xi^{2n+1}}{\sqrt{1-\xi^2}}d\xi=0, \quad \int_{-1}^{+1}\frac{\xi^{2n}}{\sqrt{1-\xi^2}}d\xi=\frac{(2n)!\,\pi}{2^{2n}(n!)^2},$$

the period is

$$T(\alpha_i, \alpha_{i+1})=\frac{2\pi}{A}\left\{1+\sum_{n=1}^{\infty}\frac{(2n)!\,P_{2n}}{2^{2n}(n!)^2}\right\}. \quad (25)$$

Since

$$\frac{(2n)!}{2^{2n}(n!)^2}=\frac{1\cdot 3\cdot 5\ldots\ldots 2n-1}{2\cdot 4\cdot 6\ldots\ldots 2n}<1$$

and the integrand of (24) converges for $|\xi|<1$, the right member of (25) is a convergent series.

In the simple elliptic case

$$f(x)=(1-x^2)(1-x^2x^2), \quad x^2<1,$$

the real period is

$$T=2\pi\left\{1+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2x^2+\ldots\ldots+\left(\frac{1\cdot 3\ldots\ldots 2n-1}{2\cdot 4\ldots\ldots 2n}\right)^2x^{2n}+\ldots\ldots\right\}. \quad (26)$$

Suppose now that $|\mu_j|>1$, but that all other $|\mu_i|<1$, and that

$$1+\lambda_j\xi+\mu_j\xi^2=\mu_j[\xi-\beta+\sqrt{-1}\,\gamma][\xi-\beta-\sqrt{-1}\,\gamma],$$

where β and γ are real. The integral in this case can be computed from -1 to β, and then from β to $+1$. Let

$$\xi = \frac{\beta - 1}{2} + \frac{\beta + 1}{2}\eta.$$

Then $\eta = -1$ when $\xi = -1$, and $\eta = +1$ when $\xi = \beta$. The quadratic factor $[(\xi - \beta)^2 + \gamma^2]$ becomes $\frac{(1+\beta)^2}{4}(1-\eta)^2 + \gamma^2$. The square root of its reciprocal is expansible as a converging power-series in η if $|\eta| < \sqrt{1 + \frac{4\gamma^2}{(1+\beta)^2}}$. Therefore the first of the two integrals is

$$T(\alpha_i, \beta) = \int_{-1}^{+1} \frac{[1 + Q_1\eta + Q_2\eta^2 + \ldots.]\, d\eta}{\sqrt{1-\eta}}.$$

For the second integral there is a similar transformation and expansion.

If two of the $|\mu_j|$ were greater than unity, the integral in x from α_i to α_{i+1} would be broken up in a similar way into the sum of three integrals which would be defined by converging series. This process of division can obviously be extended to include the case where any finite number of $|\mu_j|$ are equal to or greater than unity.

§ 5. Explicit Construction of the Periodic Solution.

Suppose it is desired to find the solution for which x oscillates between α_i and α_{i+1}. By the transformation (21) the differential equation becomes

$$\xi' = A\sqrt{(1-\xi^2)(1+\rho_1\xi)\ldots.(1+\rho_k\xi)[1+\lambda_1\xi+\mu_1\xi^2]\ldots.[1+\lambda_l\xi+\mu_l\xi^2]F(\xi)}, \quad (27)$$

where $F(\xi)$ is a power-series in ξ which converges if $|\xi| \leqq 1$.

For convenience let us make the transformation

$$\left.\begin{aligned} \rho_i &= \bar{\rho}_i\mu, \\ \lambda_i &= \bar{\lambda}_i\mu, \\ \mu_i &= \bar{\mu}_i\mu^2, \\ c_i &= \bar{c}_i\mu^i, \end{aligned}\right\} \quad (28)$$

where the c_i are coefficients of expansion of $F(\xi)$. The parameter μ will guide the computations; in numerical applications it will be taken equal to unity and the bars on the ρ_i, λ_i, μ_i and c_i will be omitted. After the transformations (28), equations (27) and (25) become

$$\left.\begin{aligned} \xi' &= A\sqrt{(1-\xi^2)\,P(\xi;\mu)}, \\ T &= \frac{2\pi}{A}\,Q(\mu), \end{aligned}\right\} \quad (29)$$

where P is a power-series in μ which converges if $|\xi| \leq 1$, provided all $|\mu_j| < 1$, and where $Q(\mu)$ is the series (25). The coefficient of μ^i in P contains ξ^i as a factor.

In order to place in evidence the period in the construction of the solution let us make the transformation

$$t - t_0 = \frac{1}{A} Q(\mu)\tau. \tag{30}$$

Therefore, the period in the variable τ is 2π. If we denote by dots the derivatives with respect to τ, we have

$$\left.\begin{aligned} \dot{\xi} &= Q(\mu)\sqrt{(1-\xi^2)P(\xi;\mu)}, \\ \ddot{\xi} &= -Q^2(\mu)\xi P(\xi;\mu) + \tfrac{1}{2}Q^2(\mu)(1-\xi^2)\frac{\partial P}{\partial \xi}(\xi;\mu). \end{aligned}\right\} \tag{31}$$

By Poincaré's extension of Cauchy's theorem* the second of these equations can be integrated as a power-series in μ, and $|\mu|$ can be taken so small that the series will converge for any preassigned interval for τ. The same result is effected by taking ρ_i, λ_i, μ_i and c_i small, and μ equal to unity.

Let us construct the actual solution of the second of (31) with the initial conditions $\xi(0) = -1$, $\dot{\xi}(0) = 0$. We have

$$\left.\begin{aligned} \xi &= \xi_0 + \xi_1\mu + \xi_2\mu^2 + \cdots, \\ Q^2 &= 1 + Q_2\mu^2 + Q_4\mu^4 + \cdots, \\ P &= 1 + P_1\xi\mu + P_2\xi^2\mu^2 + \cdots, \\ \frac{\partial P}{\partial \xi} &= P_1\mu + 2P_2\xi\mu^2 + \cdots + nP_n\xi^{n-1}\mu^n + \cdots, \end{aligned}\right\} \tag{32}$$

where the P_n and Q_n are known constants, and the ξ_n are functions of τ to be determined. Substituting (32) in (31) and equating to zero the coefficients of various powers of μ, we find

$$\left.\begin{aligned} &\ddot{\xi}_0 + \xi_0 = 0, \\ &\ddot{\xi}_1 + \xi_1 = \tfrac{1}{2}P_1 - \tfrac{1}{2}P_1\xi_0^2, \\ &\cdots\cdots\cdots\cdots\cdots, \\ &\ddot{\xi}_n + \xi_n = R_n(\xi_0, \cdots, \xi_{n-1}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}\right\} \tag{33}$$

where R_n is a known polynomial in $\xi_0, \cdots, \xi_{n-1}$.

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* *Les Methodes Nouvelles de la Mécanique Céleste*, Vol. I, p. 58.

Equations (33) can be integrated in the order in which they are written. Since the initial conditions are identities in μ, we have

$$\left.\begin{aligned} &\xi_0(0) = -1, \quad \xi_j(0) = 0, \quad j = 1, \ldots, \infty, \\ &\dot{\xi}_j(0) = 0, \quad j = 0, \ldots, \infty. \end{aligned}\right\} \tag{34}$$

With these initial conditions the solution of the first equation of (33) is

$$\xi_0 = -\cos\tau. \tag{35}$$

Then the second equation of (33) becomes

$$\ddot{\xi}_1 + \xi_1 = -\tfrac{1}{4}P_1 - \tfrac{3}{4}P_1\cos 2\tau,$$

whose solution satisfying (34) is

$$\xi_1 = -\tfrac{1}{4}P_1 + \tfrac{1}{4}P_1\cos 2\tau. \tag{36}$$

Let us suppose that $\xi_0, \ldots, \xi_{n-1}$ have been computed and that it has been found that ξ_j, $j = 0, \ldots, n-1$, is a sum of cosines of multiples of τ, the highest multiple being $j+1$, and then consider the right member of the $(n+1)$-st equation of (33). Its general term, aside from a constant coefficient, has the form

$$[R_n] = \xi_{j_1}^{i_1}\,\xi_{j_2}^{i_2}\ldots\ldots\,\xi_{j_n}^{i_n}. \tag{37}$$

It follows from (31) and (32) that the exponents and subscripts satisfy the relations

$$\left.\begin{aligned} &i_1 + i_2 + \ldots\ldots + i_n = I_n \leqq n+1, \\ &i_1 j_1 + i_2 j_2 + \ldots\ldots + i_n j_n \leqq n - (I_{n-1}). \end{aligned}\right\} \tag{38}$$

Each element of the general term being a sum of cosines, the product is a sum of cosines. The highest multiple is

$$N_n = i_1(j_1+1) + i_2(j_2+1) + \ldots\ldots + i_n(j_n+1),$$

which, by (38), has as its greatest value $n+1$. Therefore, the differential equation at the $(n+1)$-st step is

$$\ddot{\xi}_n + \xi_n = A_0^{(n)} + A_1^{(n)}\cos\tau + A_2^{(n)}\cos 2\tau + \ldots\ldots + A_{n+1}^{(n)}\cos(n+1)\tau. \tag{39}$$

The solution of (31) is periodic with the period 2π in τ for all $|\mu|$ sufficiently small. Therefore,

$$\sum_{j=0}^{\infty}\xi_j(\tau + 2\pi)\,\mu^j \equiv \sum_{j=0}^{\infty}\xi_j(\tau)\,\mu^j,$$

where the identity holds in both μ and τ. It follows from the fact that it is an identity in μ that

$$\xi_j(\tau + 2\pi) \equiv \xi_j(\tau), \; j = 0, \ldots, \infty. \tag{40}$$

Since this relation is an identity in τ, each ξ_j is separately periodic with the period 2π.

Now consider the integration of (39). Since the solution is periodic, the coefficient $A_1^{(n)}$ is zero. This condition has been secured at each step by the factor $Q^2(\mu)$ in the second of (31). With $A_1^{(n)} = 0$, the solution of (39) satisfying (34) is

$$\left.\begin{aligned} \xi_n &= \alpha_0^{(n)} + \alpha_1^{(n)} \cos\tau + \alpha_2^{(n)} \cos 2\tau + \ldots + \alpha_{n+1}^{(n)} \cos(n+1)\tau, \\ \alpha_0^{(n)} &= A_0^{(n)}, \\ \alpha_j^{(n)} &= \frac{A_j^{(n)}}{1-j^2}, \quad j = 2, \ldots, n+1, \\ \alpha_1^{(n)} &= -A_0^{(n)} + \sum_{j=2}^{n+1} \frac{A_j^{(n)}}{j^2-1}. \end{aligned}\right\} \tag{41}$$

These results have the properties assumed in the beginning, and the induction is therefore complete.

§ 6. Direct Construction of the Solution from the Integral.

The properties of the solution being known, it can conveniently be found from the first equation of (31). Let this equation be written in the form

$$\dot{\xi}^2 - Q^2(\mu)(1-\xi^2) P(\xi;\mu) \equiv F(\dot{\xi}, \xi; \mu) = 0. \tag{42}$$

This equation holds identically in μ and τ. Since ξ and $\dot{\xi}$ are expansible as converging power-series in μ, F can be expanded as a power-series in μ of the form

$$F = F_0 + F_1\mu + \ldots + F_n\mu^n + \ldots = 0.$$

Since this equation is an identity in μ, we have

$$F_n(\dot{\xi}_j, \xi_j) = 0, \quad j = 0, \ldots, n; \quad n = 0, \ldots, \infty. \tag{43}$$

Now suppose the expressions for ξ_j and $\dot{\xi}_j$ given in the first of (41) are substituted in (43). It follows from (42) that the largest value of j in F_n is $j = n$. The result of the substitution will be a sum of cosines of integral multiples of τ, the highest being $n+2$. Therefore, (43) may be written

$$F_n = B_0^{(n)} + B_1^{(n)} \cos\tau + \ldots + B_{n+2}^{(n)} \cos(n+2)\tau = 0. \tag{44}$$

Since this equation is an identity in τ, we have

$$B_j^{(n)} = 0, \quad j = 0, \ldots, n+2. \tag{45}$$

It follows from the form of (42) and the fact that in $P(\xi, \mu)$ every term except the first, which is unity, is multiplied by μ that the $B_j^{(n)}$ have the form

$$\left.\begin{aligned} B_0^{(n)} &= -2\alpha_1^{(n)} + C_0^{(n)}, \\ B_1^{(n)} &= -2\alpha_0^{(n)} - 3\alpha_3^{(n)} + C_1^{(n)}, \\ B_2^{(n)} &= +0 - 4\alpha_3^{(n)} + C_2^{(n)}, \\ B_j^{(n)} &= +(j-2)\alpha_{j-1}^{(n)} - (j+2)\alpha_{j+1}^{(n)} + C_j^{(n)}, \quad j = 2, \ldots, n, \\ B_{n+1}^{(n)} &= +(n-1)\alpha_n^{(n)} + 0 + C_{n+1}^{(n)}, \\ B_{n+2}^{(n)} &= +n\alpha_{n+1}^{(n)} + 0 + C_{n+2}^{(n)}, \end{aligned}\right\} \tag{46}$$

where the $C_j^{(n)}$, $j = 0, \ldots, n+2$, are known functions of $\alpha_i^{(0)}, \ldots, \alpha_i^{(n-1)}$. Therefore if $\alpha_i^{(0)}, \ldots, \alpha_i^{(n-1)}$ have been computed, equations (45) determine $\alpha_0^{(n)}, \ldots, \alpha_{n+1}^{(n)}$. The solution is most conveniently made by starting with the last equation and using them in order to the first, with the exception of the third, which is redundant. If the constant of integration in (42) had not been determined on the start so as to belong to the solution with the period 2π, the coefficients of the $(n+1)$-st term of its expansion as a power-series in μ would be determined by the third equation of (46).

§ 7. Direct Construction of the Solution from the Values of ξ and its Derivatives at $\tau = 0$ and $\tau = \pi$.

We may write (31) and their successive derivatives in the form

$$\left.\begin{aligned} \dot{\xi} &= R_1(\xi, \mu), \\ \ddot{\xi} &= R_2(\xi, \mu), \\ &\cdots\cdots\cdots, \\ \overset{n}{\xi} &= R_n(\xi, \mu), \end{aligned}\right\} \quad (47)$$

where the $R_j(\xi, \mu)$ are power-series in ξ and μ. The derivatives obtained in the right member are of course eliminated at the successive steps by the first equation. Since $\xi = -1$ at $\tau = 0$ and $\xi = +1$ at $\tau = \pi$ identically in μ, we have

$$\left.\begin{aligned} &\xi(0) = \sum_{j=0}^{\infty} \xi_j(0)\mu^j = -1, \\ &\xi(\pi) = \sum_{j=0}^{\infty} \xi_j(\pi)\mu^j = +1, \\ &\cdots\cdots\cdots\cdots\cdots, \\ &\overset{n}{\xi}(0) = \sum_{j=0}^{\infty} \overset{n}{\xi}_j(0)\mu^j = R_n(-1, \mu) = \sum_{j=0}^{\infty} p_n^{(j)}(-1)\mu^j, \quad n = 1, \ldots, \infty, \\ &\overset{n}{\xi}(\pi) = \sum_{j=0}^{\infty} \overset{n}{\xi}_j(\pi)\mu^j = R_n(+1, \mu) = \sum_{j=0}^{\infty} p_n^{(j)}(+1)\mu^j. \end{aligned}\right\} \quad (48)$$

Now consider the determination of the coefficients of ξ_j. Its form is given in (41), from which it follows that all odd derivatives vanish at both $\tau = 0$ and $\tau = \pi$. All $R_{2k+1}(\xi, \mu)$ also vanish at $\tau = 0$ and $\tau = \pi$, because they carry $\sqrt{1-\xi^2}$ as a factor. Hence the coefficients of equations (48) are identically

zero when n is odd. Since equations (48) hold identically in μ, we have for even values of n

$$\begin{aligned}
&\xi_j(0) = 0, \quad j = 1, \ldots, \infty, \\
&\xi_j(\pi) = 0, \\
&\overset{2n}{\xi}_j(0) = p_{2n}^{(j)}(-1), \quad j = 0, \ldots, \infty, \\
&\overset{2n}{\xi}_j(\pi) = p_{2n}^{(j)}(+1);
\end{aligned}$$

or, making use of (31),

$$\left.\begin{aligned}
&\alpha_0^{(j)} + \alpha_1^{(j)} + \ldots + \alpha_{j+1}^{(j)} = 0, \\
&\alpha_0^{(j)} - \alpha_1^{(j)} + \ldots + (-1)^{j+1}\alpha_{j+1}^{(j)} = 0, \\
&0 + \alpha_1^{(j)} + 2^{2n}\alpha_2^{(j)} + \ldots + (j+1)^{2n}\alpha_{j+1}^{(j)} = p_{2n}^{(j)}(-1), \quad n = 1, 2, \ldots, \infty, \\
&0 - \alpha_1^{(j)} + 2^{2n}\alpha_2^{(j)} - \ldots + (-1)^{j+1}(j+1)^{2n}\alpha_{j+1}^{(j)} = p_{2n}^{(j)}(+1).
\end{aligned}\right\} \quad (49)$$

Since the right members of these equations are known, the $\alpha_k^{(j)}$ are uniquely determined by this infinite set of equations, provided the determinant of the coefficients of any $j+2$ of them is distinct from zero. It is clearly necessary to use at least one of the first two equations, which alone involve $\alpha_0^{(j)}$. If those equations are taken for which $\tau = 0$, the determinant is an alternant and is distinct from zero. Therefore the solutions are uniquely defined by the properties which have been established and by these equations. This is the simplest method of computing them. The results of the computation are easily checked by substituting them in any of the infinitely many superfluous equations.

§ 8. Application to the Legendre Elliptic Functions.

The results which have been obtained in §§ 5–7 include the Legendre Elliptic Functions as simple special cases. Consider sn τ, the *sine amplitude* function, which is defined by

$$\left.\begin{aligned}
x' &= \sqrt{(1-x^2)(1-\kappa^2x^2)}, \quad x^2 < 1, \\
x'' &= -(1+\kappa^2)x + 2\kappa^2x^3.
\end{aligned}\right\} \quad (50)$$

The parameter κ^2 plays the rôle of μ above, and the solutions are in powers of κ^2. In order that the results may agree with the usual notations, we take the initial condition $x(0) = 0$. In this case we have

$$Q(\mu) = Q(\kappa^2) = 1 + (\tfrac{1}{2})^2\kappa^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\kappa^4 + \ldots + \left(\frac{1\cdot 3\ldots 2n-1}{2\cdot 4\ldots 2n}\right)^2\kappa^{2n} + \ldots \quad (51)$$

Then

$$\dot{x} = Q(\kappa^2)\sqrt{(1-x^2)(1-\kappa^2x^2)}.$$

The solution of this equation as a power-series in x^2 is

$$x = x_0 + x_2 x^2 + x_4 x^4 + \ldots\ldots \tag{52}$$

Making the computation by the methods of §§ 5–7, it is found that

$$\left.\begin{aligned}
x_0 &= \sin\tau,\\
x_2 &= \tfrac{1}{16}\,[\sin\tau + \sin 3\tau],\\
x_4 &= \tfrac{1}{16^2}\,[7\sin\tau + 8\sin 3\tau + \sin 5\tau],\\
x_6 &= \tfrac{1}{16^3}\,[67\sin\tau + 82\sin 3\tau + 16\sin 5\tau + \sin 7\tau],\\
x_8 &= \tfrac{1}{16^4}\,[738\sin\tau + 945\sin 3\tau + 230\sin 5\tau + 24\sin 7\tau + \sin 9\tau],\\
x_{10} &= \tfrac{1}{16^5}\,[8808\sin\tau + 11{,}661\sin 3\tau + 3264\sin 5\tau + 442\sin 7\tau\\
&\qquad + 32\sin 9\tau + \sin 11\tau],\\
x_{12} &= \tfrac{1}{16^6}\,[110{,}728\sin\tau + 150{,}486\sin 3\tau + 46{,}519\sin 5\tau\\
&\qquad + 7440\sin 7\tau + 718\sin 9\tau + 40\sin 11\tau + \sin 13\tau],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{53}$$

Each of these expressions carries $\sin\tau$ as a factor. Since $x(\pi/2) = 1$ for all x^2, it follows that $x_{2n}(\pi/2) = 0$, $n = 1, \ldots, \infty$. Therefore each $x_{2n}(\pi/2)$, $n = 1, \ldots, \infty$, carries $\cos\tau$ as a factor. Since $\dot{x}(\pi/2)$ also vanishes for $x^2 < 1$, it follows that each x_{2n} has $\cos^2\tau$ as a factor. After $\sin\tau\cos^2\tau$ is removed, the result can conveniently be expressed as a sum of even powers of cosines of τ. The explicit results are

$$\left.\begin{aligned}
x_0 &= \sin\tau,\\
x_2 &= \tfrac{1}{4}\sin\tau\cos^2\tau,\\
x_4 &= \tfrac{1}{64}\sin\tau\cos^2\tau\,[5 + 4\cos^2\tau],\\
x_6 &= \tfrac{1}{256}\sin\tau\cos^2\tau\,[10 + 11\cos^2\tau + 4\cos^4\tau],\\
x_8 &= \tfrac{1}{1283^2}\sin\tau\cos^2\tau\,[389 + 500\cos^2\tau + 272\cos^4\tau + 64\cos^6\tau],\\
x_{10} &= \tfrac{1}{2561^2}\sin\tau\cos^2\tau\,[1054 + 1499\cos^2\tau + 984\cos^4\tau + 368\cos^6\tau\\
&\qquad + 64\cos^8\tau],\\
x_{12} &= \tfrac{1}{10242^2}\sin\tau\cos^2\tau\,[12{,}242 + 18{,}759\cos^2\tau + 13{,}800\cos^4\tau\\
&\qquad + 6448\cos^6\tau + 1856\cos^8\tau + 256\cos^{10}\tau],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{54}$$

These expressions are very convenient for use in practical computations.

Let $y = \mathrm{cn}\,\tau$, the *cosine amplitude* function. Then we have

$$y = \sqrt{1 - x^2} = \sqrt{1 - x_0^2}\,\sqrt{1 - \frac{\left[2x_0 + \sum_{j=1}^{\infty} x_{2j}\,x^{2j}\right]\sum_{j=1}^{\infty} x_{2j}\,x^{2j}}{1 - x_0^2}}. \tag{55}$$

It follows from this equation that y is expansible as a power-series in x^2 and that the coefficient of x^{2n} is a sum of cosines of odd multiples of τ, the highest being $2n+1$. The explicit expression for y can be derived from this equation, but it is just as conveniently found from the differential equation which y satisfies, viz.,

$$\dot{y} = Q(x^2)\sqrt{(1-y^2)(1-x^2+x^2y^2)}, \tag{56}$$

where $y(0)=1$, $y(\pi/2)=0$. Forming the successive derivatives, equations corresponding to (49) are obtained. Since $y(\pi/2)\equiv 0$, it follows that each y_{2n}, $n=0, \ldots, \infty$, carries $\cos\tau$ as a factor, and since $y(0)\equiv 1$, $\dot{y}(0)\equiv 0$, it follows that each y_{2n}, $n=1, \ldots, \infty$, carries $\sin^2\tau$ as a factor. These results also follow directly from the properties of x and the form of (55). Making the computation, we find

$$\left.\begin{aligned}
y &= y_0 + y_2x^2 + y_4x^4 + \ldots,\\
y_0 &= \cos\tau,\\
y_2 &= -\tfrac{1}{4}\cos\tau\sin^2\tau,\\
y_4 &= -\tfrac{1}{64}\cos\tau\sin^2\tau\,[7+4\cos^2\tau],\\
y_6 &= -\tfrac{1}{256}\cos\tau\sin^2\tau\,[17+13\cos^2\tau+4\cos^4\tau],\\
y_8 &= -\tfrac{1}{1282}\cos\tau\sin^2\tau\,[759+660\cos^2\tau+304\cos^4\tau+64\cos^6\tau],\\
y_{10} &= -\tfrac{1}{2562}\cos\tau\sin^2\tau\,[2289+2149\cos^2\tau+1192\cos^4\tau+400\cos^6\tau\\
&\qquad +64\cos^8\tau],\\
y_{12} &= -\tfrac{1}{10242}\cos\tau\sin^2\tau\,[29{,}023+28{,}727\cos^2\tau+17{,}800\cos^4\tau\\
&\qquad +7472\cos^6\tau+1984\cos^8\tau+256\cos^{10}\tau],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{57}$$

Now let $z=\sqrt{1-x^2x^2}$, the *delta amplitude* function. The variable z satisfies the equation

$$\dot{z} = Q(x^2)\sqrt{(1-z^2)(-1+x^2+z^2)}. \tag{58}$$

Making the computations as before, we find for the explicit value of z

$$\left.\begin{aligned}
z &= 1 + z_2x^2 + z_4x^4 + \ldots,\\
z_2 &= -\tfrac{1}{2}\sin^2\tau,\\
z_4 &= -\tfrac{1}{8}\sin^2\tau\,[1+\cos^2\tau],\\
z_6 &= -\tfrac{1}{64}\sin^2\tau\,[4+5\cos^2\tau+2\cos^4\tau],\\
z_8 &= -\tfrac{1}{322}\sin^2\tau\,[40+56\cos^2\tau+32\cos^4\tau+8\cos^6\tau],\\
z_{10} &= -\tfrac{1}{1282}\sin^2\tau\,[448+677\cos^2\tau+454\cos^4\tau+176\cos^6\tau\\
&\qquad +32\cos^8\tau],\\
z_{12} &= -\tfrac{1}{2562}\sin^2\tau\,[1344+2152\cos^2\tau+1586\cos^4\tau+756\cos^6\tau\\
&\qquad +224\cos^8\tau+32\cos^{10}\tau],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{59}$$

The functions $\operatorname{sn}\tau$, $\operatorname{cn}\tau$ and $\operatorname{dn}\tau$ arise very naturally in many problems of mechanics and geometry.* The series (54), (57) and (59) are convenient for use in solving these problems, and the coefficients which are given here can be relied upon as being correct. But the point of chief interest is that if the differential equations defining the functions are of the more general type considered here, the same methods of obtaining their explicit values can be used.

§ 9. The Realm of Convergence of the Series (52), (57) and (59).

For simplicity let us first consider the series for the sine amplitude function, which is defined by

$$\left.\begin{aligned} \dot{x} &= Q(x^2)\sqrt{(1-x^2)(1-x^2x^2)}, \quad x=0 \text{ at } \tau=0, \\ \ddot{x} &= -Q^2(x^2)(1+x^2)x + 2Q^2(x^2)x^2x^3. \end{aligned}\right\} \tag{60}$$

It follows from the second of these equations and the fact that $Q(x^2)$ converges for all $|x^2|<1$ that the solution for x considered as a function of x^2 is regular for all $|x^2|<1$ and all finite values of τ, provided x does not become infinite for some value of τ. In the solutions (52) and (54) τ enters as a parameter and takes only real values. Therefore, if x does not become infinite for any real τ on the interval 0 to 2π, the solution (52), being periodic with the period 2π, converges for all $|x^2|<1$ and all finite real values of τ.

In order to establish the result that x does not become infinite for a real value of τ, we shall show that the integral

$$I=\int_0^\infty \frac{dx}{Q(x^2)\sqrt{(1-x^2)(1-x^2x^2)}}$$

is not real. It is necessary to specify the path of integration from 0 to ∞. We shall integrate from zero to $1-\varepsilon$ along the real axis; then along the circle with radius ε and center at 1 to the line joining 1 with the point $1/x$, which will in general be complex; then along the line from 1 to $1/x$ to a distance ε from $1/x$; then along the small circle ε around $1/x$ to the line from $1/x$ to ∞ whose inclination to the axis of reals is the argument of $1/x$; then along this line to ∞. This may be indicated by

$$I=I_1+I_2+I_3+I_4+I_5=\int_0^1+\int_{c(1)}+\int_1^{1/\kappa}+\int_{c(1/\kappa)}+\int_{1/\kappa}^\infty, \tag{61}$$

where $I_1, \ldots, I_5$ represent the integrals (61) in order. It is easy to show that $\lim_{\varepsilon=0} I_2=\lim_{\varepsilon=0} I_4=0$. The transformation $x=1/x\xi$ throws I_5 into the form I_1.

* See Appell and Lacour, *Fonctions Elliptiques*, Chap. V.

Therefore we have

$$I = 2\int_0^1 \frac{dx}{Q(x^2)\sqrt{(1-x^2)(1-x^2x^2)}} + \int_1^{1/\kappa} \frac{dx}{Q(x^2)\sqrt{(1-x^2)(1-x^2x^2)}}. \quad (62)$$

By the definition of $Q(x^2)$ we have

$$\int_0^1 \frac{dx}{Q(x^2)\sqrt{(1-x^2)(1-x^2x^2)}} = \frac{\pi}{2}\frac{Q(x^2)}{Q(x^2)} = \frac{\pi}{2},$$

which is real whatever the value of x^2. It is one-fourth of one period of the elliptic function. The other integral is one-half of the other period, and therefore it can not be real, since the ratio of the two periods can not be real.

Now consider any other path of integration to infinity. If none of the branch-points ± 1, $\pm 1/x$ is included between the two paths of integration, the value of the integral is the same. Suppose only the branch-point $+1$ is included between the two paths. Then the new path can be deformed, without crossing a branch-point, into the old one plus an integral from zero to $+1$ and back, after making the circuit of this point. Hence, in this case the value of the integral is $4I_1 + I_3$, which is not real. Suppose only the branch-point $1/x$ is included between the two paths. The new path can be deformed, without crossing a branch-point, into the old one plus an integral from ∞ to $1/x$ and back, after a circuit around $1/x$, along a line whose inclination to the axis of reals is the argument of $1/x$. Hence, in this case the value of the integral is $2I_1 + I_3 - 2I_1$, which is not real. It is easy to see that the integral from zero to infinity along any path is $2I_1 + I_3 + 2m_1 I_1 + 2m_3 I_3$, where m_1 and m_3 are positive or negative integers, or zero. This sum can not be real and it follows that x can not become infinite for any real finite τ. *Therefore, the series* (52) *converges for all real finite values of* τ, *provided* $|x^2| < 1$.

An analogous discussion proves the same result for the expansions of the cosine amplitude and the delta amplitude functions.

THE UNIVERSITY OF CHICAGO, *April* 27, 1911.

On Cubic Birational Space-Transformations.

By Hilda P. Hudson.

In 1871 Cremona* gave a method of determining all the birational transformations in which a given surface becomes a plane, when one representation of it on a plane is known; he added a great many examples of cubic transformations, but did not aim at completeness. For the non-singular cubic surface there exist seven transformations, which have been given by Loria† and Sturm.‡ This paper deals with singular cubic surfaces.

The method is briefly as follows. It is required to discover all the homaloidal families of cubic surface (ϕ_1) containing a given member ϕ, and having certain singularities in common with it. Suppose ϕ represented on a plane α by an auxiliary transformation T; all the cubic families here concerned have at least one common node, and T is taken to be projection from the first of these. Then to the family of curves of intersection of ϕ with (ϕ_1) there corresponds a homaloidal family of proper or degenerate curves in the plane α, satisfying certain conditions; and conversely, to every such plane family there corresponds a homaloidal family (ϕ_1), each member of which contains a certain fixed fundamental system H consisting of curves, simple points, points of contact of various orders and multiple points. The different plane homaloidal families of the earlier degrees are known,§ and by examining each in turn we are led back to all the existing families (ϕ_1) of the required kind.

The following investigation treats of cubic surfaces having 1, 2, 3, 4 nodes or a nodal line; there are further subdivisions according as the tangent cone at any node has a fixed part of order 0, 1 or 2, presenting 4, 7 or 9 conditions to the surface.

* *Rend. R. Ist. Lomb.*, (2) IV, pp. 269, 315; cited as C. See also *Math. Ann.*, IV, p. 213.

† *Atti Acc. Sci. Torino*, XXVI, p. 197; cited as L.

‡ *Die Lehre v. d. geom. Verwandtschaften*, Leipzig, 1909, IV, p. 360; cited as S.

§ Cremona, *Mem. dell' Acc. di Bologna*, (2) V, p. 117. Cayley, *Proc. Lond. Math. Soc.*, III, p. 127 = *Coll. Math. Papers*, VII, p. 189.

I. *One Node p.*

(i) Let p be a cnicnode. Then the intersection of ϕ, ϕ_1 is a $C_9 p^4$.

[I follow Cremona in using a suffix to show the order of the curve C; the small letters which immediately follow to denote points on it, or numerals to denote the number of points; and an index to show the multiplicity of each point. Where it is necessary to distinguish between branches through a binode touching the fixed and variable planes respectively (see below), this is shown by a suffix to the small letter; *e. g.*, $C_5 p_f^2 p_v$ is a quintic curve with three branches through the binode p, two touching the fixed plane and one the variable plane.]

The projection of this $C_9 p^4$ is a plane $C_5 abcdef$, where these points are the fundamental points of T, and are the traces on α of the six rays, or straight lines through p lying on ϕ; these six points lie on a conic, the trace on α of the tangent cone at p. This plane curve must be further restricted so as to become the generic curve of a homaloidal family of order 5 or less; the possible families are:

$$C_1, \quad C_2 3, \quad C_3 1^2 4, \quad C_4 3^2 3, \quad C_4 1^3 6, \quad C_5 6^2, \quad C_5 1^3 3^2 3, \quad C_5 1^4 8,$$

the figures showing the fixed base points of the families. When the family is of order less than 5, the C_5 degenerates into the variable member of the proper family and a fixed part passing through all of the six fundamental points of T which are not base points of the variable part.

There is a homaloidal family (ϕ_1) corresponding to each way of arranging this plane family and dividing the fundamental points of T between the fixed and variable parts; the other base points of the variable part give isolated points of the fundamental system H of (ϕ_1). If this variable part has a second branch through a fundamental point of T, then H contains the corresponding ray; the remaining fixed curves in H correspond to the fixed plane curves, and the former have as many branches through p as the latter have intersections (other than fundamental points) with the conic *abcdef*.

If one of the fixed plane curves is this conic, the surfaces (ϕ_1) contain all points adjacent to p on ϕ; *i. e.*, they have the same tangent cone; this possibility will be considered separately later.

If the fixed curves in H have three or more branches through p, then the existence of the node is a necessary consequence of the passage of the surface through these curves, and we have a particular case of a transformation of a non-singular surface, in which the fundamental curve has acquired a triple point. No further attention is here paid to such cases.

(ii) Let p be a binode. The conditions which a binode imposes on the coefficients of a general cubic surface are not linear unless one tangent plane at the node is fixed. Three rays lie in each plane; the C_9 of intersection of ϕ, ϕ_1 projects into a plane $C_4 def$, where d, e, f lie in the variable plane.

If the fixed curves in H have three or more branches touching the fixed plane at the node, then the passage of ϕ through H causes the degeneration of the tangent cone; if the fixed curves have two or more branches touching the other plane, this becomes fixed, giving a case treated below; the same happens if the straight line def in the projection is fixed.

The varieties of binode, called by Cayley B_4, B_5, B_6 (in which the edge, or intersection of tangent planes, is a ray), impose conditions which are not linear unless either the edge, or the other two rays in the fixed plane, be fixed. Either supposition leads merely to particular cases of other transformations.

(iii) Let p be a cnicnode with a fixed tangent cone. The projection of the intersection is any plane cubic. This includes the case of a binode with both tangent planes fixed, or any kind of unode.

II. *Two Nodes p, q.*

(i) Let p, q be cnicnodes. The intersection of ϕ, ϕ_1 consists of $pq + C_8 p^3 q^3$; the latter part projects into a plane $C_5 a^3 bcde$, for here two rays coincide in pq (the word "ray" meaning throughout a line through the first node p). Corresponding to each branch of a plane curve through a, there is a branch of a twisted curve through q. All cases in which the fixed curve has two branches through one node can be neglected as giving particular cases of previous transformations.

(ii) Let p be a binode. Since pq lies on the surface, it must lie on either the fixed or the variable tangent plane at p.

First let it lie on the variable plane. The projection is $C_4 a^3 b$, where a, a, b are the traces of the three rays in the variable plane, two of which coincide in pq. If now the curve in H has one branch through p touching the variable plane, this plane becomes fixed, and the whole tangent cone at p is fixed (see below). The line ab cannot be part of the plane curve, for the plane pab meets ϕ in pq, pq, pb only.

(iii) Next let p be a binode, and let q lie on the fixed tangent plane. Then this plane touches the surfaces all along pq, which counts twice in the intersection. The projection of the remaining C_7 is $C_4 a^2 def$. The tangent cone at q touches the fixed plane, and one other fixed curve through q would make

that node a necessity; therefore the fixed plane curve cannot pass through a. But one other fixed curve through p touching the fixed plane would cause the degeneration of the tangent cone at p. It follows that there is no fixed curve.

(iv) Let p be a cnicnode with a fixed tangent cone. Then the surfaces touch the cone all along pq, which counts twice in the intersection. The projection of the remaining C_7 is C_3a^2. Again, the fixed part cannot pass through a.

(v) Let p, q both be binodes. One tangent plane at each node contains pq, and these must be the same plane, osculating the surfaces all along pq, counting three times in the intersection and causing the degeneration of each tangent cone.

(vi) Let p be a cnicnode with a fixed tangent cone, and let q be a binode. Then the fixed plane at q contains pq, which counts three times in the intersection. The surfaces touch but do not osculate the fixed plane all along pq. The projection of the remaining C_6 is C_3a^2, with one fixed tangent at the node.

(vii) If both tangent cones are fixed, the projection is C_3a^3; no such homaloidal family exists.

III. *Three Nodes p, q, r.*

(i) Let p, q, r be cnicnodes. The intersection consists of $pq + pr + qr + C_6p^2q^2r^2$; the projection of the last part is $C_4a^2b^2cd$, for here two pairs of rays coincide in pq, pr respectively. Since any fixed branch through any node makes that node a necessity, we need only consider the cases in which a^2b^2 lie on the variable plane curve.

(ii) Let p be a binode. Then the fixed plane must contain one other node, say q, and the variable plane contains r. The intersection contains pr, qr and a line of contact pq, making the node q a necessity.

(iii) Let p be a cnicnode with a fixed tangent cone. Then the intersection contains qr and two lines of contact pq, pr, making the nodes q, r a necessity.

All other assignments of fixed planes or cones fail in the same way to give new transformations.

IV. *Four Nodes.*

The system is homaloidal without further condition, giving a particular case of transformation No. 2.

V. *A Nodal Line.*

A nodal straight line counts four times in the intersection. If the vertex of projection is any point p on the line, the remaining C_5 projects into C_5a^4bc, where b, c are the traces of the two generators through p. Or if T is Cremona's quadro-cubic transformation, the plane curve is C_4a^3.

Various particular cases, but no new transformations, arise from considering the cases in which the simple directrix coincides with the nodal line, when there is contact of one or both of the sets of sheets with a fixed plane or quadric all along the line.

VI.

In the following list the transformations given by Cremona and Sturm are included for the sake of completeness. The transformations are arranged in the first place according to the order of the reverse transformation, secondly according to the number of nodes, and finally according to the degree of the fixed parts (if any) of the tangent cones.

No.	Order.	Isolated points in H.	Multiple points; curves in H; (authorities, see references at end of list).
1	3–2	none.	nodal line; 3 generators; (C., S., Cayley, Noether).
2	3–3	"	no node; C_6; (C., L., S., Cayley, Noether, De Paolis, Kantor, Berry, Beloch).
3	"	"	node; $C_6 p^2$; (C., S., Noether, De Paolis, Del Pezzo, Kantor).
4	"	"	node, fixed cone; $C_6 p^4$; (C.).
5	"	2 points.	nodal line; 2 generators; (C., S., De Paolis, Del Pezzo, Kantor, Berry).
6	3–4	1 point.	no node; C_5; (C., L., S.).
7	"	"	node; $C_5 p^2$; (Berry, Del Pezzo).
8	"	"	binode; $C_5 p_f^2 p_v$.
9	"	"	node, fixed cone; $C_3 p^2$ + 2 rays; (Berry).
10	"	"	node p, fixed cone, binode q; $C_2 p q_v$ + line of osculation pq, surfaces touch plane.
11	"	4 points.	nodal line; 1 generator; (C., S.).
12	3–5	2 points.	no node; $C_3 + C_1$; (C., L., S.).
13	"	"	node; $C_4 p$; (C.).
14	"	"	" $C_3 p$ + ray.
15	"	"	binode; $C_4 p_f^2$.
16	"	"	" $C_2 p_f$ + ray$_f$ + ray$_v$.
17	"	"	node, fixed cone; $C_3 p^2$ + ray.
18	"	"	" " 2 rays + ray of contact, surfaces touch quadric.
19	"	"	2 nodes; $C_3 pq + pq$; (C.).
20	"	"	binode p, node q; $C_2 p_f q$ + ray$_f$ + pq.
21	"	"	node p, fixed cone, node q; 2 rays + line of contact pq, surfaces touch plane.
22	"	"	node p, fixed cone, binode q; ray + line of osculation pq, surfaces touch plane.
23	"	1 point of contact.	no node; C_4; (C., L., S.).
24	"	"	node; $C_4 p^2$.
25	"	"	binode; $C_3 p_f p_v$ + ray$_f$.
26	"	"	node, fixed cone; 4 rays.

No.	Order.	Isolated points in H.	Multiple points; curves in H; (authorities).
27	3–5	6 points.	nodal line; (C., S.)
28	3–6	3 points.	no node; $C_1 + C_1$ of contact; (C., L., S.).
29	"	"	node; plane C_3; (C.).
30	"	"	" C_2 + ray.
31	"	"	" C_1 + ray of contact; (S.).
32	"	"	" ray of osculation, surfaces touch quadric.
33	"	"	binode; $C_2 p_f$ + ray$_v$.
34	"	"	node, fixed cone; $C_3 p^2$.
35	"	"	" " ray + ray of contact, surfaces touch quadric.
36	"	"	2 nodes; $C_2 p + pq$; (C.).
37	"	"	binode p, node q; $C_2 p_f q + pq$.
38	"	"	" " 2 rays$_f$ + pq.
39	"	"	node p, fixed cone, node q; ray + line of contact pq, surfaces touch plane.
40	"	"	node p, fixed cone, binode q; line of osculation pq, surfaces touch plane.
41	"	1 point of contact, 1 point.	no node; $C_1 + C_1 + C_1$; (L., S., Bonicelli).
42	"	" "	node; $C_3 p$.
43	"	" "	" C_1 + 2 rays.
44	"	" "	binode; $C_3 p_f p_v$.
45	"	" "	" $C_2 p_f$ + ray$_f$.
46	"	" "	" ray$_v$ + ray$_f$ of contact, surfaces touch quadric.
47	"	" "	node, fixed cone; 3 rays.
48	"	" "	2 nodes; $C_1 q + pq$ + ray.
49	"	" "	binode p, node q in fixed plane; ray$_v$ + line of contact pq, surfaces touch plane.
50	"	" "	3 nodes; $pq + pr + qr$; (C.).
51	"	1 point of osculation.	no node; C_3; (C., L., S.).
52	"	"	node; $C_3 p^2$.
53	3–7	4 points.	node, fixed cone; ray of contact, surfaces touch quadric.
54	"	"	2 nodes; $C_1 + pq$; (C.).
55	"	"	binode p, node q; ray$_f$ + pq.
56	"	"	node p, fixed cone, node q; line of contact pq, surfaces touch plane.
57	"	1 point of contact, 2 points.	node; C_2; (C.).
58	"	" "	binode; $C_2 p_f$.
59	"	" "	node, fixed cone; 2 rays.
60	"	2 points of contact.	node; C_1 + ray; (C.).
61	"	"	binode; ray$_f$ + ray$_v$.
62	"	"	" ray$_f$ of contact, surfaces touch quadric
63	"	"	2 nodes; ray + pq.
64	"	"	binode p, node q in fixed plane; line of contact pq, surfaces touch plane.
65	"	1 point of osculation, 1 point.	node; $C_2 p$; (C.).
66	"	" "	binode; 2 rays$_f$.
67	3–8	5 points.	binode p, node q; pq.
68	"	1 point of contact, 3 points.	node, fixed cone; ray.
69	"	2 points of contact, 1 point.	binode; ray$_v$.
70	"	1 point of osculation, 2 points.	node; C_1; (C.).
71	"	" "	binode; ray$_f$.
72	3–9	1 point of contact, 4 points.	node, fixed cone.
73	"	3 points of contact.	binode.
74	"	1 point of osculation, 3 points.	"
75	"	1 point of contact of 3rd order, 2 points.	node; (C., S., Beloch).

REFERENCES.

(See foot-note, p. 203.)

Beloch, *Ann. di Mat.*, (3) XVI, p. 64.

Berry, *Camb. Phil. Trans.* Cases of No. 1 are used in Vol. XIX, p. 281; of No. 2, XIX, p. 255; of No. 5, XIX, pp. 275, 282; XX, pp. 109, 111; of No. 7, XVIII, p. 334; XIX, p. 279; of No. 9, XIX, p. 281.

Bonicelli, *Giorn. di Mat.*, XL, p. 184.

Cayley, *Coll. Math. Papers*, VII, p. 229.

Del Pezzo, *Rendic. R. Acc. Napoli*, (3) II, p. 288. Cases of Nos. 3, 7 are included in § 28, and of No. 5 in § 3.

De Paolis, *Rendic. R. Acc. Linc.*, (4) I, pp. 735, 754. Cases of No. 2 are included in Cl. I, 2, II, 1, III, 2 (1°, 2°), 3 (1°), 4 (1°, 3°); of No. 3 in I, 1 (1°, 2°); of No. 5 in III, 1, 3 (2°).

Kantor, AMER. J. OF MATH., XIX, p. 1.

Noether, *Math. Ann.*, III, p 547.

VII.

All the quartic homaloidal families of Nos. 6–11 have multiple lines. Those of No. 11 have a fundamental system consisting of a triple straight line, three generators and a point; they are described by Cremona (*Ann. di Mat.*, 2 (V), p. 158). Those of No. 6 have a double conic and simple quintic; they are described by Sturm (*loc. cit.*, p. 371).

The C_5p^2 of No. 7 meets four rays, pc, pd, pe, pf, and with these makes up the total intersection of ϕ with a cubic cone. The line l on ϕ which meets the remaining rays pa, pb is a 3-secant of C_5 and with it makes up the total intersection of ϕ with a quadric, which is the locus of l as ϕ varies within the homaloidal family. If ϕ contains any one other point of l, it contains the whole line; *i. e.*, all the points of l present the same condition to ϕ and correspond to the same point in the second space, which is therefore a point of a fundamental line of the quartic family (ϕ_4'); this fundamental line is simple because l is linear, and linear because ϕ contains only one line l. The same argument applied to the rays $pc, \ldots, pf$ shows that (ϕ_4') also has a fundamental simple quartic. Now if t is the fundamental isolated point on ϕ, the plane tpa meets ϕ in pa and a conic through t, p and three points on C_5; this conic presents only one condition to ϕ, and since ϕ contains two such conics, there is a fundamental nodal conic on (ϕ_4'). This with the simple quartic and straight line makes up the fixed part C_{13}' of the intersection of two of the quartic surfaces.

The $C_5 p_f^2 p_v$ of No. 8 meets one ray pf in the variable plane and with it makes up the total intersection of ϕ with a quadric cone. This ray gives rise to a fundamental straight line on (ϕ_4'); but all the cubic surfaces which contain pf have the same variable tangent plane at p, viz., that which contains pf and the tangent to the third branch of C_5 at p; the line on (ϕ_4') is therefore a line of contact. The three rays in the fixed plane give rise to a simple fundamental cubic on (ϕ_4'); and the planes through t and the remaining two rays to a nodal conic. The C_{13}' of fixed intersection of (ϕ_4') consists therefore of a conic counted four times, a straight line counted twice and a cubic.

In No. 9, the four rays which are not fixed give rise to a simple fundamental quartic on (ϕ_4'); the straight line on ϕ which meets the two fixed rays gives rise to a simple straight line; and the two planes through t and the fixed rays lead separately to nodal straight lines.

In No. 10, there are three rays at p which do not coincide in pq; these give rise to a simple cubic. The one other ray at q in the fixed plane gives rise to a simple straight line. The plane tpq meets ϕ in pq and a conic with a fixed tangent at p and also passing through t, q, which gives rise to a nodal straight line. All the cubic surfaces which pass through this conic have the same variable tangent plane at q, and the two sheets of ϕ_4' both touch the same fixed plane all along the nodal line, which counts eight times in the intersection. To make up the C_{13}' there is needed another simple straight line. This corresponds to pq; for pq counts three times in the intersection of ϕ, ϕ_1, and one of the three coincident lines may be thought of as separated from the others, and not meeting the variable C_4 of intersection of ϕ, ϕ_1, but constituting a fundamental line of the special kind.

VIII. *Composition.*

Of the 75 transformations enumerated above, there are 26 which can be compounded of quadro-quadric transformations (Nos. 1, 5, 11, 16, 20, 21, 22, 27, 30, 31, 33, 36, 37, 38, 41, 43, 45, 46, 48, 49, 50, 54, 55, 57, 58, 61); 16 others can be compounded of quadro-quadric transformations and others of the list (Nos. 9, 10, 17, 18, 26, 34, 35, 39, 40, 47, 59, 60, 63, 64, 65, 66); the remaining 33 are independent from the point of view of composition.

BRYN MAWR COLLEGE, PA., *April*, 1911.

*On the Definition of Multiplication of Irrational Numbers.**

By Oswald Veblen.

In developing the theory of the irrational real numbers it is usual to start with the set of all rational numbers and to let each cut in this system define an irrational number. A cut (x, y) may be defined as a pair of sets of rational numbers $[x]$ and $[y]$ such that every rational number is either an x or a y and such that $x < y$ for every x and every y. The order relations of the irrational numbers are determined by the definition: $(x_1, y_1) < (x_2, y_2)$ if and only if there exist an x_2 and a y_1 such that $y_1 < x_2$. The sum of the two numbers defined by cuts (x_1, y_1) and (x_2, y_2) is defined as the number associated with the cut $(x_1 + x_2,\ y_1 + y_2)$. The product of two positive irrationals, (x_1, y_1) and (x_2, y_2), is defined as the number associated with the cut $(x_1 x_2',\ y_1 y_2)$, where $[x_2']$ is the set of all positive numbers in $[x_2]$; the product of two negative irrationals and of a positive and a negative irrational is defined similarly.

These definitions of addition and multiplication are stated in the form of arbitrary rules. They are, however, designed so that all the algebraic laws of operation (associative, distributive, commutative, etc.)† shall be satisfied, so that whenever two rational numbers are added or multiplied the result is the ordinary sum or product of the two rationals, and also so that when a rational number a is added or multiplied to an irrational (x, y) the results are the irrational numbers defined by the cuts $(a + x,\ a + y)$ and $(a x,\ a y)$. The question arises: *Is it possible to define the addition and multiplication of irrationals so as to satisfy these requirements, and yet so as to give a different result from the ordinary one when irrationals are combined with irrationals?*

* Presented to the American Mathematical Society, Sept. 13, 1911.

† Of course, if we were to require that addition and multiplication shall preserve order relations (*e. g.*, $ab > 0$, whenever $a > 0$, $b > 0$, etc.), it is obvious that the ordinary definitions would result. But the conditions we are inquiring about merely require that order relations shall be preserved when rational numbers are combined with rationals or irrationals, and our question is whether it is possible to infer anything about the case in which irrationals are combined with irrationals.

The answer is that it is possible. In fact, we can impose the further requirement that addition of irrationals shall proceed by the rule

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

This result depends on the theorem * that there exists a well-ordered sequence of irrational numbers

$$a_1, a_2, \ldots, a_\omega, a_{\omega+1}, \ldots, \tag{1}$$

such that every real number is expressible uniquely in the form

$$\alpha_0 + \alpha_1 a_{i_1} + \alpha_2 a_{i_2} + \ldots + \alpha_n a_{i_n}, \tag{2}$$

where there are a finite number of terms and the α's are rational.

The ordinary rule for multiplying irrationals described in the first paragraph determines a multiplication table for the a's, that is to say, a set of rules of the form

$$a_i a_j = \beta_0 + \beta_1 a_{i_1} + \beta_2 a_{i_2} + \ldots + \beta_m a_{i_m}. \tag{3}$$

The product of any two irrationals is obtained by expressing them in the form (2), multiplying according to the rule for multiplying polynomials and reducing the result according to the multiplication table.

Now suppose we denote by

$$a'_1, a'_2, a'_3, \ldots, a'_\omega, a'_{\omega+1}, \ldots \tag{4}$$

the a's arranged in another order of the same type as (1). Such an order would be obtained, for example, by interchanging a_1 and a_2 and leaving the other a's unaltered. Every real number is expressible uniquely in the form

$$\alpha_0 + \alpha_1 a'_{i_1} + \ldots + \alpha_n a'_{i_n}. \tag{5}$$

A multiplication table for the a''s is given by the rule

$$a'_i a'_j = \beta_0 + \beta_1 a'_{i_1} + \ldots + \beta_m a'_{i_m},$$

whenever

$$a_i a_j = \beta_0 + \beta_1 a_{i_1} + \ldots + \beta_m a_{i_m}.$$

A new rule for multiplying any two real numbers is now determined, namely: Express each in the form (5), multiply the two polynomials, and reduce according to the new multiplication table.

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* Hamel, *Mathematische Annalen*, Vol. LX, p. 459. Of course, this result is open to all the objections that can be made to the proposition that the continuum can be rearranged as a well-ordered set.

The number system determined in this fashion satisfies all the laws of operation of the ordinary real number system, for by letting each a' correspond to the a with the same subscript we set up a simple isomorphism between the two number systems. Moreover, the product of an irrational by a rational or of two rationals is the same in this system as in the ordinary one.

We may insure that the two number systems are distinct by selecting the a's in the first place so that $a_1 = \sqrt{2}$ and $a_2 = \sqrt{3}$, and then choosing the a''s so that $a_1' = a_2$.

It is not difficult to set up a number system in which both multiplication and addition of irrationals are modified. It is not possible, however, to obtain one in which multiplication is the same as in the ordinary system while the rule for addition is altered for the irrational numbers. For by the distributive law,

$$a + b = a\left(1 + \frac{1}{a}b\right).$$

Since we are assuming that addition of rationals to irrationals is unaltered, $\left(1 + \frac{1}{a}b\right)$ is given by the usual rule. Since multiplication is unaltered, $a\left(1 + \frac{1}{a}b\right)$ is determined by the usual rule. Hence, $a + b$ is the same as in the ordinary system.

A Correction.

The number system which has been constructed in the article above may be used to point out an omission in the article on "A Set of Assumptions for Projective Geometry" by myself and Professor J. W. Young in Vol. XXX of this Journal. On page 364 we state an Assumption of Continuity as follows:

C. If there exists any non-modular net of rationality, at least one point Q of some line l and at least one net of rationality R on l containing Q is such that associated with every singly open cut $K(Q)$ in R is a point X_k such that: 1) X_k is on l; 2) if two cuts $K_1(Q)$ and $K_2(Q)$ are distinct, the points X_{k_1} and X_{k_2} are distinct; 3) if two cuts $K_1(Q)$ and $K_2(Q)$ are projective, the points X_{k_1} and X_{k_2} form a homologous pair.

This, together with the assumptions of alignment and extension and the assumption that there is not more than one chain on a line, we stated to be sufficient to determine the real projective geometry.

If we construct an analytic projective space based on the number system described above, we find, however, that it satisfies all of these assumptions. This is obvious with the exception, perhaps, of the third part of Assumption C. If the net R of that assumption be taken as $R\,(0\,1\,\infty)$, any projectivity which transforms this net into itself may be written

$$x' = \frac{a\,x + b}{c\,x + d},$$

where a, b, c, d are rational. Such a transformation gives the same result in the number system based on the a's as in the ordinary one, and hence C, 3) is satisfied. Therefore, in order that our set of assumptions for projective geometry be categorical, Assumption C must be strengthened by a provision that order relations are preserved also by projective transformations among irrational points.

I have noted the following other errata in the paper on Projective Geometry:

Part of the proof of Theorem 14, page 355, is omitted; the part lacking is, however, well known.

On page 359, second line of Theorem 19, read "net" instead of "plane".

On the second line of Theorem 20 insert "of the net" after "(line)".

*Minimal Surfaces in Euclidean Four-Space.**

By Luther Pfahler Eisenhart.

Introduction.

By definition a minimal curve in Euclidean four-space, Σ_4, is one whose coordinates x, y, z, t satisfy the condition

$$dx^2 + dy^2 + dz^2 + dt^2 = 0.$$

Recently we have observed† that such a curve can be defined in terms of a unique parameter, so that its coordinates are expressible in the form

$$\left.\begin{aligned} x &= \phi - u\phi' + \psi', \quad & iy &= \phi - u\phi' - \psi', \\ z &= \psi - u\psi' - \phi', \quad & it &= -\psi + u\psi' - \phi', \end{aligned}\right\} \quad (1)$$

where $i = \sqrt{-1}$.

If we have two such minimal curves in Σ_4, the locus of the mid-points of all joins of points on the two curves is a two-dimensional spread in Σ_4 which, being analogous to a minimal surface in ordinary space as defined by Lie,‡ we call a *minimal surface in* Σ_4. This paper is a study of minimal surfaces in Σ_4 and sets forth many results analogous to the established theorems concerning ordinary minimal surfaces.

As introductory to the subsequent study, § 1 deals with minimal curves in Euclidean four-space defined in the form (1), and in particular the transformations of u, ϕ, ψ corresponding to rotation of the coordinate axes and to reflection of the curve in a hyperplane. The section closes with a discussion of certain plane minimal curves.

In § 2 certain properties of the general two-dimensional spread in four-space are stated, generally without proof, and the general equations are given

* Read before the American Mathematical Society, September 12, 1911.

† "A Fundamental Parametric Representation of Space Curves," *Annals of Mathematics*, Series II, Vol. XIII (1911), pp. 17–35.

‡ "Beiträge zur Theorie der Minimalflächen," *Mathematische Annalen*, Vol. XIV (1879), p. 339.

for future use. These results are taken from memoirs by Karl Kommerell* and Levi,† and the reader is referred to these papers for the proofs of the theorems and formulas.

A minimal surface may be defined as one for which the first variation of the area-integral vanishes. In § 3 this definition is shown to be equivalent to the one previously used, and the nature of the characteristic is observed. With the introduction in § 4 of the normal parameters and functions for the minimal curves on the surface the formulas assume elegant forms, from which follows a direct discussion of the properties of the surface in the neighborhood of a point. The changes of parameters produced by the rotation of the surfaces and certain results of this investigation are studied in § 5.

Kommerell‡ has studied the surfaces in four-space defined by an equation of the form

$$z + it = f(x + iy),$$

where f is analytic in the domain considered, and he has shown that these surfaces are minimal. They are considered briefly in § 6 from the point of view of this paper.

When the normal functions ϕ, ψ and ϕ_0, ψ_0 are algebraic functions of u and u_0 respectively, the minimal surface is algebraic. The proof of the converse is given in § 7. It is shown in § 8 that with each minimal surface there are associated a group of applicable minimal surfaces which are analogous to the *associated minimal surfaces* of ordinary space. We use the same term for these surfaces and obtain a number of theorems which set forth other relations between the surfaces of such a family.

§ 1. *Minimal Curves. Rotations and Reflections.*

From equations (1) we have

$$\frac{dx + i\,dy}{dz + i\,dt} = -\frac{dz - i\,dt}{dx - i\,dy} = u. \tag{2}$$

When a minimal curve is defined in terms of a general parameter, equations (2) enable one to determine the *normal parameter* u in terms of which the equations of the curve assume the form (1). From its method of derivation this parameter

* "Die Krümmung der zweidimensionalen Gebilde im ebenen Raum von vier Dimensionen," Thesis, Tübingen, 1897; also, "Riemann'sche Flächen im ebenen Raum von vier Dimensionen," *Mathematische Annalen*, Vol. LX (1905), pp. 548–596.

† "Saggio sulla theoria delle superficie a due dimensioni immerse in un iperspazio," Thesis, Pisa, 1905.

‡ *Loc. cit.*, p. 548.

u is necessarily unique. Moreover, the *normal functions* ϕ and ψ are determined by the quadratures

$$\left.\begin{aligned} \frac{d\phi}{du} &= -\tfrac{1}{2}(z+it), \\ \frac{d\psi}{du} &= \tfrac{1}{2}(x-iy). \end{aligned}\right\} \quad (2')$$

It is important for us to study the effect upon the form (1) of a rotation of coordinate axes. It is a known fact that the most general rotation in Euclidean four-space may be decomposed into two *simple rotations* each of which leaves a plane two-spread unaltered point for point, and these two invariant planes are perpendicular to one another with only one point in common.* The equations defining the change of coordinates due to a *simple rotation* may be written

$$\left.\begin{aligned} Bx_1 &= (1+f^2-a^2+g^2-b^2+h^2-c^2)x + 2(a-bh+cg)y \\ &\quad + (b-cf+ah)z + 2(c-ag+bf)t, \\ By_1 &= 2(-a+cg+ah)x + (1+f^2-a^2+b^2-g^2+c^2-h^2)y \\ &\quad + 2(h+fg-ab)z + 2(-g+hf-ac)t, \\ Bz_1 &= 2(-b-cf+ah)x + 2(-h+fg-ah)y \\ &\quad + (1+g^2-f^2+c^2-h^2+a^2-b^2)z + 2(f+gh-bc)t, \\ Bt_1 &= 2(-c+bf-ag)x + 2(g+fh-ac)y \\ &\quad + 2(-f+gh-bc)z + (1+h^2-c^2+a^2-f^2+b^2-g^2)t, \end{aligned}\right\} \quad (3)$$

where a, b, c, f, g, h, B are constants satisfying the conditions

$$\left.\begin{aligned} af+bg+ch &= 0, \\ B &= 1+a^2+b^2+c^2+f^2+g^2+h^2. \end{aligned}\right\} \quad (4)$$

In this rotation the invariant plane is defined by

$$\left.\begin{aligned} (a^2+b^2+c^2)x - (a-bh+cg)y - (b-cf+ah)z \\ - (c-ag+bf)t = 0, \\ (a-cg-bh)x + (a^2+g^2+h^2)y - (h+fg-ab)z \\ + (g-hf+ac)t = 0. \end{aligned}\right\} \quad (5)$$

The simple rotation, whose invariant plane is perpendicular to (5) and has only one point in common with it, is defined by equations similar to (3) in which the constants a', b', c', f', g', h' are given by

$$a' = ef, \quad b' = eg, \quad c' = eh, \quad f' = ea, \quad g' = eb, \quad h' = ec, \quad (6)$$

where $e = \pm 1$.

* Cf. Cole, "On Rotations in Space of Four Dimensions," AMERICAN JOURNAL OF MATHEMATICS, Vol. XII (1890), pp. 199–210; also Cayley, "Sur quelques propriétés des déterminants gauches," *Crelle*, Vol. XXXII (1846), pp. 119–123.

The minimal curve given by (1) may be defined by equations of the same form for x_1, y_1, z_1, t_1 in terms of three functions, u_1, ϕ_1, ψ_1. From (1) and (2) it follows that

$$u_1 = \frac{dx_1 + i\,dy_1}{dz_1 + i\,dt_1},$$
$$\phi_1 = \tfrac{1}{2}[x_1 + iy_1 - u_1(z_1 + it_1)],$$
$$\psi_1 = \tfrac{1}{2}[z_1 - it_1 + u_1(x_1 - iy_1)].$$

By means of (1) and (3) these may be reduced to

$$u_1 = \frac{[1 - i(a-f)]u + [(b-g) + i(h-c)]}{[-(b-g) + i(h-c)]u + [1 + i(a-f)]}, \tag{7}$$

and

$$\left.\begin{aligned} \phi_1 &= \frac{[1 - i(a+f)]\phi + [(g+b) + i(c+h)]\psi}{[-(b-g) + i(h-c)]u + [1 + i(a-f)]}, \\ \psi_1 &= \frac{-[(g+b) - i(c+h)]\phi + [1 + i(a+f)]\psi}{[-(b-g) + i(h-c)]u + [1 + i(a-f)]}. \end{aligned}\right\} \tag{8}$$

In order to obtain the equations of a general rotation, that is, one which does not leave a plane invariant, we must combine the foregoing with a similar transformation suggested by (6). We consider the two cases separately.

If we denote by u_2, ϕ_2, ψ_2 the functions arising from u_1, ϕ_1, ψ_1 when the constants have the values (6) with $e=1$, the equations of the transformation are readily obtained from (7) and (8). Applying the two transformations consecutively, we have

$$\left.\begin{aligned} u_2 &= u, \\ \phi_2 &= \frac{2}{B}\left\{\left[1 - \frac{B}{2} - i(a+f)\right]\phi + \left[g + b + i(c+h)\right]\psi\right\}, \\ \psi_2 &= \frac{2}{B}\left\{-\left[(g+b) - i(c+h)\right]\phi + \left[1 - \frac{B}{2} + i(a+f)\right]\psi\right\}. \end{aligned}\right\} \tag{9}$$

Proceeding in like manner with the case $e=-1$ in (6), we obtain

$$\left.\begin{aligned} u_2 &= \frac{\left[1 - \frac{B}{2} - i(a-f)\right]u + \left[b - g + i(h-c)\right]}{-\left[(b-g) - i(h-c)\right]u + \left[1 - \frac{B}{2} + i(a-f)\right]}, \\ \phi_2 &= \frac{B\phi}{U_1}, \qquad \psi_2 = \frac{B\psi}{U_1}, \end{aligned}\right\} \tag{10}$$

where U_1 denotes the denominator of the expression for u_2.

Conversely, it can be shown that the minimal curve whose normal parameter and functions are

$$u_1 = u, \qquad \phi_1 = \frac{a_1\phi + a_2\psi}{A}, \qquad \psi_1 = \frac{a_3\phi + a_4\psi}{A}, \tag{11}$$

where a_1, a_2, a_3, a_4 are constants and

$$A^2 = a_1 a_4 - a_2 a_3, \tag{12}$$

differs from the curve (2) by the rotation defined by

$$\left.\begin{aligned}
x_1 &= \frac{1}{2A}\left[(a_1+a_4)x + (a_1-a_4)iy + (a_2-a_3)z - e(a_2+a_3)it\right],\\
y_1 &= \frac{1}{2A}\left[(a_4-a_1)ix + (a_1+a_4)y - (a_2+a_3)iz + e(a_3-a_2)t\right],\\
z_1 &= \frac{1}{2A}\left[(a_3-a_2)x + (a_2+a_3)iy + (a_1+a_4)z + e(a_1-a_4)it\right],\\
t_1 &= \frac{1}{2A}\left[e(a_2+a_3)ix + e(a_2-a_3)y + e(a_4-a_1)iz + (a_1+a_4)t\right],
\end{aligned}\right\} \tag{13}$$

where $e = 1$.

In like manner the curve for which

$$u_1 = \frac{a_1 u + a_2}{a_3 u + a_4}, \qquad \phi_1 = \frac{A\phi}{a_3 u + a_4}, \qquad \psi_1 = \frac{A\psi}{a_3 u + a_4} \tag{14}$$

differs from (2) by the rotation defined by (13), where $e = -1$.

It should be remarked that a necessary and sufficient condition that the rotation be real is that a_1, a_4 and a_2, $-a_3$ be pairs of conjugate imaginaries, which condition is satisfied in the case of (9) and of (10).

Having thus considered the change in the normal parameter due to a rotation, we consider next the effect of a reflection in one of the coordinate hyperplanes. For example, the reflection in the plane $t = 0$ of the curve (1) gives a curve whose equations are the same as (2) except the last, which merely differs in sign. If we write the equations of the reflected curve in the same form as (1), but in terms of u_1, ϕ_1, ψ_1, it is readily found that*

$$\left.\begin{aligned}
u_1 &= \frac{\phi''}{\psi''}, \qquad \psi_1 = \frac{\psi'\phi'' - \phi'\psi''}{\psi''},\\
\phi_1 &= \phi - u\phi' + (u\psi' - \psi)\frac{\phi''}{\psi''}.
\end{aligned}\right\} \tag{15}$$

* This result coincides with the transformation from one normal parameter to the other of a twisted curve in three-space. Cf. *Annals of Mathematics*, *loc. cit.*, p. 19.

The reflections in the other coordinate hyperplanes lead to similar results. In a more general way one may find the transformations corresponding to a reflection in any hyperplane whatsoever.

A particular minimal curve which is of interest in the subsequent study is that for which

$$\phi = c\psi, \tag{16}$$

where c denotes a constant. In this case equations (1) reduce to

$$\left.\begin{aligned} x &= c\psi + (1 - cu)\psi', & iy &= c\psi - (1 + cu)\psi', \\ z &= \psi - (u + c)\psi', & it &= -\psi + (u - c)\psi'. \end{aligned}\right\} \tag{17}$$

From these equations we find that the curve lies in the plane which is the intersection of the two isotropic hyperplanes

$$\left.\begin{aligned} c(x - iy) + (z + it) &= 0, \\ x + iy - c(z - it) &= 0. \end{aligned}\right\} \tag{18}$$

For the minimal curve (17) equations (15) reduce to

$$u_1 = c, \qquad \phi_1 = 0, \qquad \psi_1 = 0.$$

Hence, the curve

$$\left.\begin{aligned} x &= c\psi + (1 - cu)\psi', & iy &= c\psi - (1 + cu)\psi', \\ z &= \psi - (u + c)\psi', & it &= \psi - (u - c)\psi', \end{aligned}\right\} \tag{19}$$

can not be defined by equations of the form (1). Conversely, it follows from (2) that a minimal curve whose equations can not be put in the form (1) lies in a plane whose equations are

$$x + iy - c(z + it) = a, \qquad c(x - iy) + (z - it) = b, \tag{20}$$

where a, b, c are constants.

§ 2. *Two-Dimensional Spreads in Four-Space and Their Properties.*

The general two-dimensional spread or surface in four-space may be defined by equations of the form

$$x = \phi_1(u, v), \quad y = \phi_2(u, v), \quad z = \phi_3(u, v), \quad t = \phi_4(u, v), \tag{21}$$

where the functions ϕ_i are analytic within the domain considered.

The tangent at the point (u, v) to the curve on the surface determined by a relation between u and v has equations of the form

$$\left.\begin{aligned} X - x &= \lambda(x_{10}\,du + x_{01}\,dv), & Y - y &= \lambda(y_{10}\,du + y_{01}\,dv), \\ Z - z &= \lambda(z_{10}\,du + z_{01}\,dv), & T - t &= \lambda(t_{10}\,du + t_{01}\,dv), \end{aligned}\right\} \tag{22}$$

where λ denotes a function of u and v and

$$x_{ij} = \frac{\partial^{i+j} x}{\partial u^i \, \partial v^j}.$$

If we consider these equations for all values of du and dv, they define a plane which we call the *tangent plane* at the point. This tangent plane is the plane common to the one-parameter family of tangent hyperplanes whose equation is

$$a(X-x)+b(Y-y)+c(Z-z)+d(T-t)=0, \tag{23}$$

with

$$ax_{10}+by_{10}+cz_{10}+dt_{10}=0, \qquad ax_{01}+by_{01}+cz_{01}+dt_{01}=0. \tag{24}$$

The plane which is normal to the tangent plane at (u, v) and has only this point in common with it is called the *normal plane* to the surface at the point. Its equations are

$$\left.\begin{aligned}(X-x)x_{10}+(Y-y)y_{10}+(Z-z)z_{10}+(T-t)t_{10}=0,\\ (X-x)x_{01}+(Y-y)y_{01}+(Z-z)z_{01}+(T-t)t_{01}=0.\end{aligned}\right\} \tag{25}$$

The hyperplane whose equation is a linear combination of these equations is called a *normal hyperplane.* There is a one-parameter family of normal hyperplanes at each point of a surface.

The normal planes at the points (u, v) and $(u+du, v+dv)$ meet in the point determined by the planes (25) and

$$\left.\begin{aligned}\Sigma(X-x)(x_{20}\,du+x_{11}\,dv)-\Sigma x_{10}^2\,du-\Sigma x_{10}x_{01}\,dv=0,\\ \Sigma(X-x)(x_{11}\,du+x_{02}\,dv)-\Sigma x_{01}x_{10}\,du-\Sigma x_{01}^2\,dv=0.\end{aligned}\right\} \tag{26}$$

Solving these four equations for $X-x$, $Y-y$, ..., $T-t$, we have

$$\left.\begin{aligned}X-x&=\frac{(ED'_x-FD_x)\,du^2+(ED''_x-GD_x)\,du\,dv+(FD''_x-GD'_x)\,dv^2}{e\,du^2+2f\,du\,dv+g\,dv^2},\\ Y-y&=\frac{(ED'_y-FD_y)\,du^2+(ED''_y-GD_y)\,du\,dv+(FD''_y-GD'_y)\,dv^2}{e\,du^2+2f\,du\,dv+g\,dv^2},\\ Z-z&=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ T-t&=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\end{aligned}\right\} \tag{27}$$

where

$$D_x=\begin{vmatrix}y_{10}&z_{10}&t_{10}\\ y_{01}&z_{01}&t_{01}\\ y_{20}&z_{20}&t_{20}\end{vmatrix}, \quad D'_x=\begin{vmatrix}y_{10},&\cdots\\ y_{01},&\cdots\\ y_{11},&\cdots\end{vmatrix}, \quad D''_x=\begin{vmatrix}y_{10},&\cdots\\ y_{01},&\cdots\\ y_{02},&\cdots\end{vmatrix}, \tag{28}$$

and $D_y, \ldots, D''_t$ denote similar functions, and where

$$e=\begin{vmatrix}x_{10},&\cdots\\ x_{01},&\cdots\\ x_{20},&\cdots\\ x_{11},&\cdots\end{vmatrix}, \quad 2f=\begin{vmatrix}x_{10},&\cdots\\ x_{01},&\cdots\\ x_{20},&\cdots\\ x_{02},&\cdots\end{vmatrix}, \quad g=\begin{vmatrix}x_{10},&\cdots\\ x_{01},&\cdots\\ x_{11},&\cdots\\ x_{02},&\cdots\end{vmatrix}. \tag{29}$$

If dv/du be eliminated from equations (27), we find that all the points thus determined in the normal plane at the point (u, v) generate a conic, which Kommerell* has called the "characteristic". In certain ways this conic is the analogue of the Dupin indicatrix of an ordinary surface in the study of the geometry of the surface in the neighborhood of the point.

We have seen that through each point (u, v) of a surface S there passes a single parameter family of hyperplanes normal to S, and that these hyperplanes have in common a unique plane — the normal plane to the surface at the point. Each of these normal hyperplanes meets the surface in a curve whose center of first curvature lies in the normal plane. Kommerell† and Levi‡ have shown that the locus of these centers of curvature is the first pedal curve of the characteristic with respect to the point (u, v) of the surface, that is, a lemniscate. Hence there are four maxima and minima radii of normal curvature. They are called the *principal radii of normal curvature*, and the corresponding directions of the normal hyperplane are called the *principal directions*. The curves on the surface determined by these directions are called the *lines of curvature*.

Kommerell shows§ that the lines of curvature are characterized by the property that the corresponding point of the characteristic lies in the corresponding osculating plane of the curve. By means of this result it may be shown that the equation of the lines of curvature is

$$\begin{aligned}\Sigma (D_x\,du^2 + 2\,D'_x\,du\,dv + D''_x\,dv^2)\,[(E\,D'_x - F\,D_x)\,du^2 \\ + (E\,D''_x - G\,D_x)\,du\,dv + (F\,D''_x - G\,D'_x)\,dv^2] = 0,\end{aligned} \tag{30}$$

and that the principal radii of curvature are given by

$$\rho^2 = \frac{(E\,G - F^2)\,(E\,du^2 + 2\,F\,du\,dv + G\,dv^2)^2}{\Sigma\,(D_x\,du^2 + 2\,D'_x\,du\,dv + D''_x\,dv^2)^2}, \tag{31}$$

when the values dv/du from (30) are substituted in this equation. ‖

§ 3. *Definition of Minimal Surfaces.*

As in the case of ordinary surfaces, we put

$$E = \Sigma\,x_{10}^2, \qquad F = \Sigma\,x_{10}\,x_{01}, \qquad G = \Sigma\,x_{01}^2, \tag{32}$$

and

$$H = \sqrt{E\,G - F^2}, \tag{33}$$

where the summation extends to terms in x, y, z, t. The element of area of the

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* *Loc. cit.*, p. 22; p. 560.
† *Loc. cit.*, p. 50; p. 563.
‡ *Loc. cit.*, pp. 67, 68.
§ *Loc. cit.*, p. 37; p. 566.
‖ Cf. Levi, *loc. cit.*, p. 58.

surface is given by $H\,du\,dv$, and the area of a portion of the surface has the value

$$\sigma = \iint H\,du\,dv. \tag{34}$$

By definition a *minimal surface* is one which minimizes the integral (34) for a given contour curve. We proceed to the determination of the conditions to be satisfied by the equations of a surface in order that it be a minimal surface.

The first variation of (34) may be written

$$\begin{aligned}\delta\sigma &= \int\int \sum \left(\frac{\partial X}{\partial x_{10}}\delta x_{10} + \frac{\partial H}{\partial x_{01}}\,\delta x_{01}\right) du\,dv \\ &= -\int\int \sum \left[\frac{\partial}{\partial u}\frac{\partial H}{\partial x_{10}} + \frac{\partial}{\partial v}\frac{\partial H}{\partial x_{01}}\right] \delta x\,du\,dv.\end{aligned}$$

In order that $\delta\sigma$ be zero, the expression in the brackets, and similar expressions in y, z, t, must be zero. We note that

$$\frac{\partial H}{\partial x_{10}} = \frac{1}{H}(x_{10}G - x_{01}F), \qquad \frac{\partial H}{\partial x_{01}} = \frac{1}{H}(x_{01}E - x_{10}F).$$

Hence the vanishing of the four parentheses requires that x, y, z, t satisfy the equation

$$G\theta_{20} - 2F\theta_{11} + E\theta_{02} + \theta_{10}\left[\frac{\partial}{\partial v}\left(\frac{F}{H}\right) - \frac{\partial}{\partial u}\left(\frac{G}{H}\right)\right] + \theta_{01}\left[\frac{\partial}{\partial u}\left(\frac{F}{H}\right) - \frac{\partial}{\partial v}\left(\frac{E}{H}\right)\right] = 0.$$

Thus far the parametric curves have been perfectly general. Let us assume that they are the lines of length zero, or minimal lines, so that $E = G = 0$. In this case the above equation reduces to $\theta_{11} = 0$. Since the converse of this result is evidently true, we have the theorem:*

The general minimal surface in four-space is a surface of translation whose generating curves are minimal and conversely.

That is, the general minimal surface is defined by

$$\left.\begin{aligned} x &= f_1(u) + \phi_1(v), & y &= f_2(u) + \phi_2(v), \\ z &= f_3(u) + \phi_3(v), & t &= f_4(u) + \phi_4(v), \end{aligned}\right\} \tag{35}$$

where

$$\sum_i f_i'^2 = 0, \qquad \sum_i \phi_i'^2 = 0, \tag{36}$$

the primes indicating differentiation with respect to the argument.

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* It should be remarked that this result may be extended without change to the case of a minimal two-spread in n-space. Cf. Levi, *loc. cit.*, pp. 90–92.

For a minimal surface referred to its minimal lines and defined by (35), we have

$$E = G = 0, \qquad F = \Sigma f_i' \phi', \tag{37}$$

and from (28) and (29) it follows that

$$D_x' = D_y' = D_z' = D_t' = 0, \qquad e = g = 0. \tag{38}$$

From this last result and the form of equations (27) follow the theorems:

The asymptotes of the characteristic at any point on a minimal surface correspond to the directions of the minimal curves on the surface through the point.

The characteristic at each point of a minimal surface is an ellipse or a circle. (Cf. § 6.)

When the conditions (37) and (38) are satisfied, the equations (27) reduce to the form

$$X - x = \frac{F}{2f}\left(D_x'' \frac{dv}{du} - D_x \frac{du}{dv}\right). \tag{39}$$

In like manner from (28) we have

$$\left.\begin{aligned} D_x^2 &= F(2x_{10}x_{01} - F)\Sigma x_{20}^2 + F^2 x_{20}^2 + x_{10}^2\left(\frac{\partial F}{\partial u}\right)^2 - 2F\frac{\partial F}{\partial u} x_{10}x_{20}, \\ D_x D_x'' &= F(2x_{10}x_{01} - F)\Sigma x_{20}x_{02} + F^2 x_{20}x_{02} \\ &\quad + \frac{\partial F}{\partial u}\frac{\partial F}{\partial v}(F - x_{10}x_{01}) - F\frac{\partial F}{\partial u} x_{10}x_{02} - F\frac{\partial F}{\partial v} x_{01}x_{20}, \\ D_x''^2 &= F(2x_{10}x_{01} - F)\Sigma x_{02}^2 + F^2 x_{02}^2 + x_{01}^2\left(\frac{\partial F}{\partial v}\right)^2 - 2F\frac{\partial F}{\partial v} x_{01}x_{02}. \end{aligned}\right\} \tag{40}$$

From these and similar expressions for $D_y^2, \ldots, D_t''^2$, we obtain

$$\left.\begin{aligned} \Sigma D_x^2 &= -F^2 \Sigma x_{20}^2, \qquad \Sigma D_x''^2 = -F^2 \Sigma x_{02}^2, \\ \Sigma D_x D_x'' &= F\left(\frac{\partial F}{\partial u}\frac{\partial F}{\partial v} - F\Sigma x_{20}x_{02}\right). \end{aligned}\right\} \tag{41}$$

§ 4. *Normal Parameters and Functions.*

From (35) and (36) and the results of § 1 it follows that any minimal surface, whose minimal curves do not lie in planes of the type (20), may be defined by equations of the form:

$$\left.\begin{aligned} x &= \tfrac{1}{2}[\phi - u\phi' + \psi' + \phi_0 - u_0\phi_0' + \psi_0'], \\ iy &= \tfrac{1}{2}[\phi - u\phi' - \psi' - \phi_0 + u_0\phi_0' + \psi_0'], \\ z &= \tfrac{1}{2}[\psi - u\psi' - \phi' + \psi_0 - u_0\psi_0' - \phi_0'], \\ it &= -\tfrac{1}{2}[\psi - u\psi' + \phi' - \psi_0 + u_0\psi_0' - \phi_0'], \end{aligned}\right\} \tag{42}$$

where the parameters u and u_0 and the functions ϕ, ϕ_0, ψ, ψ_0 are determined by equations similar to (2) and (2′).

It should be observed that the functions ϕ, ϕ , ψ, ψ_0 are determined only to within additive integral functions of the first order, provided that translations of the surface are disregarded. In fact, if these functions be replaced in (42) by the respective functions

$$\phi + ua_1 + a_2, \quad \phi_0 + u_0 b_1 + b_2, \quad \psi + uc_1 + c_2, \quad \psi_0 + u_0 d_1 + d_2,$$

it is found that the right-hand members of the above equations are increased by the respective constants

$$\tfrac{1}{2}(a_2 + c_1 + b_2 + d_1), \quad \tfrac{1}{2}(a_2 - c_1 - b_2 + d_1),$$
$$\tfrac{1}{2}(c_2 - a_1 + d_2 - b_1), \quad -\tfrac{1}{2}(c_2 + a_1 - d_2 + b_1).$$

Conversely, by such a change of functions any translation can be effected.

In terms of these functions the coefficients E, F, G have the form

$$E = G = 0, \quad F = \tfrac{1}{2}(1 + uu_0)(\phi''\phi_0'' + \psi''\psi_0''). \tag{43}$$

Moreover, the expressions (41) are reducible to

$$\left.\begin{aligned} \Sigma D_x^2 &= F^2 U, \quad \Sigma D_x''^2 = F^2 U_0, \\ \Sigma D_x D_x'' &= -F\left[\frac{(1+uu_0)^2}{4} UU_0 + \frac{F^2}{(1+uu_0)^2}\right], \end{aligned}\right\} \tag{44}$$

where, for the sake of brevity, we have put

$$U = \phi''\psi''' - \phi'''\psi'', \quad U_0 = \phi_0''\psi_0''' - \phi_0'''\psi_0''. \tag{45}$$

From these results it follows that the equation (30) of the lines of curvature on a minimal surface defined by equations (42) reduces to the simple form

$$U du^4 - U_0 du_0^4 = 0. \tag{46}$$

If we put

$$\int \sqrt[4]{U}\, du = \bar{u} + i\bar{v}, \quad \int \sqrt[4]{U_0}\, du_0 = \bar{u} - i\bar{v}, \tag{47}$$

it follows from (46) that the lines of curvature are defined by

$$\left.\begin{aligned} \bar{u} &= \text{const.}, \quad \bar{v} = \text{const.} \\ \bar{u} + \bar{v} &= \text{const.}, \quad \bar{u} - \bar{v} = \text{const.} \end{aligned}\right\} \tag{48}$$

Hence we have the theorem:

When equations of a real minimal surface are written in the form (42), *the determination of the line of curvature reduces to the quadrature*

$$\int \sqrt[4]{\phi''\psi''' - \phi'''\psi''}\, du. \tag{49}$$

If K denotes the Gaussian curvature* of the quadratic form which is the

* Cf. E., p. 155. A reference of this kind is to the author's Differential Geometry.

square of the linear element of the surface, when the coefficients have the form (43), then

$$K = -\frac{1}{F^3}\left[\frac{F^2}{(1+uu_0)^2} + \frac{(1+uu_0)^2}{4}\,UU_0\right]. \tag{50}$$

The formula (31) gives the radius of first curvature of the curve in which the surface is cut by the normal hyperplane whose direction is determined by the corresponding value of dv/du.* When the values from (43) and (44) are substituted in this expression, we have

$$\frac{1}{\rho^2} = \frac{1}{2F^3}\left[\frac{(1+uu_0)^2}{4}\,UU_0 + \frac{F^2}{(1+uu_0)^2}\right] - \frac{1}{4F}\left[U\left(\frac{du}{du_0}\right)^2 + U_0\left(\frac{du_0}{du}\right)^2\right]. \tag{51}$$

One shows without difficulty that the direction which makes the angle $\pi/4$ with the direction defined by

$$\frac{du_0}{du} = M(u, u_0)$$

is given by

$$\frac{du_0}{du} = iM(u, u_0).$$

Hence if ρ' denotes the radius of normal curvature in a direction making the angle $\pi/4$ with a direction whose radius of normal curvature is ρ, we have†

$$\frac{1}{\rho^2} + \frac{1}{\rho'^2} = -K. \tag{52}$$

A particular case of this is afforded by the normal curvatures in the directions of two lines of curvature inclined at the angle $\pi/4$ to one another.

One finds from (51) that the principal radii of normal curvature are given by

$$\rho^2 = \frac{2F^3}{\left[\frac{1+uu_0}{2}\sqrt{UU_0} - \frac{\varepsilon F}{1+uu_0}\right]^2}, \tag{53}$$

where $\varepsilon = \pm 1$ and the corresponding directions are given by

$$\sqrt{U}\,du^2 - \varepsilon\sqrt{U_0}\,du_0^2 = 0.$$

The necessary and sufficient condition that the lines of curvature be indeterminate is that $U = U_0 = 0$. In this case ρ is the same for all directions, and

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* Levi, *loc. cit.*, p. 58.

† Cf. Hovestadt, Programm des Münster'schen Realgymnasiums (1880); also Kommerell, *loc. cit.*, p. 33.

it may be seen also from (39) that the characteristic is a circle with the point (u, u_0) of the surface for center, and only in this case. Hence we have the theorem:

A necessary and sufficient condition that the characteristic be a circle is that

$$\phi''\psi''' - \phi'''\psi'' = 0, \qquad \phi_0''\psi_0''' - \phi_0'''\psi_0'' = 0; \tag{53'}$$

in this case the center is at the point (u, u_0) of the surface, so that the locus of the centers of normal curvature for the point is the same circle, and the lines of curvature are undetermined.

§ 5. *Change of Normal Parameters. Rotations.*

The particular value of the form of equations (42) lies in the fact that if u_0 is the conjugate-imaginary of u and the functions ϕ_0 and ψ_0 are conjugate to ϕ and ψ respectively, the surface is real. In view of results of § 1 it is evident that the equations of minimal curves on the surface, and consequently the equations of the surface, may be given other similar forms. For example, the equations of the curve of parameter u_0 may be expressed in terms of a different parameter u_1 such that its equations shall be of the same form as the curves of parameter u. The conditions necessary and sufficient to this end are

$$u_1\phi_1'' - \psi_1'' = (u_0\phi_0'' - \psi_0'')\frac{du_0}{du_1}, \qquad u_1\phi_1'' + \psi_1'' = -(u_0\phi_0'' + \psi_0'')\frac{du_0}{du_1},$$

$$u_1\psi_1'' + \phi_1'' = (u_0\psi_0'' + \phi_0'')\frac{du_0}{du_1}, \qquad u_1\psi_1'' - \phi_1'' = -(u_0\psi_0'' - \phi_0'')\frac{du_0}{du_1},$$

which may be shown to be equivalent to

$$u_1 = -\frac{1}{u_0}, \qquad \phi_1 = \frac{\psi_0 - c_1}{u_0} + c_2, \qquad \psi_1 = \frac{c_3 - \phi_0}{u_0} + c_4, \tag{54}$$

where c_1, c_2, c_3, c_4 denote constants. Evidently these are necessary and sufficient conditions that the two generating curves shall be superposable by a translation. Furthermore, if we put the expressions (54) for u_1, ϕ_1, ψ_1 in equations (7) and (8), or for u_2, ϕ_2, ψ_2 in (9) and (10), and solve for u_0, ϕ_0, ψ_0, we find the necessary and sufficient conditions which these latter quantities must satisfy in order that the two generating curves be congruent. These conditions are readily found and will not be written out here.

The results of § 1 dealing with the effect upon the normal parameter and the normal functions of rotation of the coordinate axes, or what is the same thing a

rotation of the curve itself, may be extended to minimal surfaces. In fact it can be shown that the minimal surface whose normal parameters and functions are in the following relation to those of the surface (42),

$$\left.\begin{aligned} &u_1 = u, \quad u_{10} = u_0, \quad \phi_1 = \frac{a_1\phi + a_2\psi}{A}, \quad \psi_{10} = \frac{a_4\phi_0 - a_3\psi_0}{A}, \\ &\psi_1 = \frac{a_3\phi + a_4\psi}{A}, \quad \psi_{10} = \frac{-a_2\phi_0 + a_1\psi_0}{A}, \quad A = a_1a_4 - a_2a_3, \end{aligned}\right\} \tag{55}$$

may be obtained from the surface (42) by the rotation defined by equations (13) when $e = 1$.

In like manner the surface for which

$$\left.\begin{aligned} &u_1 = \frac{a_1u + a_2}{a_3u + a_4}, \quad u_{10} = \frac{a_4u_0 - a_3}{-a_2u_0 + a_1}, \quad A = a_1a_4 - a_2a_3, \\ &\phi_1 = \frac{A\phi}{a_3u + a_4}, \quad \phi_{10} = \frac{A\phi_0}{-a_2u_0 + a_1}, \\ &\psi_1 = \frac{A\psi}{a_3u + a_4}, \quad \psi_{10} = \frac{A\psi_0}{-a_2u_0 + a_1} \end{aligned}\right\} \tag{56}$$

differs from the surface (42) by the rotation defined by the equations (13) when $e = -1$.

The second of the foregoing results may also be stated as follows:

If there exists between two minimal surfaces a relation such that their normal parameters are linear fractional functions of one another, as (56), *either of the surfaces may be so rotated that the normal parameters become equal; in this case the new and old functions ϕ and ψ of the rotated surface are in the relations* (56).

§ 6. *Surfaces of Riemann.*

Kommerell* has given the name Riemann surface to the two-dimensional spread in plane four-space, defined by

$$z + it = f(x + iy), \tag{57}$$

where f is a function of the complex variable $x + iy$ analytic in a determined

Loc. cit., p. 548.

domain of this variable. It may be shown* that the most general Riemann surface is defined by

$$x = x, \qquad y = y, \qquad z = u(x, y), \qquad t = v(x, y),$$

where u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Kommerell† has shown furthermore that all of these surfaces are minimal surfaces, as defined in § 3.

We shall now determine in what manner these surfaces are characterized by the normal functions ϕ and ψ. To this end we observe that the Jacobian of the functions $x + iy$ and $z + it$ as given by (42) is

$$(1 + uu_0)\phi''\psi_0''.$$

Hence if we disregard translations, we have the theorem:

The necessary and sufficient condition that the minimal surface defined by (42) *be real and satisfy the condition* (57) *is that*

$$\phi = \phi_0 = 0 \quad \text{or} \quad \psi = \psi_0 = 0.$$

If $\phi = \phi_0 = 0$, equations (42) become

$$\left.\begin{aligned} x &= \tfrac{1}{2}(\psi + \psi_0'), & y &= \frac{i}{2}(\psi - \psi_0'), \\ z &= \tfrac{1}{2}(\psi - u\psi' + \psi_0 - u_0\psi_0'), & t &= \frac{i}{2}(\psi - u\psi' - \psi_0 + u_0\psi_0'). \end{aligned}\right\} \quad (58)$$

From (58) it is evident that the curves $u_0 = \text{const.}$ lie in planes parallel to the plane

$$x + iy = 0, \qquad z + it = 0,$$

and the curves $u = \text{const.}$ in planes parallel to the plane

$$x - iy = 0, \qquad z - it = 0.$$

In § 5 we have seen the effect of a rotation of axes upon the equations of a minimal surface. Comparing equations (55) and (56) with the preceding results of this section, we have the theorem:

The necessary and sufficient condition that a minimal surface defined by (42)

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* *Loc. cit.*, p. 571. † *Loc. cit.*, p. 586.

be a real Riemann surface is that the functions ϕ, ϕ_0, ψ, ψ_0 *satisfy the conditions*

$$a\phi + b\psi = 0, \qquad a_0\phi_0 + b_0\psi_0 = 0, \tag{59}$$

where a, b *are constants, and* a_0, b_0 *are their respective conjugate imaginaries.*

From this result and (18) it follows that

The minimal surfaces of Riemann are characterized by the fact that their minimal curves lie in planes whose equations are of the form

$$a(z+it) - b(x-iy) = c, \qquad a(x+iy) + b(z-it) = d,$$

and

$$a_0(z-it) - b_0(x+iy) = c_0, \qquad a_0(x-iy) + b_0(z+it) = d_0,$$

where c, d *are constants and* c_0, d_0 *are their conjugate imaginaries.*

Since equations (59) are equivalent to (53′), we have the following theorem:

*A necessary and sufficient condition that the characteristic at each point of a surface be a circle is that the surface be of the Riemann type; the center of the circle is at the corresponding point of the surface.**

§ 7. *Algebraic Minimal Surfaces.*

If we put

$$\left.\begin{aligned} J_{12} &= \frac{\partial(x,y)}{\partial(u,u_0)}, \quad J_{13} = \frac{\partial(x,z)}{\partial(u,u_0)}, \quad J_{14} = \frac{\partial(x,t)}{\partial(u,u_0)}, \\ J_{23} &= \frac{\partial(y,z)}{\partial(u,u_0)}, \quad J_{24} = \frac{\partial(y,t)}{\partial(u,u_0)}, \quad J_{34} = \frac{\partial(z,t)}{\partial(u,u_0)}, \end{aligned}\right\} \tag{60}$$

we have

$$I^2 = J_{12}^2 + J_{13}^2 + \ldots + J_{34}^2 = \Sigma x_{10}^2 \cdot \Sigma x_{01}^2 - (\Sigma x_{10}x_{01})^2.$$

If we put further

$$P_{ij} = -P_{ji} = \frac{J_{ij}}{I} \quad \begin{pmatrix} i = 1, \ldots, 4, \\ j = 1, \ldots, 4, \end{pmatrix} \quad i \neq j\Big),$$

we find that

$$\left.\begin{aligned} P_{12}^2 + P_{13}^2 + P_{14}^2 + P_{23}^2 + P_{24}^2 + P_{34}^2 &= 1, \\ P_{12}P_{34} + P_{14}P_{23} - P_{13}P_{24} &= 0. \end{aligned}\right\} \tag{61}$$

We may call the functions P_{ij} the *direction-cosines of the tangent plane.*

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* Kommerell, *loc. cit.*, p. 580, shows that each surface of Riemann possesses this property, but not that it is characteristic.

When the expressions (2) are substituted in the above equations and we take

$$I = -iF = -\frac{i}{2}(1 + u u_0)\Phi_1,$$

where

$$\Phi_1 = \phi'' \phi_0'' + \psi'' \psi_0'', \quad \Phi_2 = \phi'' \psi_0'' + \phi_0'' \psi'', \quad \Phi_3 = \phi'' \psi_0'' - \phi_0'' \psi'',$$

we find

$$\left.\begin{aligned} P_{12} &= \frac{\psi'' \psi_0'' - u u_0 \phi'' \phi_0''}{(1 + u u_0)\Phi_1}, & P_{13} &= \frac{i}{2}\left[\frac{\Phi_3}{\Phi_1} + \frac{u - u_0}{u u_0 + 1}\right], \\ P_{14} &= \frac{1}{2}\left[\frac{\Phi_2}{\Phi_1} - \frac{u + u_0}{1 + u u_0}\right], & P_{23} &= \frac{1}{2}\left[\frac{\Phi_2}{\Phi_1} + \frac{u + u_0}{1 + u u_0}\right], \\ P_{24} &= -\frac{i}{2}\left[\frac{\Phi_3}{\Phi_1} + \frac{u_0 - u}{1 + u u_0}\right], & P_{34} &= \frac{u u_0 \psi'' \psi_0'' - \phi'' \phi_0''}{(1 + u u_0)\Phi_1}. \end{aligned}\right\} \quad (62)$$

From these equations we obtain the following:

$$\frac{P_{23} - P_{14}}{1 + P_{12} - P_{34}} = \frac{u + u_0}{2}, \quad \frac{P_{13} + P_{24}}{1 + P_{12} - P_{34}} = \frac{i(u - u_0)}{2}. \quad (63)$$

We are now in a position to establish the theorem:

The necessary and sufficient condition that the minimal surface defined by (42) *be a real algebraic surface is that the functions* ϕ, ϕ_0, ψ, ψ_0 *be algebraic.*

From the form of equations (42) the sufficiency of this condition is evident. In order to establish its necessity, we make use of the following theorem of Weierstrass:*

Given a function $\Phi(\xi + i\eta)$ *and let* $\Psi(\xi, \eta)$ *denote the real part of* Φ; *if in a certain domain an algebraic relation exists between* Ψ, ξ *and* η, *then* Φ *is an algebraic function of* $\xi + i\eta$ *for that domain.*

In applying this theorem, we observe that, since

$$\frac{\partial(x, y)}{\partial(u', u_0')} = \frac{\partial(x, y)}{\partial(u, u_0)} \cdot \frac{\partial(u, u_0)}{\partial(u', u_0')},$$

the direction-cosines are absolute invariants under a change of parameters. Hence, equations (63) give the relation between any set of parameters and the particular ones u, u_0 which lead to the form (42).

* *Monatsberichte der Berliner Akademie* (1867), pp. 511–518; also E., p. 261.

An algebraic minimal surface may be defined by two algebraic equations of the form

$$z = \psi_1(x, y), \qquad t = \psi_2(x, y). \tag{64}$$

If x, y be taken for parameters in finding the functions P_{ij}, and if the real and imaginary parts of u are given by

$$u = \xi + i\eta, \qquad u_0 = \xi - i\eta,$$

then equations (63) are expressible in the form

$$\phi_1(x, y) = \xi, \qquad \phi_2(x, y) = \eta,$$

where ϕ_1 and ϕ_2 are algebraic. Eliminating y and x from these two equations, we have respectively algebraic equations of the form

$$F_1(x, \xi, \eta) = 0, \qquad F_2(y, \xi, \eta) = 0.$$

Moreover, by means of (64) we get two other algebraic equations

$$F_3(z, \xi, \eta) = 0, \qquad F_4(t, \xi, \eta) = 0.$$

Consequently (cf. 42) the real part of each of the functions

$$\left.\begin{aligned} f_1 &= \phi - u\phi' + \psi', & f_2 &= \phi - u\phi' - \psi', \\ f_3 &= \psi - u\psi' - \phi', & f_4 &= \psi - u\psi' + \phi' \end{aligned}\right\} \tag{65}$$

is an algebraic function of ξ and η; hence, by the above Weierstrass theorem the functions f_1, f_2, f_3, f_4 defined by (65) are algebraic functions of u. But

$$\phi = \frac{1}{2}(f_1 + f_2) - \frac{u}{2}(f_3 - f_4),$$

$$\psi = \frac{1}{2}(f_3 + f_4) + \frac{u}{2}(f_1 - f_2).$$

Therefore, the functions ϕ and ψ are algebraic functions of u, and in like manner ϕ_0 and ψ_0 are algebraic functions of u_0.

§ 8. *Associate Minimal Surfaces. Formulas of Schwarz.*

Just as in the case of ordinary minimal surfaces in three-space, if a minimal surface in plane four-space is defined by (35), the equations of the form

$$x_\alpha = e^{i\alpha} f_1 + e^{-i\alpha}\phi_1, \ldots, \tag{66}$$

where α denotes a constant, define a family of applicable minimal surfaces which

we say are *associate* to (35) and to one another. We denote by S_α the surface whose equations are (66) for a definite value of α.

If $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ denote the coordinates of the *adjoint surface* defined by (66), when $\alpha = \pi/2$, the equations (66) may be written

$$x_\alpha = \cos\alpha\, x + \sin\alpha\, \bar{x}, \ldots\ldots \tag{67}$$

Hence, the applicability of a family of associate minimal surfaces may be expressed in the form:

*A minimal surface admits of a continuous deformation into a series of minimal surfaces, and in the deformation each point of the surface describes an ellipse whose plane passes through a fixed point which is the center of the ellipse.**

From equations (35) and (66) we find that *tangents to corresponding curves on S and S_α make the angle α with one another.*†

Referring to equations (42) we see that the surfaces associate to S are determined by the functions

$$\phi e^{ia}, \quad \phi_0 e^{-ia}, \quad \psi e^{ia}, \quad \psi_0 e^{-ia}. \tag{68}$$

From this result and the expressions (62) for the direction-cosines of the tangent plane to S it follows that

The tangent planes at corresponding points of a family of associate minimal surfaces are parallel.

From (29) it follows that the function f has the same value at corresponding points on S and S_α. Moreover, from (28) we have that each of the functions D for S_α is equal to the product of e^{ia} and the corresponding D for S, and each D'' for S_α, the product of e^{-ia} and the corresponding D'' for S.

We consider now the point on the characteristic of S, at the point (u, u_0), corresponding to a given direction

$$\frac{du_0}{du} = M(u, u_0). \tag{69}$$

If $X_\alpha, Y_\alpha, Z_\alpha, T_\alpha$ denote the coordinates of the point on the characteristic for S_α at (u, u_0) corresponding to the direction

$$\frac{du_0}{du} = M(u, u_0)e^{ia}, \tag{70}$$

* Cf. E., p. 264.

† Cf. Levi, *loc. cit.*, p. 94.

we have from equations (39) and similar ones for S_α

$$X_\alpha - x_\alpha = X - x, \qquad Y_\alpha - y_\alpha = Y - y, \qquad \ldots\ldots \tag{71}$$

Since the tangent planes to S and S_α at (u, u_0) are parallel, so also are the normal planes. From this result and equations (71) we have the theorem:

The characteristics at corresponding points of a family of associate surfaces are equal ellipses, similarly placed in parallel planes.

We shall say that two points on the characteristics which are in the relation (71) are *congruent*, since they can be made to coincide by a translation of either surface such that corresponding points on the two surfaces coincide also. One finds without difficulty that the curves (70) on S make the angle $\alpha/2$ with the curves (69). Hence, we have the theorem:

Let P and P_α be congruent points on corresponding characteristics of S and an associate S_α; to the direction on S_α determining P_α corresponds on S a direction making the angle $\alpha/2$ with the direction determining P.

From the definition of lines of curvature of a surface, we have as a corollary of the preceding theorem:

*The curves on S corresponding to lines of curvature on S_α make the constant angle $\alpha/2$ with the lines of curvature on S.**

This follows likewise from (46) and the equations of the lines of curvature on S_α, namely

$$e^{2i\alpha} U du^4 - e^{-2i\alpha} U_0 du_0^4 = 0.$$

In fact, in terms of the functions u and v (cf. 48), the finite equations of the lines of curvature on S_α are

$$\cos\frac{\alpha}{2}\bar{u} - \sin\frac{\alpha}{2}\bar{v} = \text{const.},$$

$$\sin\frac{\alpha}{2}\bar{u} + \cos\frac{\alpha}{2}\bar{v} = \text{const.},$$

$$\cos\frac{\alpha}{2}(\bar{u} - \bar{v}) - \sin\frac{\alpha}{2}(\bar{u} + \bar{v}) = \text{const.},$$

$$\sin\frac{\alpha}{2}(\bar{u} - \bar{v}) + \cos\frac{\alpha}{2}(\bar{u} + \bar{v}) = \text{const.}$$

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*Cf. E., p. 264.

The foregoing results may be obtained also by means of the formula (51) of the radius of normal curvature.

A minimal surface and its adjoint possess the property of orthogonality of linear elements, expressed by

$$dx\,d\bar{x} + dy\,d\bar{y} + dz\,d\bar{z} + dt\,d\bar{t} = 0. \tag{72}$$

Since the tangent plane to $S_{\pi/2}$ is parallel to the tangent plane of S, we have

$$\begin{Vmatrix} dx_1 & dy_1 & dz_1 & dt_1 \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} & \dfrac{\partial t}{\partial u} \\ \dfrac{\partial x}{\partial u_0} & \dfrac{\partial y}{\partial u_0} & \dfrac{\partial z}{\partial u_0} & \dfrac{\partial t}{\partial v} \end{Vmatrix} = 0. \tag{73}$$

From (72), (73) and § 7 we obtain

$$\frac{dx_1}{P_{12}dy + P_{13}dz + P_{14}dt} = \frac{dy_1}{P_{23}dz + P_{24}dt + P_{21}dx}$$

$$= \frac{dz_1}{P_{34}dt + P_{31}dx + P_{32}dy} = \frac{dt_1}{P_{41}dx + P_{42}dy + P_{43}dz} = \lambda, \tag{74}$$

where λ denotes the common value. By means of (62), (42) and (67), it may be shown that $\lambda = 1$. Hence we have in (74) a set of formulas analogous to the celebrated *formulas of Schwarz* for ordinary minimal surfaces.*

As in the case of ordinary minimal surfaces, these formulas enable one to find a minimal surface passing through a given curve in space and tangent to a given plane at each point. In fact, it may be shown that if x, y, z, t as functions of a real parameter v define a curve in space, and P_{ij} as functions of v determine planes passing through corresponding points of the curve, one at each point, and if ω denotes a complex variable, and $R(\theta)$ the real part of an analytic function θ of ω, then the equations

$$x_1 = R[x(\omega) + i\int_v^\omega P_{12}dy + P_{13}dz + P_{14}dt$$

and similar expressions for y_1, z_1, t_1 define the minimal surface desired.

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* Cf. E., p. 264.

As a consequence of this result we have the theorems analogous to theorems in the theory of ordinary minimal surfaces:*

If a straight line lies in a minimal surface, the surface is symmetric with respect to this line.

If a hyperplane is normal to a surface at all of its points of intersection, the surface is symmetric with respect to the plane.

PRINCETON UNIVERSITY, *June*, 1911.

* Cf. E., pp. 266, 267.

*A Contribution to the Foundations of Fréchet's Calcul Fonctionnel.**

By T. H. Hildebrandt.

Preface.

During the last few years there has been manifest a tendency in the direction of generalization of current analysis. Two memoirs bearing on this subject have recently made their appearance: in the first instance, M. Fréchet's Paris thesis, *Sur quelques Points du Calcul Fonctionnel;*† and secondly, E. H. Moore's *Introduction to a Form of General Analysis.*‡ We consider briefly the contents and direction of generalization of each.

Fréchet's work may be divided into two parts: (1) a theory of continuous functions on an abstract set, and (2) a generalization of the theory of linear point sets. The first of these was no doubt suggested by the analogies between theorems on continuous function of a single variable, of n variables, of lines, of curves, etc. The element of generality enters in the consideration of a class or set Ω of elements q, which are not specifically defined. For the class there is postulated the existence of a notion of limit of a sequence of elements, satisfying a number of conditions which are properties of the limit of a sequence of real numbers. In terms of such a limit, it is possible to define a sequentially continuous function, and hence to construct a theory of sequentially continuous functions. To attain the second end, there is postulated for the class Ω the existence of a *voisinage* or distance function δ of pairs of elements, there being a value of δ for every pair of elements of the class. This distance function δ is subjected to a number of conditions, generalizations of properties of its real variable analogue, the absolute value of the difference between two numbers. In terms of such a δ, a limit is definable, and a theory of sets, concerning derived, closed, etc., sets, is obtainable.

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* This paper is in the form sent to the editors in April, 1910, with the addition of § 15 (6) and (7) and a few changes in § 17, due to the article by Fréchet: Rend. di Pal., XXX, 1-26. Cf. also Hedrick: *Trans. Am. Math. Soc.*, XII, 285-294, which contains some more general and some less general theorems than the present paper.

† Reprinted in *Rendiconti del Circulo di Palermo*, Vol. XXII, pp. 1-64.

‡ Published in *The New Haven Mathematical Colloquium*, New Haven, 1910.

The *General Analysis* of Moore may be termed "*a theory of classes of functions on an unconditioned range.*" The subject under consideration is a system $(\mathfrak{A}; \mathfrak{P}; \mathfrak{M})$, consisting of the class $\mathfrak{A}$ of real numbers a, the class $\mathfrak{P}$ of unconditioned elements p, and the class $\mathfrak{M}$ of real-valued, single-valued functions μ of p, p ranging over the class $\mathfrak{P}$; *i. e.*, of functions on $\mathfrak{P}$ to $\mathfrak{A}$. The theory treats of properties of the classes $\mathfrak{M}$, the properties in question being common to the following special classes of functions:

(1) The class of all unipartite numbers x, *i. e.*, a function of the variable p having only one value.

(2) The class of all n-partite numbers $(x_1, \ldots, x_n)$, functions of the variable index p; $p = 1, 2, \ldots, n$.

(3) The class of all absolutely convergent series $x_1, x_2, \ldots, x_n, \ldots$, where $\Sigma|x_n|$ is convergent, functions of the variable index p; $p = 1, 2, 3, \ldots, n, \ldots$.

(4) The class of all continuous functions on the interval $0 \leqq p \leqq 1$ of the real number system.

The first part of the memoir treats of certain closure and dominance properties of general reference, *i. e.*, independent of the nature of the parameter p. The second part treats of properties of special reference, in connection with the question of composition of classes of functions, one of the classes being on an unconditioned range. In particular, three properties, K_1, K_2, and Δ, are treated, K_1 and K_2 relating to the relations

$$K_{pm} \text{ and } K_{p_1p_2m},$$

which in turn depend upon a development* of the class $\mathfrak{P}$. The property Δ also relates to the development of $\mathfrak{P}$. For the real variable p, the first of these two relations is the inequality $p \geqq m$, and the second the inequality $|p_1 - p_2| \leqq \frac{1}{m}$.

The present work concerns itself with the Fréchet point of view. It had its inception in an attempt to replace the distance function δ of Fréchet by a weaker condition on the class Ω. The fact that in most instances the δ appears in connection with an inequality of the type

$$\delta_{q_1q_2} \leqq \frac{1}{m}$$

suggested the adoption of the second K-relation of Moore, $K_{q_1q_2m}$, in the place of

* Real variable analogue, the set of divisions of an interval into n equal parts, $n = 1, 2, 3, \ldots$.

the δ. By stating, in the case of every theorem, the precise conditions on K sufficient to carry the argument, and extending this idea to the case in which the class $\mathfrak{Q}$ is subjected only to the condition of the existence of a limit, a two-fold result was obtained: (a) *that an unconditioned limit suffices for the theorems on sequentially continuous functions obtained by Fréchet, and* (b) *that it is possible to obtain the theory of sets of elements with a distance function δ, subjected to weaker conditions than those imposed by Fréchet.* To show that the conditions in question were weaker, the complete existential theory,* of the properties of the K-relation, and as a consequence of the corresponding properties of the δ, was constructed.

The first part treats of the limit L, the K-relation, and their properties. Instead, however, of considering the existence of an L and a K-relation as a property or condition of the class $\mathfrak{Q}$, we use the notion of a system, the class $\mathfrak{Q}$ together with L being a system $(\mathfrak{Q}; L)$, and, with the K-relation, being a system $(\mathfrak{Q}; K)$. In the second part the most important theorems of Fréchet are taken up. It might be regarded in the light of a proof of the above results.

I.

Systems $(\mathfrak{Q}; L)$ $(\mathfrak{Q}; K)$ $(\mathfrak{Q}; \delta)$: Definitions, Properties and Interrelations.

1. *Introductory.* We consider in this paper properties of and functions on a class $\mathfrak{Q}$ of *general* elements q. The elements q are general in that nothing is specified as to their nature, that is, as to whether they be numbers, points on a line, points in n-dimensional space, sequences, real-valued functions, etc. However, we suppose the elements q of the class $\mathfrak{Q}$ to be well-defined, individually and in their totality. Further, that there exists at least one element in the class. These suppositions do not limit the generality of the class $\mathfrak{Q}$.

The class $\mathfrak{Q}$ enters the theory through two properties L and K, which together with the class $\mathfrak{Q}$ form systems $(\mathfrak{Q}; L)$ and $(\mathfrak{Q}; K)$ respectively, the nature of which is to be specified in the sequel.

2. *Notational.*† We shall denote throughout this paper, *classes*, that is collections, sets, aggregates, ensembles, etc., by capital German letters, *e. g.* $\mathfrak{Q}$; $\mathfrak{R}$; $\mathfrak{M}$, etc., in particular, $\mathfrak{Q}$ a class of elements q, $\mathfrak{R}$ a class of elements r,

* Cf. Moore, *loc. cit.*, p. 82. Also below, § 9.

† For an extensive treatment of the ideas underlying subjects touched on in this paragraph see Moore, *loc. cit.*, §2, pp. 15 ff.

$\mathfrak{M}$ a class of functions μ, etc. The notation for the class always corresponds to the notation for the elements.

The brackets [] will be used to denote *a class of*, and in particular a class of the elements included within. Thus $[q]$ stands for a class of elements q, and we have therefore

$$\Omega = [q].$$

As special classes of frequent occurrence we have the classes of *all real numbers; all numbers greater than unity; all positive numbers; all positive integers; all positive and negative integers;* denoted variously by:

$$\mathfrak{A},\ [a];\quad [c];\quad [d],\ [e];\quad [n],\ [l],\ [k];\quad [m],$$

respectively.

Real-valued single-valued functions are denoted by Greek letters.

The fact that an element x *belongs to the class* Ω, is expressed:

$$x^{\Omega}.$$

Since we have $\Omega = [q]$, evidently:

The statement x^{Ω} is equivalent to the statement x is a q.

This abbreviation is extended to the case of classes of elements, and we have:

$$\mathfrak{R}^{\Omega},$$

denoting the fact that *the class* $\mathfrak{R}$ *is a subclass of the class* Ω.

To state that the element x *has the property* P, we use the notation:

$$x^{P}.$$

The concept of belonging to a class may be said to be a special case of such a general property, and from this point of view, the notations are in agreement. The notation is also extended to the case of classes, and we have:

$$\Omega^{P},$$

denoting that *the class* Ω *has the property* P. If the property P is not holding we prefix the negative sign. Thus

$$\Omega^{-P}$$

specifies that *the class* Ω *does not have the property* P.

Finally, in the statement of propositions and proofs it has been found convenient for the sake of clearness and brevity to use some of the symbols used be Peano* and Moore,† in particular the following:

* Peano: Formulario Matematico, Editio V[1], 1906. † *Loc. cit.*, p. 150.

$=$ to denote *logical equality*,
$\neq$ " *logical diversity*,
$\equiv$ " *definitional identity*,
$\supset$ " *it is true that;* ( ) *implies* ( ); *if* ( ), *then* ( ),
$\subset$ " ( ) *is implied by* ( ),
$\backsim$ " ( ) *is equivalent to* ( ); ( ) *implies and is implied by* ( ),
$\exists$ " *there exists*,
$\ni$ " *such that*,
$\cdot$ " *and*,
$\cdot$, $:$, $\because$ *as signs of punctuation, the largest number of punctuation dots being around the principal implication*,
$[\,]$ to denote *a class of*,
$\{\}$ " *the sequence of.*

3. *Definition of Limit. The Systems* $(\mathfrak{Q}; L)$. *Properties of Limit in* $(\mathfrak{Q}; L)$. We assume the following definition of limit:

Limit is a relation between sequences of elements and single elements.

The nature of the relation is not specified permanently; it may vary with the type of element considered. If such a relation is holding between a sequence of elements, and a single element, the single element is said to be the limit of the sequence of elements.

In case there is defined a limit relation for the class $\mathfrak{Q}$, we say that the class $\mathfrak{Q}$ has the property L, and in this way we obtain a system $(\mathfrak{Q}; L)$. The limit L in such a system might be considered as drawn up in the form of a table which for every sequence of the class specifies the corresponding single element, if such an one exists. We shall suppose that:*

$$\underset{n}{L} q_n = q$$

states the fact that the limit relation L is holding between the sequence $\{q_n\}$ and the element q.

The limit L in the class $\mathfrak{Q}$ may have one or more of the following properties:

(1) *Limit is unique.*

(2) *If a sequence has a limit, any subsequence of the sequence taken in the same order has the same limit.*†

---

* Throughout this paper we denote $\underset{n=\infty}{L}$ by $\underset{n}{L}$.

† This is a statement of two distinct facts: (1) that the limit of the subsequence exists, and (2) that it is the same as the limit of the original sequence. Similarly below, in the case of properties (3), (4), (5).

(3) *If a sequence has a limit, any sequence obtained by removing a finite number of elements at the beginning of the sequence, has the same limit.*

(4) *If a sequence has a limit, any sequence obtained by prefixing a finite number of terms to the sequence, has the same limit.*

(5) *If a sequence has a limit, any sequence obtained by a rearrangement of the sequence has the same limit.*

(6) *If all of the terms of a sequence are identical, the sequence has a limit, which is the repeated element.*

We denote by L^1; L^{23}; L^{-34}; etc., a limit L having the property (1); the properties (2), and (3); not the property (3), but the property (4); etc., respectively. The limit of the real number system is an L^{123456}. The limit used by Fréchet is an L^{126}. We lay down permanently no properties of L, but in the various theorems in a system $(\mathfrak{Q}; L)$ specify explicitly sufficient properties of the L.

There exists but one relation between these properties of L. We have:

$$L^2 \cdot \supset \cdot L^3,$$

i. e., if L has the property (2) then it also has the property (3). That this relation is holding is at once evident. That it is the only relation between these properties may be shown by the exhibition of limits having the properties (1), (2), (3), (4), (5), (6), or their negatives, in all the 2^6 conceivable combinations, not excluded by the above relation. By employing the method of § 10 for the construction of limits, it is possible with little difficulty to obtain the various combinations desired.

4. *The K-relation.* We shall subsequently show that it is possible to obtain, relative to the system $(\mathfrak{Q}; L)$, the L being unconditioned, a theory of sequentially continuous functions. However, some of the theorems in the theory of point sets are not holding, even though we suppose the limit L to have the six properties of § 3. The necessity of a system more restricted, and therefore less general, than the system $(\mathfrak{Q}; L)$ is thus apparent.

We define:

The K-relation is a relation between pairs of elements, and positive and negative integers.

The nature of the K-relation will evidently depend upon the nature of the elements considered, and the situation in which it is to be employed. Relative to a class $\mathfrak{Q}$, we may consider the K-relation as drawn up in the form of a table

which specifies for every combination of a pair of elements q_1, q_2 with an integer m, *i. e.*, for every $q_1 q_2 m$, whether or not the relation holds. We denote the fact that the K-relation is holding (not holding) between $q_1 q_2 m$ by

$$K_{q_1 q_2 m} \quad (-K_{q_1 q_2 m}).$$

If we join the K-relation to a class $\mathfrak{Q}$, we obtain a system $(\mathfrak{Q}; K)$. We shall, however, make a restrictive postulate relative to this system, viz.:

In the system $(\mathfrak{Q}; K)$, *for every pair of elements* q_1, q_2 *of the class* $\mathfrak{Q}$ *there exists at least one integer* m *such that*

$$K_{q_1 q_2 m}.^*$$

The K-relation may have one or more of the following properties:†

(1) $$K_{q_1 q_2 m'} \cdot m \leqq m' \cdot \supset \cdot K_{q_1 q_2 m},$$

i. e., if the K-relation holds between q_1, q_2, and m', then it also holds between q_1, q_2, and m, where m is any integer less than m'.

(2) $$K_{q_1 q_2 m} \cdot \supset \cdot K_{q_2 q_1 m},$$

i. e., if the K-relation holds between q_1, q_2, and m, then it also holds if the q's are interchanged, *i. e.*, between q_2, q_1, and m. The relation $K_{q_1 q_2 m}$ is symmetrical in the arguments q_1, q_2.

---

* Fréchet (*loc. cit.*, p. 18) introduces in his work a "voisinage" or distance function on pairs of elements, and supposes that to every pair of elements q_1, q_2 of the class $\mathfrak{Q}$ there corresponds a real number a, the value of the function. We shall denote this function by δ, so that δ is a function on $\mathfrak{Q}\mathfrak{Q}$ to $\mathfrak{A}$. The presence of a δ in the class $\mathfrak{Q}$ gives us a system $(\mathfrak{Q}; \delta)$. The idea of the above relation was suggested by the frequent occurrence of inequalities of the form

$$\delta_{q_1 q_2} \leq \frac{1}{m}.$$

However, to permit the use of quantities on the right-hand side of this inequality which are greater than unity, it has been found convenient to substitute the inequality

$$\delta_{q_1 q_2} \leq \frac{1}{c^m},$$

where c is a real number greater than unity, and m takes on both positive and negative integral values. We take up the relation between the Fréchet δ, and the K-relation in §6.

† These properties are analogues of the properties which Fréchet presupposes relative to the δ, viz.:

$$\delta_{q_1 q_2} \geq 0; \quad \delta_{q_1 q_2} = \delta_{q_2 q_1}; \quad \delta_{q_1 q_2} = 0 \cdot \supset \cdot q_1 = q_2;$$

i. e., if $\delta_{q_1 q_2}$ is zero, then $q_1 = q_2$, and if $q_1 = q_2$, then $\delta_{q_1 q_2}$ is zero:

$$\exists \phi_e \ni \left(\lim_{e=0} \phi_e = 0 \cdot \delta_{q_1 q_2} \leq e \cdot \delta_{q_2 q_3} \leq e \cdot \supset \cdot \delta_{q_1 q_3} \leq \phi_e \right),$$

that is, there exists a real-valued single-valued function of e which approaches zero as e approaches zero, and is such that if $\delta_{q_1 q_2} \leq e$ and $\delta_{q_2 q_3} \leq e$, then $\delta_{q_1 q_2} \leq \phi_e$.

(3) $$K_{q_1q_2m}(m) \cdot \supset \cdot q_1 = q_2,$$

i. e., if the relation $K_{q_1q_2m}$ holds for all values of m*, then $q_1 = q_2$.

(4) $$q_1 = q_2 \cdot \supset \cdot K_{q_1q_2m}(m),$$

the converse of (3).

(5) $$\exists\, \phi_m \ni \left(\lim_{m=\infty} \phi_m = \infty \cdot K_{q_1q_2m}\, K_{q_2q_3m} \cdot \supset \cdot K_{q_1q_3\phi_m} \right),$$

i. e., there exists a function ϕ of m, with integral values ϕ_m, approaching infinity with m, and such that if $K_{q_1q_2m}$ and $K_{q_2q_3m}$ then $K_{q_1q_3\phi_m}$.†

As properties equivalent to (5) if the symmetry (2) is holding, we notice:

(6) $$\exists\, \phi_m \ni \left(\lim_{m=\infty} \phi_m = \infty \cdot K_{q_2q_1m}\, K_{q_2q_3m} \cdot \supset \cdot K_{q_1q_3\phi_m} \right),$$

which reads like (5). Further:

(7) $$\exists\, \phi_m \ni \left(\lim_{m=\infty} \phi_m = \infty \cdot K_{q_1q_2m}\, K_{q_3q_2m} \cdot \supset \cdot K_{q_1q_3\phi_m} \right),$$

(8) $$\exists\, \phi_m \ni \left(\lim_{m=\infty} \phi_m = \infty \cdot K_{q_2q_1m}\, K_{q_3q_2m} \cdot \supset \cdot K_{q_1q_3\phi_m} \right).$$

The functions ϕ of (5) to (8) need not be identical. We denote them by ϕ^5, ϕ^6, ϕ^7, ϕ^8, respectively. In addition to ϕ^{5678}, we shall need the following:

(a) $\phi^9_m = \phi^7_{m'}$, where m' is the lesser of ϕ^6_m and m,
(b) $\phi^{10}_m = \phi^8_{m'}$, where m' is the lesser of ϕ^6_m and m,
(c) $\phi^{11}_m = \phi^8_{m''}$, where m'' is the lesser of ϕ^7_m and m.

Evidently each of these three ϕ's approaches infinity with m.

We speak of K^1, K^{23}, K^{-34}, etc., as K-relations having the property (1), the properties (2) and (3), not the property (3) but the property (4), etc.

5. *Relations between the Properties of the K-relation.* Before taking up the relations between the above eight properties we prove the following lemmas:

(1) $$K^{167} \cdot K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_1\phi^9_m},$$

i. e., if the K-relation has the properties (1), (6), and (7), of § 4, then the K-relation also holds between q_2, q_1 and ϕ^9_m, where ϕ^9_m is defined in § 4, in terms of ϕ^6_m and ϕ^7_m, whose existence is assured by the presence of K^{67}.‡

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* Denoted by (m). Cf. Moore, *loc. cit.*, p. 27.

† In addition to the Fréchet property mentioned above compare also:

$$|a_1 - a_2| \leq 1/c^m \cdot |a_2 - a_3| \leq 1/c^m \cdot \supset \cdot |a_1 - a_3| \leq 1/c\phi_m.$$

‡ This lemma and the succeeding ones aim at replacing the symmetry property (2) by a number of properties, collectively weaker, but for our purposes equally effective.

Suppose $K_{q_1q_2m}$. Then since K^6 we have: $K_{q_2q_2\phi^6{}_m}$. Let m' be the lesser of ϕ^6_m and m. Then since K^1 we have

$$K_{q_1q_2m'} \text{ and } K_{q_2q_2m'},$$

and so if we apply K^7, we obtain:

$$K_{q_2q_1\phi^7{}_{m'}}, \text{ } i.\ e., \text{ } K_{q_2q_1\phi^9{}_m}.$$

In a similar manner we show:

(2) $$K^{168} \cdot K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_1\phi^{10}{}_m}.$$

(3) $$K^{178} \cdot K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_1\phi^{11}{}_m}.$$

We have further:

(4) $$K^{46} \cdot K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_1\phi^6{}_m},$$

(5) $$K^{47} \cdot K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_1\phi^7{}_m},$$

(6) $$K^{48} \cdot K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_1\phi^8{}_m},$$

the proof of which is easily evident.

We then have the following propositions relating to the interrelations between the properties of K:

(7) $$K^2 :\supset: K^5 \cdot \supset \cdot K^6 \cdot \supset \cdot K^7 \cdot \supset \cdot K^8,$$

i. e., if the K-relation has the property (2), then the properties (5), (6), (7), and (8) are equivalent. For we evidently can choose:

$$\phi^5_m = \phi^6_m = \phi^7_m = \phi^8_m.$$

(8) $$K^4 :\supset: K^6 \cdot \supset \cdot K^7;$$

i. e., if the K-relation has the property (4), then properties (6) and (7) are equivalent. We show first that if (6) is present, then (7) is also. Suppose

$$K_{q_1q_2m} \text{ and } K_{q_3q_2m}.$$

Then by (4) above, we have:

$$K_{q_2q_1\phi^6{}_m} \text{ and } K_{q_2q_3\phi^6{}_m}.$$

Hence, applying K^6:

$$K_{q_1q_3\phi^6{}_{\phi^6{}_m}}; \text{ } i.\ e., \text{ } \phi^7_m = \phi^6_{\phi^6{}_m}.$$

Similarly if K^7 we show that

$$\phi^6_m = \phi^7_{\phi^7{}_m},$$

(9) $$K^4 :\supset: K^7 \cdot \supset \cdot K^8.$$

In a manner analogous to the preceding, it appears that:

$$\phi^8_m = \phi^7_{\phi^7_m},$$

$$K^4 : \supset : K^8 \cdot \supset \cdot K^5. \tag{10}$$

Evidently

$$\phi^5_m = \phi^8_{\phi^8_m}.$$

$$K^{14} : \supset : K^8 \cdot \supset \cdot K^7. \tag{11}$$

The proof is the result of an application of proposition (5), and the method of proposition (1).

The last four propositions may be gathered into a single one, and we thus get:

$$K^{14} : \supset : K^6 \cdot \supset \cdot K^7 \cdot \supset \cdot K^8 \cdot \supset \cdot K^5; \tag{12}$$

i. e., under the hypothesis that K has the properties (1) and (4), the properties (6), (7), and (8) are equivalent, and any one implies (5).

$$K^{167} \cdot \supset \cdot K^5. \tag{13}$$

This is a result of proposition (1), and the method of (1). It appears that

$$\phi^5_m = \phi^6_{m''},$$

where m'' is the lesser of ϕ^9_m and m.

In an entirely similar manner we have:

$$K^{167} \cdot \supset \cdot K^8; \; K^{168} \cdot \supset \cdot K^{57}; \; K^{178} \cdot \supset \cdot K^{56}. \tag{14}$$

We can write these as a series of continued implications and equivalences as follows:

$$K^1 : \supset : K^{67} \cdot \supset \cdot K^{78} \cdot \supset \cdot K^{68} \cdot \supset \cdot K^5; \tag{15}$$

i. e., if K has the property (1), then the combinations (6) (7), (7) (8), (6) (8) are equivalent, and any combination implies property (5).

We shall show later, in § 9, that these constitute all of the relations which hold between the properties of K, the class $\mathfrak{Q}$ being general. It is too much of digression to take up this discussion at this point.

6. *The Relation between the Fréchet δ and the K-relation.* In the foot-note in § 4 we referred to the fact that Fréchet employs a distance function δ, having certain properties. It is possible to separate some of these properties,

and we thus obtain the following, of which the last seven are analogous to the properties of the K-relation above, with corresponding numbers:

(0)* $\qquad \delta_{q_1 q_2} \geqq 0$ for every pair q_1, q_2.

(2) $\qquad \delta_{q_1 q_2} = \delta_{q_2 q_1}$.

The δ is a symmetrical function of its arguments.

(3) $\qquad \delta_{q_1 q_2} = 0 \cdot \supset \cdot q_1 = q_2$.

If for a pair of values $q_1 q_2$ the δ function has the value zero, the two members of the pair are identical.

(4) $\qquad q_1 = q_2 \cdot \supset \cdot \delta_{q_1 q_2} = 0$.

(5)† $\qquad \exists\, \phi_e \ni \left(\underset{e=0}{L} \phi_e = 0 \cdot \delta_{q_1 q_2} \leqq e \cdot \delta_{q_2 q_3} \leqq e \cdot \supset \cdot \delta_{q_1 q_3} \leqq \phi_e \right)$.

(6) $\qquad \exists\, \phi_e \ni \left(\underset{e=0}{L} \phi_e = 0 \cdot \delta_{q_2 q_1} \leqq e \cdot \delta_{q_2 q_3} \leqq e \cdot \supset \cdot \delta_{q_1 q_3} \leqq \phi_e \right)$.

(7) $\qquad \exists\, \phi_e \ni \left(\underset{e=0}{L} \phi_e = 0 \cdot \delta_{q_1 q_2} \leqq e \cdot \delta_{q_3 q_2} \leqq e \cdot \supset \cdot \delta_{q_1 q_3} \leqq \phi_e \right)$.

(8) $\qquad \exists\, \phi_e \ni \left(\underset{e=0}{L} \phi_e = 0 \cdot \delta_{q_2 q_1} \leqq e \cdot \delta_{q_3 q_2} \leqq e \cdot \supset \cdot \delta_{q_1 q_3} \leqq \phi_e \right)$.

We denote by δ^2, δ^{34}, etc., a δ having the properties (2), the properties (3) and (4), etc.

In a system $(\mathfrak{Q}; \delta)$ we are able to define a K, and thus obtain a system $(\mathfrak{Q}; K)$ as follows:

$$K_{q_1 q_2 m} \equiv \delta_{q_1 q_2} \leqq \frac{1}{2^m} ;$$

i. e., the K-relation holds between q_1, q_2 and m, if $\delta_{q_1 q_2} = \frac{1}{2^m}$. Denote the K-relation thus defined by K_δ. Then it follows at once:

---

* This property, not analogous to any properties of the K, plays a rôle only in that it avoids the persistent use of the absolute value of the δ. From any given δ we can obtain an equally effective δ having the property (0), by simply taking the absolute of the given δ. We shall suppose in the sequel that this has been done, and that we are operating with a δ having the property (0).

† *I. e.*, there exists a real-valued single-valued function ϕ_e which approaches zero as e approaches zero, such that if $\delta_{q_1 q_2} \leqq e$ and $\delta_{q_2 q_3} \leqq e$ then $\delta_{q_1 q_3} \leqq \phi_e$. Moreover, we are able to say that there exists a function ϕ which is bounded for all finite values of e. For suppose it were not so. Then there would exist

(a) $\{ e_n \}$, e_0, such that $e_n < e_0$;

(b) $\{ q_{1n} \}$, $\{ q_{2n} \}$, $\{ q_{3n} \}$, such that $\delta_{q_{1n} q_{2n}} \leq e_n$ and $\delta_{q_{2n} q_{3n}} \leq e_n$ and $\delta_{q_{1n} q_{3n}} > n$.

But

$$\delta_{q_{1n} q_{2n}} \leqq e_0 \cdot \delta_{q_{2n} q_{3n}} \leqq e_0 \cdot \supset \cdot \delta_{q_{1n} q_{3n}} \leqq \phi_{e_0},$$

and we thus have a contradiction. We shall therefore suppose that the ϕ chosen is bounded for all finite values of e. Similarly for $\phi^6 e$, $\phi^7 e$, $\phi^8 e$ below.

(1) K_δ^1; *i. e*, the K-relation defined from the δ has the property (1) of § 4. We have further:

(2) If δ has any of the properties (2), (3), (4), (5), (6), (7), (8) above, the K_δ has the corresponding properties. In full:

$$\delta^2 \cdot \supset \cdot K_\delta^2;\ \delta^3 \cdot \supset \cdot K_\delta^3;\ \delta^4 \cdot \supset \cdot K_\delta^4;\ \delta^5 \cdot \supset \cdot K_\delta^5;\ \delta^6 \cdot \supset \cdot K_\delta^6;$$
$$\delta^7 \cdot \supset \cdot K_\delta^7;\ \delta^8 \cdot \supset \cdot K_\delta^8.$$

The first three of these cause no particular difficulty. As for $\delta^5 \cdot \supset \cdot K_\delta^5$, it is necessary to construct a ϕ^5. Let:

$$E_m = \overline{B}\left(\phi_e \mid \frac{1}{2^{m-1}} < e \leqq \frac{1}{2^m}\right);$$

i. e., the least upper bound of the values of ϕ_e, while e lies between $\frac{1}{2^{m-1}}$ and $\frac{1}{2^m}$. In accordance with the foot-note on p. 247, this will evidently exist. Then if

$$\frac{1}{2^{m'-1}} < E_m \leqq \frac{1}{2^{m'}},$$

we set $\phi_m = m'$. In this way we define a function ϕ with integral values ϕ_m, and evidently

$$\underset{m=\infty}{L}\, \phi_m = \infty.$$

A similar construction holds for (6), (7), and (8).

We thus have the result that the Fréchet theory is a special case of a K-theory.

On the other hand, in a system $(\Omega; K)$ where the K-relation has the property (1), of § 4, we can define a δ, and hence obtain a system $(\Omega; \delta)$, in the following manner:

(a).* If $K_{q_1 q_2 m}(m)$, then $\delta_{q_1 q_2} = 0$.

(b). If not (*a*), then set $\delta_{q_1 q_2} = \frac{1}{2^{m'}}$, where†

$$m' = \overline{B}(m \ni K_{q_1 q_2 m}).$$

If we define a δ corresponding to a K-relation having the property (1), in this way we are able to state a proposition analogous to proposition (2) above.

---

* If $K_{q_1 q_2 m}$ holds for all values of m, then $\delta_{q_1 q_2} = 0$.

† m' is the least upper bound of the values of m for which the K-relation holds between $q_1 q_2 m$.

As a consequence, whatever theorems involving the properties of § 4 are holding in a system $(\Omega; K)$, are also holding in a system $(\Omega; \delta)$; and conversely, the theorems holding in a $(\Omega; \delta)$ will also hold in a $(\Omega; K)$, provided the K-relation in question has the property (1).

It seems then that the K-relation with the property (1) of § 4 is not more general than a δ. We can, however, regard the K-relation in the light of an operation which exhibits a δ having rational values only, corresponding to any given δ. If, moreover, the K-relation is sufficient for our theory, then we have shown, incidentally, that the essential part of the δ is a rational part. We shall therefore use the K-relation in preference to the δ.

7. *Limit defined in terms of the K-relation.* In terms of the K-relation we may define limit as follows:*

$$\underset{n}{L}q_n = q \mathrel{:\equiv:} m \mathrel{:\supset:} \exists\, n_m \,.\, n \geqq n_m \mathrel{.\supset.} K_{q_n q m}.$$

Evidently such a limit is an L. We shall denote it by L_K. It follows then that any theory obtainable in a system $(\Omega; L)$ is also holding in a system $(\Omega; K)$ with $L = L_K$.

We have the following propositions relative to the properties of L_K:

(1) $$K \mathrel{.\supset.} L_K^{2345},$$

where (2), (3), (4), (5) are the properties of L of § 3. This is at once evident from the definition of L_K.

(2) $$K^4 \mathrel{.\supset.} L_K^6,$$

which is also an immediate consequence of the definition of L_K.

(3) $$K^{136} \mathrel{.\supset.} L_K^1;$$

i. e., if K^{136}, then the L_K produces a unique limit. This may be shown as follows: suppose

$$\underset{n}{L}q_n = q', \text{ and } \underset{n}{L}q_n = q''.$$

---

* $\underset{n}{L}q_n = q$ is by definition the same as the statement: For every m there exists an integer n_m such that for every n greater than n_m we have $K_{q_n q m}$. Of course, this is not the only possible definition of limit in a K-situation; *e. g.*, we might define:

$$\underset{n}{L}q_n = q \mathrel{:\equiv:} m \mathrel{:\supset:} \exists\, n_m \,.\, n = n_m \mathrel{.\supset.} K_{q_n q m}.$$

The above definition, however, is the analogue of the Fréchet definition:

$$\underset{n}{L}q_n = q \mathrel{.\equiv.} \underset{n}{L}\delta_{q_n q} = 0.$$

Then by the definition of L_K we have:

$$m \cdot \supset \cdot \exists\, n_m \ni n \geqq n_m \cdot \supset \cdot K_{q_n q' m} \text{ and } K_{q_n q'' m},$$

the n_m being the greater of the values of n_m for q' and q''. Then K^6 gives us:

$$K_{q' q'' \phi_m}(m).$$

Now since $\underset{m}{L}\phi_m = \infty$, and K has the property (1), we have:

$$K_{q' q'' m}(m),$$

and so K^3 gives us

$$q' = q''.$$

Usually in a situation in which it is a question of arguing the fact that a sequence approaches an element as a limit, we do not obtain a statement which is identical with the above definition, but rather one of which the statement of the definition is a consequence. One of the cases of frequent occurrence is covered by the following lemma:*

$$(4) \qquad K^1 \cdot \{q_n\}, q :\supset: \exists\, \psi_n \ni (\underset{n}{L}\psi_n = \infty \cdot K_{q_n q \psi_n}) \cdot \supset \cdot \underset{n}{L} q_n = q.$$

For since $L\psi_n = \infty$, we have:

$$m :\supset: \exists\, n_m \ni n \geqq n_m \cdot \supset \cdot \psi_n > m.$$

Since K^1, this n_m will also serve as the n_m of the definition of limit.

As a special case of this we might consider this proposition when for every n we have

$$q_n = q'.$$

Then we have:

$$(5) \quad K^1 \cdot q \cdot q_n = q'(n) :\supset: \exists\, \psi_n \ni (\underset{n}{L}\psi_n = \infty \cdot K_{q_n q \psi_n}) \cdot \supset \cdot K_{q' q m}(m).$$

This lemma was used in the proof of (3) above.

8. *On the Composition of Classes of Elements, and of K-relations.* Suppose two classes $\mathfrak{Q}' = [q']$, and $\mathfrak{Q}'' = [q'']$. We derive from these classes a product or composite class:

$$\mathfrak{Q}'\mathfrak{Q}'' = [(q', q'')];$$

i. e., a class $\mathfrak{Q} = [q]$ whose elements $q = (q', q'')$ are bipartite, q' and q'' independently ranging over the class $\mathfrak{Q}'$ and $\mathfrak{Q}''$ respectively. In practice, when there is no possibility of misinterpretation, we denote the element (q', q'') by $q'\,q''$.

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* *I. e.*, if the K-relation has the property (1) of §4, and the sequence $\{q_n\}$ and the element q are such that there exists a function ψ with integral values ψ_n, approaching infinity with n, and such that $K_{q_n q \psi_n}$, then the sequence has q as a limit.

Similarly, from a finite number of classes $\mathfrak{Q}'$, $\mathfrak{Q}''$, ..., $\mathfrak{Q}^n$ we obtain a composite class:

$$\mathfrak{Q}'\mathfrak{Q}''\ldots\mathfrak{Q}^n = [q'q''\ldots q^n].$$

Returning to the case $\mathfrak{Q}'\mathfrak{Q}''$, suppose there exist in $\mathfrak{Q}'$ and $\mathfrak{Q}''$ K-relations: K' and K'', respectively; *i e.*, suppose we are dealing with the systems $(\mathfrak{Q}'; K')$ and $(\mathfrak{Q}''; K'')$. Then the existence of a system $(\mathfrak{Q}; K)$ is of interest. This depends upon the possibility of defining a K-relation from the K' and K''. We construct such a K-relation as follows:

Suppose $q_1 = q_1'q_1''$ and $q_2 = q_2'q_2''$. Then

$K_{q_1q_2m}$ if there exist m' and m'' such that $K'_{q'_1q'_2m'}$ and $K''_{q''_1q''_2m''}$, and m is the smaller of m' and m''.

Evidently such a K-relation will satisfy the condition that for every pair of elements q_1 and q_2, of the class $\mathfrak{Q}$, there exists an m such that $K_{q_1q_2m}$, in so far as such a condition is holding in the systems $(\mathfrak{Q}'; K')$ and $(\mathfrak{Q}''; K'')$.*

We consider how the presence of properties (1)–(8) of § 4 of the K' and K'' affects the presence of the corresponding properties of the K. We have:

$$K'^P \cdot K''^P \cdot\supset\cdot K^P,$$

where $P = (1), (2), (4), (5), (6), (7), (8)$, but

$$K'^{13} \cdot K''^{13} \cdot\supset\cdot K^3.$$

The proof of these propositions is very simple.

We can then state the result that if we have a theory in a system $(\mathfrak{Q}'; K')$, which concerns itself with properties (1)–(8) of § 4, a corresponding theory is holding in a composite class built up of two such systems.

A similar construction and result relative to the composite class and the composite K will evidently hold if we are dealing with any finite number of systems $(\mathfrak{Q}; K)$.

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* The above definition is not the only one possible. Others which suggest themselves are:

(a) $K_{q_1q_2m}$ if both $K'_{q'_1q'_2m}$ and $K''_{q''_1q''_2m}$.

(b) $K_{q_1q_2m}$ if either $K'_{q'_1q'_2m}$ or $K''_{q''_1q''_2m}$.

(c) $K_{q_1q_2m}$ if there exist m' and m'' such that $K'_{q'_1q'_2m'}$ and $K''_{q''_1q''_2m''}$ and m is the larger of m' and m''.

(d) $K_{q_1q_2m}$ if there exist m' and m'' such that $K'_{q'_1q'_2m'}$ and $K''_{q''_1q''_2m''}$ and $m = m' + m''$.

(e) $K_{q_1q_2m}$ if there exist m' and m'' such that $K'_{q'_1q'_2m'}$ and $K''_{q''_1q''_2m''}$ and $m = m' \times m''$.

All of these except the first have the defect that the theorem K'^3 and $K''^3 \cdot\supset\cdot K^3$ is not holding, and moreover it is not possible to find a simple condition on K' and K'' which will carry with it the presence of the property (3) in K. In the case of the first, we do not have a system $(\mathfrak{Q}; K)$ as defined in § 4, unless K' and K'' have the property (1) of § 4.

9. *On the Complete Existential Theory of the Properties of the K-relation.* We conclude this part by the construction of a complete existential theory of the properties (1)–(8) of the K-relation.

*Definition.** *The complete existential theory* of a set of properties P_1, P_2, ..., P_n of systems Σ, consists of (a) the body of interrelations between the properties, and (b) a body of systems Σ, there being a system Σ for each of the conceivable combinations of properties P_1, ..., P_n and their negatives $-P_1$, ..., $-P_n$, not excluded by the interrelations of (a).

In case a system Σ is obtainable for each of the 2^n combinations of P_1, ..., P_n and their negatives $-P_1$, ..., $-P_n$, the set of properties P_1, ..., P_n is said to be *completely independent and consistent.*

Evidently, if a set of n properties P_1, ..., P_n are completely independent and consistent, they are also independent in the ordinary sense, *i. e.*, it is possible to find systems Σ having the properties:

$$-P_1, P_2, \ldots, P_n;\quad P_1, -P_2, P_3, \ldots, P_n;\quad \ldots;\quad P_1, P_2, \ldots, P_{n-1}, -P_n.$$

We propose to consider the eight properties of the K-relation given in § 4. In § 5 we have derived a set of relations between these properties. These relations exclude 176 of the $2^8 = 256$ combinations of the properties (1)–(8), and their negatives, leaving 80 to be discussed. To complete the existential theory it is necessary to obtain a K-relation for each of these eighty combinations of properties of K. We take up these eighty K-relations relative to classes $\mathfrak{Q}$ of the following types:

(a) A class $\mathfrak{Q}$ consisting of one element.
(b) A class $\mathfrak{Q}$ consisting of two elements.
(c) A class $\mathfrak{Q}$ consisting of three elements.
(d) A class $\mathfrak{Q}$ consisting of four elements.
(e) A class $\mathfrak{Q}$ consisting of a finite number of elements.
(f) A class $\mathfrak{Q}$ consisting of a denumerable infinitude of elements.
(g) A class $\mathfrak{Q}$ consisting of a non-denumerable infinitude of elements of the power of the interval 0....1. These are denoted by

$$\mathfrak{Q}^{I}, \mathfrak{Q}^{II_2}, \mathfrak{Q}^{II_3}, \mathfrak{Q}^{II_4}, \mathfrak{Q}^{II_\infty}, \mathfrak{Q}^{III}, \mathfrak{Q}^{IV},$$

respectively.

In the following table, which gives the eighty combinations of the presence and absence of the properties of K, let + stand for the presence and — for the absence of the property. The Roman numerals in the last column give the type of class $\mathfrak{Q}$ of smallest dimension in which it is possible to determine a K-relation having the combination of properties in question.

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* Cf. Moore, *loc. cit.*, p. 82.

TABLE I.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)			(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	
1	+	+	+	+	+	+	+	+	I, II_2	41	−	−	+	−	+	−	+	+	II_2
2	+	+	+	+	−	−	−	−	III	42	−	−	+	−	−	+	+	+	II_3
3	−	+	+	+	+	+	+	+	II_2	43	−	−	+	−	+	+	−	−	II_2
4	−	+	+	+	−	−	−	−	II_3	44	−	−	+	−	+	−	+	−	II_2
5	+	−	+	+	+	+	+	+	II_2	45	−	−	+	−	−	+	+	−	II_3
6	+	−	+	+	+	−	−	−	III	46	−	−	+	−	+	−	−	+	II_2
7	+	−	+	+	−	−	−	−	III	47	−	−	+	−	−	+	−	+	II_3
8	+	+	−	+	+	+	+	+	II_2	48	−	−	+	−	−	−	+	+	II_3
9	+	+	−	+	−	−	−	−	II_3	49	−	−	+	−	+	−	−	−	II_2
10	+	+	+	−	+	+	+	+	I, II_2	50	−	−	+	−	−	+	−	−	II_3
11	+	+	+	−	−	−	−	−	III	51	−	−	+	−	−	−	+	−	II_3
12	−	−	+	+	+	+	+	+	II_2	52	−	−	+	−	−	−	−	+	II_3
13	−	−	+	+	+	−	−	+	II_2	53	−	−	+	−	−	−	−	−	II_2
14	−	−	+	+	+	−	−	−	II_2	54	−	+	−	−	+	+	+	+	II_2
15	−	−	+	+	−	−	−	−	II_3	55	−	+	−	−	−	−	−	−	II_2
16	−	+	−	+	+	+	+	+	II_3	56	+	−	−	−	+	+	+	+	II_3
17	−	+	−	+	−	−	−	−	II_3	57	+	−	−	−	+	+	−	−	II_2
18	−	+	+	−	+	+	+	+	I, II_2	58	+	−	−	−	+	−	+	−	II_2
19	−	+	+	−	−	−	−	−	II_2	59	+	−	−	−	+	−	−	+	II_2
20	+	−	−	+	+	+	+	+	II_3	60	+	−	−	−	+	−	−	−	II_3
21	+	−	−	+	+	−	−	−	II_2	61	+	−	−	−	−	+	−	−	II_3
22	+	−	−	+	−	−	−	−	II_3	62	+	−	−	−	−	−	+	−	II_3
23	+	−	+	−	+	+	+	+	II_2	63	+	−	−	−	−	−	−	+	II_3
24	+	−	+	−	+	+	−	−	III	64	+	−	−	−	−	−	−	−	II_3
25	+	−	+	−	+	−	+	−	III	65	−	−	−	−	+	+	+	+	II_2
26	+	−	+	−	+	−	−	+	III	66	−	−	−	−	+	+	+	−	II_4
27	+	−	+	−	+	−	−	−	III	67	−	−	−	−	+	+	−	+	II_3
28	+	−	+	−	−	+	−	−	III	68	−	−	−	−	+	−	+	+	II_3
29	+	−	+	−	−	−	+	−	III	69	−	−	−	−	−	+	+	+	II_3
30	+	−	+	−	−	−	−	+	III	70	−	−	−	−	+	+	−	−	II_2
31	+	−	+	−	−	−	−	−	III	71	−	−	−	−	+	−	+	−	II_2
32	+	+	−	−	+	+	+	+	II_3	72	−	−	−	−	−	+	+	−	II_3
33	+	+	−	−	−	−	−	−	II_2	73	−	−	−	−	+	−	−	+	II_2
34	−	−	−	+	+	+	+	+	II_2	74	−	−	−	−	−	+	−	+	II_3
35	−	−	−	+	+	−	−	+	II_3	75	−	−	−	−	−	−	+	+	II_3
36	−	−	−	+	+	−	−	−	II_2	76	−	−	−	−	+	−	−	−	II_2
37	−	−	−	+	−	−	−	−	II_3	77	−	−	−	−	−	+	−	−	II_3
38	−	−	+	−	+	+	+	+	II_2	78	−	−	−	−	−	−	+	−	II_3
39	−	−	+	−	+	+	+	−	II_3	79	−	−	−	−	−	−	−	+	II_3
40	−	−	+	−	+	+	−	+	II_2	80	−	−	−	−	−	−	−	−	II_2

(a) The class $\mathfrak{Q}$ consisting of a single element. Let the element be q. Then to define the K-relation it is necessary to state the values of m for which K_{qqm} holds. Then we have:

(1) $$(\mathfrak{Q}^{\mathrm{I}}; K) \cdot \supset \cdot K^{235678}.$$

There is no difficulty about seeing that K^{25678}. As for K^3 we either have

$$K_{qqm}(m) \quad \text{or} \quad \exists\, m \,\mathfrak{z}\, -K_{qqm}.$$

In the first case K^3, and in the second K^3 *vacuously.** We have further:

(2) $$(\mathfrak{Q}^{\mathrm{I}}; K) :\supset: K^4 \cdot \supset \cdot K^1.$$

For if K^4 we must have $K_{qqm}(m)$, and so K^1.

These two propositions exclude all but the following three combinations:

1. $+++++++++$; 10. $+++-++++$; 18. $-++-++++$.

For 1, let K_{qqm} hold for every m; for 10, let K_{qqm} hold for $m \leqq m_1$; and for 18, let K_{qqm} hold for $m \geqq m_1$.

The case in which the class $\mathfrak{Q}$ consists of only one element not being of frequent occurrence, we give K-relations satisfying the combinations 1, 10, 18 under the next head also.

(b) The class $\mathfrak{Q}$ consisting of two elements, q_1 and q_2. In this case we must assign values for which

$$K_{q_1q_1m}, \; K_{q_1q_2m}, \; K_{q_2q_1m}, \text{ and } K_{q_2q_2m}$$

hold. We arrange the combinations possible in II_2 in tabular form, giving first the combination of properties, then the values of m for which the K-relations hold, and finally the functions ϕ^5, ϕ^6, ϕ^7, and ϕ^8, in case we have K^5, K^6, K^7 or K^8.

We thus obtain 34 systems in $\mathfrak{Q}^{\mathrm{II}_2}$. We are not able to obtain more on account of certain relations which exist between these properties in case the class $\mathfrak{Q}$ consists of two elements only. These relations, suggesting themselves in the attempt to obtain a K-relation satisfying combinations of properties which they exclude, are:

(1) $$K^{13} \cdot \supset \cdot K^{5678}.$$

* This term and concept, introduced by Moore (*Transactions of the American Mathematical Society*, III, p. 489, foot-note), is of considerable importance in this type of discussion. We shall make use of it frequently, especially as far as the property (3) is concerned.

TABLE II.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	$K_{q_1q_1m}$	$K_{q_1q_2m}$	$K_{q_2q_1m}$	$K_{q_2q_2m}$	ϕ
1	+	+	+	+	+	+	+	+	(m)	$m \leqq m_1$	$m \leqq m_1$	(m)	$\phi_m = m$
3	—	+	+	+	+	+	+	+	(m)	$m \geqq m_1$	$m \geqq m_1$	(m)	$\phi_m = m$
5	+	—	+	+	+	+	+	+	(m)	$m \leqq m_1$	$m \leqq m_2 < m_1$	(m)	$\phi_m = m$
8	+	+	—	+	+	+	+	+	(m)	(m)	(m)	(m)	$\phi_m = m$
10	+	+	+	—	+	+	+	+	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$\phi_m = m$
12	—	—	+	+	+	+	+	+	(m)	$m \geqq m_1$	$m \geqq m_2 > m_1$	(m)	$\phi_m = m$
13	—	—	+	+	+	—	—	+	(m)	m odd	m even	(m)	$\phi^5_m = m$; $\phi^8_m = m+1$
14	—	—	+	+	+	—	—	—	(m)	$m \geqq m_1$	$m < m_1$	(m)	$\phi^5_m = m$
18	—	+	+	—	+	+	+	+	$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_1$	$\phi_m = m$
19	—	+	+	—	—	—	—	—	$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_1$	$m < m_1$	
21	+	—	—	+	+	—	—	—	(m)	(m)	$m \leqq m_1$	(m)	$\phi^5_m = m$
23	+	—	+	—	+	+	+	+	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_2 < m_1$	$m \leqq m_1$	$\phi_m = m$: $m > m_1$ $\phi_m = m_2$: $m < m_1$
33	+	+	—	—	—	—	—	—	$m \leqq m_1$	(m)	(m)	$m \leqq m_1$	
34	—	—	—	+	+	+	+	+	(m)	(m)	$m \geqq m_1$	(m)	$\phi_m = m$: $m \geqq m_1$ $\phi_m = m_1$: $m < m_1$
36	—	—	—	+	+	—	—	—	(m)	(m)	m_1	(m)	$\phi^5_m = m$
38	—	—	+	—	+	+	+	+	$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_2 > m_1$	$m \geqq m_1$	$\phi_m = m$: $m \geqq m_2$ $\phi_m = m_2$: $m < m_2$
40	—	—	+	—	+	+	—	+	m odd, m_1	m_1	m even, m_1	m_1	$\phi_{m_1} = m_1$; $\phi^{58}_m = m$; $\phi^6_m = m+1$
41	—	—	+	—	+	—	+	+	m odd, m_1	m even, m_1	m_1	m_1	$\phi_{m_1} = m_1$; $\phi^{58}_m = m$; $\phi^7_m = m+1$
43	—	—	+	—	+	+	—	—	$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	$m < m_1$	$\phi^{56}_m = m$
44	—	—	+	—	+	—	+	—	$m < m_1$	$m < m_1$	$m \geqq m_1$	$m \geqq m_1$	$\phi^{57}_m = m$
46	—	—	+	—	+	—	—	+	$m < m_1$	$m \geqq m_2 > m_1$	$m_1 < m < m_2$	$m < m_1$	$\phi^{58}_m = m$
49	—	—	+	—	+	—	—	—	$m \geqq m_1$	$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	$\phi^5_m = m$
53	—	—	+	—	—	—	—	—	$m < m_1$	$m \geqq m_1$	$m \geqq m_1$	$m < m_1$	
54	—	+	—	—	+	+	+	+	$m \geqq m_1$	(m)	(m)	$m \geqq m_1$	$\phi_m = m$: $m \geqq m_1$ $\phi_m = m_1$: $m < m_1$
55	—	+	—	—	—	—	—	—	$m < m_1$	(m)	(m)	$m \geqq m_1$	
57	+	—	—	—	+	+	—	—	$m \leqq m_1$	(m)	$m \leqq m_1$	(m)	$\phi^{56}_m = m$
58	+	—	—	—	+	—	+	—	(m)	(m)	$m \leqq m_1$	$m \leqq m_1$	$\phi^{57}_m = m$
59	+	—	—	—	+	—	—	+	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$\phi^{58}_m = m$
65	—	—	—	—	+	+	+	+	$m \geqq m_1$	(m)	$m \geqq m_1$	$m \geqq m_1$	$\phi_m = m$: $m \geqq m_1$ $\phi_m = m_1$: $m < m_1$
70	—	—	—	—	+	+	—	—	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \geqq m_1$	ϕ^{56}; cf. 65
71	—	—	—	—	+	—	+	—	$m \geqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	ϕ^{57}; cf. 65
73	—	—	—	—	+	—	—	+	$m \leqq m_1$	(m)	$m = m_1$	$m \leqq m_1$	ϕ^{58}; cf. 65
76	—	—	—	—	+	—	—	—	$m \geqq m_1$	(m)	$m < m_1$	$m \geqq m_1$	ϕ^5; cf. 65
80	—	—	—	—	—	—	—	—	$m < m_1$	(m)	$m \geqq m_1$	$m \geqq m_1$	

This proposition holds also when $\mathfrak{Q}$ is any class of finite dimensions. We prove it by constructing the ϕ. Evidently *

$$\overline{m}' = \overline{B}\,(m \ni K_{q_i q_k m},\ i \neq k;\quad i, k = 1, 2, \ldots, n)$$

exists. Also, if K does not have the property (4), there will exist:

$$\overline{m}'' = \overline{B}\,(m \ni K_{q_i q_i m},\ i \neq j \ni K_{q_j q_j m}\ (m)).$$

Let $\overline{m}$ be the greater of these two, which, in case K^4, will evidently be $\overline{m}'$. Let further:

$$\underline{m}' = \underline{B}\,(\overline{B}\,(m \ni K_{q_i q_k m},\ i \neq k) \mid i, k = 1, 2, \ldots, n),$$

$$\underline{m}'' = \underline{B}\,(\overline{B}\,(m \ni K_{q_i q_i m},\ i \neq j \ni K_{q_j q_j m}\ (m)) \mid i)$$

and $\underline{m}$ the smaller of these. This will exist on account of the finiteness of the class $\mathfrak{Q}$. Then we construct:

$$\phi_m = \underline{m} \text{ when } m \leqq \overline{m}, \text{ and } \phi_m = m \text{ when } m > \overline{m}.$$

It is easy to see that this ϕ serves the desired purpose.

(2) $$K^{24-3} \cdot \supset \cdot K^{15678}.$$

Since K^4 we have

$$K_{q_1 q_1 m}(m), \text{ and } K_{q_2 q_2 m}(m).$$

Further, from K^{2-3},

$$K_{q_1 q_2 m}(m), \text{ and } K_{q_2 q_1 m}(m).$$

Hence K^{15678}.

(3) $$K^4 \cdot \supset \cdot K^5.$$

For if K^4 we can choose $\phi_m = m$.

(4) $$K^{1-2} \cdot \supset \cdot K^5.$$

We can construct a ϕ in a manner similar to that used in proposition (1).

(5) $$K^{167-3} \cdot \supset \cdot K^{2458}.$$

If K^{-3}, we have either $K_{q_1 q_2 m}(m)$ or $K_{q_2 q_1 m}(m)$. Suppose $K_{q_1 q_2 m}(m)$. Then

$$K^{16} \cdot \supset \cdot K_{q_2 q_2 m}(m),\ K^{17} \cdot \supset \cdot K_{q_2 q_1 m}(m), \text{ and } K^{16} \cdot \supset \cdot K_{q_1 q_1 m}(m).$$

And so we have K^{2458}.

(6) $$K^{48-3} \cdot \supset \cdot K^{67}.$$

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* $\overline{m}'$ is the least upper bound of the values of m for which $K_{q_i q_k m}$ holds, where $i \neq k$, and both i and k range over the values 1, 2, ..., n.

By K^4 we have $K_{q_1q_1m}(m)$, and $K_{q_2q_2m}(m)$. Also by K^{-3} we have either $K_{q_1q_2m}(m)$ or $K_{q_2q_1}(m)$. But in either case the presence of K^8 permits us to take

$$\phi^6_m = \phi^7_m = \phi^8_m.$$

(7) $$K^6 \cdot \supset \cdot K^5.$$

K^5 affects the following pairs of K-relations:

$$K_{q_1q_1m}, K_{q_1q_1m}; \quad K_{q_1q_1m}, K_{q_1q_2m}; \quad K_{q_1q_2m}, K_{q_2q_1m}; \quad K_{q_1q_2m}, K_{q_2q_2m};$$
$$K_{q_2q_1m}, K_{q_1q_1m}; \quad K_{q_2q_1m}, K_{q_1q_2m}; \quad K_{q_2q_2m}, K_{q_2q_1m}; \quad K_{q_2q_2m}, K_{q_2q_2m}.$$

Of these K^6 relates to the following pairs:

$$K_{q_1q_1m}, K_{q_1q_1m}; \quad K_{q_1q_1m}, K_{q_1q_2m}; \quad K_{q_2q_2m}, K_{q_2q_1m}; \quad K_{q_2q_2m}, K_{q_2q_2m}.$$

Moreover, by K^6 we have:

$$K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_2\phi^6m}, \text{ and } K_{q_2q_1m} \cdot \supset \cdot K_{q_1q_1\phi^6m}.$$

So that

$$K_{q_1q_2m}, K_{q_2q_1m} \cdot \supset \cdot K_{q_1q_1m\,\phi^5_m}$$

and

$$K_{q_2q_1m}, K_{q_1q_2m} \cdot \supset \cdot K_{q_2q_2\phi^5_m}.$$

There thus remain to be treated only:

$$K_{q_1q_2m} \text{ with } K_{q_2q_2m} \text{ and } K_{q_2q_1m} \text{ with } K_{q_1q_1m}.$$

We build our ϕ^5 as follows: for m's common only to $K_{q_1q_2m}$ and $K_{q_2q_2m}$, and common only to $K_{q_2q_1m}$ and $K_{q_1q_1m}$, let $\phi^5_m = m$. For all other values of m let $\phi^5_m = \phi^6_m$. It is not difficult to show that this is an effective ϕ^5_m.

(8) $$K^7 \cdot \supset \cdot K^5.$$

The demonstration of this is analogous to that of (7).

(9) $$K^8 \cdot \supset \cdot K^5.$$

We show that it is possible to choose $\phi^5_m = \phi^8_{\phi^6_m}$, by taking up each individual pair to which K^5 applies.

(10) $$K^{67} \cdot \supset \cdot K^8.$$

The proof is similar to that of (7).

(11) $$K^{68-8} \cdot \supset \cdot K^7.$$

By writing down all pairs to which K^7 applies, and using the fact that since

K^{-3}, we have either $K_{q_1q_2m}(m)$ or $K_{q_2q_1m}(m)$, we find without much difficulty that $\phi^7_m = \phi^6_{\phi^8{}_{\phi^6{}_m}}$.

(12) $$K^{78-3} \cdot \supset \cdot K^6.$$

The proof of this is similar to that of (11). We find that $\phi^6_m = \phi^7_{\phi^8{}_{\phi^7{}_m}}$.

(13) $$K^{1-2-3} \cdot \supset \cdot K^4 \text{ or } K^6 \text{ or } K^7 \text{ or } K^8.$$

Since K^{-3}, we have either $K_{q_1q_2m}(m)$ or $K_{q_2q_1m}(m)$, but not both, since K^{-2}. Suppose $K_{q_1q_2m}(m)$. Then $K^{1-2} \cdot \supset \cdot \exists\, m_1 \ni K_{q_2q_1m}$ for $m \leqq m_1$ only. There are then four possibilities relative to $K_{q_1q_1m}$ and $K_{q_2q_2m}$:

(α) $K_{q_1q_1m}(m)$, and $K_{q_2q_2m}(m)$. Then K^4.

(β) $K_{q_1q_1m}(m)$, and $K_{q_2q_2m}$ $m \leqq m_2$. Then K^7.

(γ) $K_{q_1q_1m}$ for $m \leqq m_2$ and $K_{q_2q_2m}(m)$. Then K^6.

(δ) $K_{q_1q_1m}$ for $m \leqq m_2$ and $K_{q_2q_2m}$ for $m \leqq m_3$. Then K^8.

These relations make impossible the systems not included in the above enumeration of the systems possible in Ω^{II_2}. In particular, (1) excludes 2, 6, 7, 11, 24, 25, 26, 27, 28, 29, 30, 31; (2) excludes further 9, 16, 17; (3) excludes 4, 5, 22, 37; (4) excludes 61, 62, 63, 64; (5) excludes 20, 32, 56; (6) excludes 35; (7) excludes 42, 45, 47, 50, 69, 72, 74, 77; (8) excludes 48, 51, 75, 78; (9) excludes 52, 79; (10) excludes 39, 66; (11) excludes 67; (12) excludes 68; (13) excludes 60. This totals 46, the number of systems which have not been obtained thus far.

(c) The class Ω consisting of three elements; *i. e.*, Ω^{II_3}. We pass now to the case in which the class Ω consists of three elements. We do not take up the combinations of properties of the K-relation found to exist in a $(\Omega^{II_2}; K)$. As a matter of fact, we shall show later how to modify the systems obtained in Ω^{II_2}, so as to procure systems in Ω^{II_3} having the same combinations of properties.

As regards the relations which hold between the properties of the K-relation when Ω consists of three elements, we have already seen that the relation

(1) $$K^{13} \cdot \supset \cdot K^{5678}$$

holds when Ω is of finite dimension, and hence when Ω has only three elements. There is but one other proposition holding in Ω^{II_3}:

(2) $$K^{567-3} \cdot \supset \cdot K^8.$$

This is proved by considering all of the pairs of K-relations to which K^8 applies.

If we assume that $K_{q_1q_2m}(m)$, one relation of this type holding on account of K^{-3}, it is easily apparent that we have $\phi^8 = \phi^6_{\phi^7\phi^5}$, excepting for the following four pairs of K-relations:

$K_{q_1q_3m} \cdot K_{q_1q_1m}$; $K_{q_1q_3m} \cdot K_{q_2q_1m}$; $K_{q_2q_3m} \cdot K_{q_1q_2m}$; $K_{q_2q_3m} \cdot K_{q_2q_2m}$, for which we have as a possible ϕ^8: $\phi^8 = \phi^7_{\phi^6\phi^5}$. We then build a ϕ^8 as follows: Take $\phi^8 = \phi^6_{\phi^7\phi^5}$ excepting for values of m for which both $K_{q_1q_3m}$ and $K_{q_1q_1m}$ or both $K_{q_2q_3m}$ and $K_{q_2q_2m}$, and not $K_{q_3q_1m}$ or $K_{q_3q_3m}$, and for which both $K_{q_1q_3m}$ and $K_{q_2q_1m}$ or both $K_{q_2q_3m}$ and $K_{q_1q_2m}$, and not $K_{q_3q_1m}$ or $K_{q_3q_2m}$ or $K_{q_3q_3m}$, hold. For these values of m we take $\phi^8 = \phi^7_{\phi^6\phi^5}$. It is not difficult to show that this ϕ is uniquely determined for all values of m and that $\underset{m}{L}\phi_m = \infty$.

This proposition excludes but one further possibility: 66. There thus remain thirty-three combinations of properties of K which it is possible to obtain in a class Ω^{II_3}, in addition to those already obtained in Ω^{II_2}. These systems are given in Table III.

(d) The class Ω consisting of four elements. Passing to the case in which the class Ω consists of four elements, we obtain only one additional system, viz., one satisfying the combination:

66. $- - - - + + + -$.

The following K-relation will satisfy this combination of properties:

$K_{q_1q_1m}$: m_2, m_3, $m \geqq m_1$;	$K_{q_1q_2m}$: (m);	$K_{q_1q_3m}$: m_2, m_6;	$K_{q_1q_4m}$: m_3;
$K_{q_2q_1m}$: m_2, m_3, $m \geqq m_1$;	$K_{q_2q_2m}$: m_2, m_3, $m \geqq m_1$;	$K_{q_2q_3m}$: m_2, m_4;	$K_{q_2q_4m}$: m_3, m_6;
$K_{q_3q_1m}$: m_2;	$K_{q_3q_2m}$: m_2;	$K_{q_3q_3m}$: m_2;	$K_{q_3q_4m}$: m_4;
$K_{q_4q_1m}$: m_3;	$K_{q_4q_2m}$: m_3, m_5;	$K_{q_4q_3m}$: m_7;	$K_{q_4q_4m}$: m_2, m_3.

We suppose the $m_2, \ldots, m_7$ to be distinct, and less than m_1. The values of the ϕ-functions are as follows:

	m_2	m_3	m_4	m_6	$m \geqq m_1$	$m < m_1$
ϕ^5	m_2	m_3	m_6	m_3	m	m
ϕ^6	m_2	m_3	m_2	m_2	m	m_1
ϕ^7	m_2	m_3	m_2	m_3	m	m_1

In this situation K^{-8}, because the ϕ^8_m is not uniquely defined for the value m_4, a result of considering $K_{q_1q_2m_4}$ with $K_{q_2q_3m_4}$ and $K_{q_2q_3m_4}$ with $K_{q_3q_4m_4}$.

TABLE III.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	$K_{q_1q_1m}$	$K_{q_1q_2m}$	$K_{q_1q_3m}$	$K_{q_2q_1m}$	$K_{q_2q_2m}$
4	—	+	+	+	—	—	—	—	(m)	$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	(m)
9	+	+	—	+	—	—	—	—	(m)	(m)	$m \leqq m_1$	(m)	(m)
15	—	—	+	+	—	—	—	—	(m)	$m \geqq m_1$	$m < m_1$	$m \geqq m_2$	(m)
16	—	+	—	+	+	+	+	+	(m)	(m)	$m \geqq m_1$	(m)	(m)
17	—	+	—	+	—	—	—	—	(m)	(m)	$m < m_1$	(m)	(m)
20	+	—	—	+	+	+	+	+	(m)	(m)	$m \leqq m_1$	(m)	(m)
22	+	—	—	+	—	—	—	—	(m)	(m)	$m \leqq m_1$	(m)	(m)
32	+	+	—	—	+	+	+	+	(m)	(m)	$m \leqq m_1$	(m)	(m)
35	—	—	—	+	+	—	—	+	(m)	(m)	m_2	(m)	(m)
37	—	—	—	+	—	—	—	—	(m)	(m)	$m < m_1$	(m)	(m)
39	—	—	+	—	+	+	+	—	m_1, m_2, m_3	m_1, m_2	m_3	m_2	m_2
42	—	—	+	—	—	+	+	+	m_4	m_1	m_2	m_3	m_4
45	—	—	+	—	—	+	+	—	m_1, m_2	m_1, m_2	m_3	m_2	m_2
47	—	—	+	—	—	—	+	+	m_5	m_1	m_2	m_3, m_4	m_5
48	—	—	+	—	—	+	—	+	m_5	m_1	m_2	m_3	m_5
50	—	—	+	—	—	+	—	—	$m \leqq m_1$	$m > m_1$	$m \leqq m_1$	$m \leqq m_1$	$m > m_1$
51	—	—	+	—	—	—	+	—	$m \geqq m_1$	$m > m_1$	$m \leqq m_1$	$m > m_1$	$m > m_1$
52	—	—	+	—	—	—	—	+	$m \leqq m_1$	$m \geqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$
56	+	—	—	—	+	+	+	+	(m)	(m)	$m \leqq m_1$	(m)	(m)
60	+	—	—	—	+	—	—	—	$m \leqq m_1$	(m)	(m)	$m \leqq m_1$	$m \leqq m_1$
61	+	—	—	—	—	+	—	—	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	(m)
62	+	—	—	—	—	—	+	—	(m)	(m)	$m \leqq m_1$	(m)	(m)
63	+	—	—	—	—	—	—	+	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$
64	+	—	—	—	—	—	—	—	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$
67	—	—	—	—	+	+	—	+	$m = 2m', m_1$	(m)	m_1	$m = 2m', m_1$	$m = 2m', m_1$
68	—	—	—	—	+	—	+	+	$m = 2m', m_1$	(m)	$m = 2m'+1, m_1$	$m = 2m', m_1$	$m = 2m', m_1$
69	—	—	—	—	—	+	+	+	$m = 2m', m_1$	(m)	m_1	$m = 2m', m_1$	$m = 2m', m_1$
72	—	—	—	—	—	+	+	—	$m = 2m', m_1$	(m)	m_1	$m = 2m', m_1$	$m = 2m', m_1$
74	—	—	—	—	—	+	—	+	$m = 3m', m_1$	(m)	m_1	$m = 3m', m_1$	$m = 3m', m_1$
75	—	—	—	—	—	—	+	+	$m = 3m', m_1$	(m)	$m = 3m'+2, m_1$	$m = 3m', m_1$	$m = 3m', m_1$
77	—	—	—	—	—	+	—	—	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$m \geqq m_1$
78	—	—	—	—	—	—	+	—	$m \geqq m_1$	(m)	$m \leqq m_1$	(m)	(m)
79	—	—	—	—	—	—	—	+	$m \leqq m_1$	(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$

TABLE III—*Continued.*

$K_{q_2q_3m}$	$K_{q_3q_1m}$	$K_{q_3q_2m}$	$K_{q_3q_3m}$	Φ
$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	(m)	
(m)	$m \leqq m_1$	(m)	(m)	
$m \geqq m_1$	$m < m_1$	$m \geqq m_2$	(m)	
$m \geqq m_1$	$m \geqq m_1$	$m \geqq m_1$	(m)	$\Phi_m = m$
$m \geqq m_1$	$m < m_1$	$m \geqq m_1$	(m)	
$m \leqq m_1$	$m \leqq m_2$	$m \leqq m_2$	(m)	$m_2 < m_1$; $\Phi_m = m_2 : m \leqq m_1$; $\Phi_m = m : m > m_1$
(m)	$m \leqq m_2$	(m)	(m)	
$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$\Phi_m = m$
m_2	$m_3 < m_2$	m_3	(m)	$\Phi^{58}_m = m$; $\Phi^{5}_{m_2} = m_3$; $\Phi^{8}_{m_3} = m_2$
$m \geqq m_1$	$m < m_1$	$m \geqq m_2$	(m)	
m_4	m_1, m_3	m_1	m_3	$\Phi^{5}_{m} = m$; $\Phi^{6}_{m_1} = m_2$; $\Phi^{6}_{m_2} = m_2$; $\Phi^{6}_{m_3} = m_3$; $\Phi^{6}_{m_4} = m_3$; $\Phi^{7}_{m_1} = m_3$; $\Phi^{7}_{m_2} = m_2$; $\Phi^{7}_{m_3} = m_3$; $\Phi^{7}_{m_4} = m_2$
m_1	m_1	m_5	m_4	$\Phi^{6}_{m_1} = \Phi^{6}_{m_2} = m_4$; $\Phi^{8}_{m_1} = m_1$
m_1	m_4	m_5	m_2	$\Phi^{67}_{m_1} = \Phi^{67}_{m_2} = m_2$
m_1, m_4	m_1	m_6	m_5	$\Phi^{8}_{m_1} = m_1$; $\Phi^{7}_{m_1} = \Phi^{7}_{m_4} = m_5$
m_1, m_2	m_1	m_4	m_5	$\Phi^{8}_{m_1} = m_1$; $\Phi^{6}_{m_1} = \Phi^{6}_{m_2} = m_5$
$m > m_1$	$m \leqq m_1$	$m > m_1$	$m \geqq m_1$	$\Phi^{6}_{m} = m_1 : m \leqq m_1$; $\Phi^{6}_{m} = m : m < m_1$
$m > m_1$	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	Cf. 50
$m \geqq m_1$	$m \geqq m_1$	$m \leqq m_1$	$m \leqq m_1$	Cf. 50
$m \leqq m_1$	$m \leqq m_2 < m_1$	$m \leqq m_2 < m_1$	$m \leqq m_3 < m_2$	$\Phi_m = m_3 : m \leqq m_1$; $\Phi_m = m : m > m_1$
(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$\Phi^{5}_{m} = m$
(m)	$m \leqq m_1$	(m)	(m)	$\Phi^{6}_{m} = m$
(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	$\Phi^{7}_{m} = m$
(m)	(m)	$m \leqq m_1$	$m \leqq m_1$	$\Phi^{8}_{m} = m$
(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	
m_1	m_1	$m = 2m'+1, m_1$	$m = 2m', m_1$	$\Phi_{m_1} = m_1$; $\Phi^{58}_{m} = m$; $\Phi^{6}_{m} = 2m$
m_1	m_1	m_1	$m = 2m', m_1$	Cf. 67
$m = 2m'+1, m_1$	$m = 2m'+1, m_1$	m_1	$m = 2m', m_1$	Cf. 67
$m = 2m'+1, m_1$	m_1	m_1	$m = 2m', m_1$	$\Phi_{m_1} = m_1$; $\Phi^{67}_{m} = 2m$
$m = 3m'+1, m_1$	$m = 3m'+1, m_1$	$m = 3m'+2, m_1$	$m = 3m', m_1$	$\Phi_{m_1} = m_1$; $\Phi^{6}_{m} = 3m$; $\Phi^{8}_{m} = m$
$m = 3m'+1, m_1$	$m = 3m'+1, m_1$	m_1	$m = 3m', m_1$	$\Phi_{m_1} = m_1$; $\Phi^{7}_{m} = 3m$; $\Phi^{8}_{m} = m$
(m)	$m \leqq m_1$	$m \geqq m_1$	(m)	$\Phi_m = m_1 : m \leqq m_1$; $\Phi_m = m : m > m_1$
(m)	$m \leqq m_1$	$m \leqq m_1$	$m \leqq m_1$	Cf. 77
(m)	$m \geqq m_1$	$m \leqq m_1$	$m \leqq m_1$	Cf. 77

(e) The class $\mathfrak{Q}$ containing any finite number of elements. It is not possible to obtain any additional systems when we pass to the general finite case.

(f) The class $\mathfrak{Q}$ consisting of a denumerable infinitude of elements. We are thus forced to proceed to the case in which the class $\mathfrak{Q}$ is infinite. The simplest type of infinity is the denumerable type, and so we consider this first. It is possible to obtain systems $(\mathfrak{Q}^{III}; K)$ which have the twelve combinations of properties of the K-relation not yet considered. Suppose the elements to be q_i, $i = 1, 2, \ldots$. Then the K-relations may be defined as follows:

2. $+ + + + - - - - : K_{q_iq_jm} : i \neq j \neq 1 \text{ or } 2 : m \leqq 2$
$i = j : (m)$
$i \text{ or } j = 1 \text{ or } 2 : m \leqq i + j.$

For from $K_{q_1q_j1+j}$ with $K_{q_jq_21+j}$ would follow $K_{q_1q_2\phi^5_{1+j}}$, which is contrary to $K_{q_1q_23}$. Since K^2 we have also K^{-6-7-8}.

6. $+ - + + + - - - : K_{q_iq_jm} : i = j : (m)$
$: i < j : m \overline{\leqq} 2$
$: i > j : m \overline{\leqq} i.$

(1) K^5. We show that $\phi^5_m = m$. Suppose $K_{q_iq_jm}$ and $K_{q_jq_km}$. Then if $i = j$ or $j = k$, we evidently have $\phi^5_m = m$. There remain to be considered thus:

(α) $i > j$, $j > k$. Then since $K_{q_iq_jm}$ holds for $m \leqq i$, $K_{q_jq_km}$ for $m \leqq j$, and $K_{q_iq_km}$ for $m \leqq i$, we evidently have $\phi^5_m = m$.

(β) $i < j$, $j > k$. Then since $K_{q_iq_jm}$ holds for $m \leqq 2$, and $K_{q_jq_km}$ for $m \leqq j$, while $K_{q_iq_km}$ at least for $m \leqq 2$, we also have $\phi^5_m = m$. Similarly it appears that this ϕ^5_m will work for the other two possibilities: $i > j$ and $j < k$, and $i < j$ and $j < k$.

(2) On the other hand we have K^{-6}. For suppose $i > 2$. Then if K^6 we should have from $K_{q_iq_1i}$ and $K_{q_iq_2i}$, $K_{q_1q_2\phi_i}$. But by hypothesis we have at most $K_{q_1q_22}$. Hence K^{-6}.

Since, further, K^6, K^7, K^8 are equivalent under K^{14} by § 5 (12), we have K^{-7} and K^{-8}.

7. $+ - + + - - - - : K_{q_iq_jm}$ $i = j : (m)$
$i \neq j;\ i - j \neq 1 : m \leqq 2$
$i - j = 1 : m \leqq 2i + 1.$

For $K_{q_iq_{i+1}2i+1}$ with $K_{q_{i+1}q_{i+2}2i+1}$ would give us by K^5: $K_{q_iq_{i+2}\phi_{2i+1}}$, and we have at most $K_{q_iq_{i+2}2}$. Hence K^{-5}. Again by K^6 we should have from $K_{q_iq_{i+1}2i+1}$ with $K_{q_iq_i2i+1}$: $K_{q_{i+1}q_i\phi_{2i+1}}$, and we have at most: $K_{q_{i+1}q_i2}$. Hence K^{-6}, and also, by § 5 (12), K^{-7} and K^{-8}.

11. $+\,+\,+\,-\,-\,-\,-\,-\,-$: $K_{q_iq_jm}$ $\quad i \neq 1$ and $j \neq 1$: $m \leqq 2$,
$\quad\quad\quad

30. $+ - + - - - - + : K_{q_i q_j m} : m \leqq 2$ except $K_{q_i q_{i+1} m}$ for $i = 3n+1$ or $3n+2$, in which case $m \leqq 2n+1$, and $K_{q_i q_{i-2} m}$ for $i = 3n$, for which $m \leqq 2i - 2$.

We have in this case $\phi^8_m = m - 1$.

31. $+ - + - - - - - : K_{q_i q_j m} : m \leqq i + j$.

We have thus obtained a situation for each of the above combinations of properties.

(g). However, the classes $\mathfrak{Q}$, to which we apply the K-relation and its properties, are generally of the denumerably infinite, or non-denumerably infinite type. In the above set of systems, however, there are considered only twelve which include a class $\mathfrak{Q}$ which is denumerably infinite, and no system in which the class $\mathfrak{Q}$ is non-denumerably infinite. The question naturally arises whether these situations are possible also in these two types of classes. We obtain the desired result by attempting to extend the given situations to classes of greater dimension, preserving the properties of the K-relation. The following scheme of extension will cover a large number of cases:

(α) Extension of $\mathfrak{Q}^{\text{II}_n}$ to $\mathfrak{Q}^{\text{II}_l}$, $l > n$.

Let the elements of $\mathfrak{Q}^{\text{II}_l}$ be $q_1, q_2, \ldots, q_l$. We put this set into correspondence with a set of n elements, $q'_1, q'_2, \ldots, q'_n$, by supposing q'_j to correspond to q_i, if $j \equiv (i-1) \bmod n$. This assigns to every member of the set $q_1, \ldots, q_l$, a definite correspondent of the set $q'_1, q'_2, \ldots, q'_n$. We suppose then that $K_{q_i q_j m}$ holds for values of m for which $K_{q'_{i'} q'_{j'} m}$ holds, where $q'_{i'}$ corresponds to q_i and $q'_{j'}$ to q_j. We then have

(1) $$K^P \text{ in } \mathfrak{Q}^{\text{II}_n} \cdot \supset \cdot K^P \text{ in } \mathfrak{Q}^{\text{II}_l},$$

where P is one of the properties: (1), (-1), (2), (-2), (-3), (4), (-4), (5), (-5), (6), (-6), (7), (-7), (8), (-8). There is no difficulty about seeing this. Hence it includes all of the properties and their negatives, excepting (3). If by K^{-4}, identically, we mean that for no i do we have $K_{q_i q_i m}$, then we can say:

(2) $$K^{3-4} \text{ in } \mathfrak{Q}^{\text{II}_n} \cdot \supset \cdot K^{3-4} \text{ in } \mathfrak{Q}^{\text{II}_l}.$$

As a result we have then that if, in the combination K^{-34}, we have K^{-4} identically, all cases excepting those for which we have K^{34} may be extended as above, and the result will be a K in a $\mathfrak{Q}^{\text{II}_l}$ which will have the same combination of properties as the K in $\mathfrak{Q}^{\text{II}_n}$. A review of the above examples will reveal

the fact that in all cases in which we have K^{3-4}, we have K^{-4} identically. Hence this mode of extension will apply to all of the above systems excepting eleven, for which we have K^{34}.

(β) A similar scheme enables us to pass from $\mathfrak{Q}^{\mathrm{II}_n}$ to $\mathfrak{Q}^{\mathrm{III}}$. Suppose $\mathfrak{Q}^{\mathrm{III}}$ to consist of the elements: $q_1, \ldots, q_k, \ldots$. We put q_k in correspondence with $q'_{k'}$ if $k \equiv (k'-1) \bmod n$. Then if we define our K-relation as in (α), we shall have the same results.

(γ) As for passing from $\mathfrak{Q}^{\mathrm{II}_n}$ to $\mathfrak{Q}^{\mathrm{IV}}$, where $\mathfrak{Q}^{\mathrm{IV}}$ is a class of the non-denumerably infinite type, which is of the same power as the continuum, the interval $0 \leqq x \leqq 1$. Let us suppose that our elements are q_x, where $0 \leqq x \leqq 1$; *i. e.*,

$$\mathfrak{Q}^{\mathrm{IV}} = [q_x;\ 0 \leqq x \leqq 1].$$

To set up a correspondence with $\mathfrak{Q}^{\mathrm{II}_n}$, divide the interval $0 \leqq x \leqq 1$ into n equal parts and set

$$q_x \text{ into correspondence with } q'_i \text{ if } \frac{i-1}{n} < x \leqq \frac{i}{n}$$

and

$$q_x \text{ into correspondence with } q'_1 \text{ if } \quad 0 \leqq x \leqq \frac{1}{n}.$$

If we define our K-relation as in (α), the same results will also hold.

(δ) Finally, to extend $\mathfrak{Q}^{\mathrm{III}}$ to $\mathfrak{Q}^{\mathrm{IV}}$, set

$$q_x \text{ into correspondence with } q'_n \text{ if } \frac{1}{2^{n-1}} \leqq x \leqq \frac{1}{2^n},$$

and

$$q_0 \text{ into correspondence with } q'_1.$$

This produces results similar to the above.

We have thus shown that it is possible to obtain all but eleven of the above combinations of properties in $\mathfrak{Q}^{\mathrm{III}}$ and $\mathfrak{Q}^{\mathrm{IV}}$. It remains to consider these. We discuss each of these cases separately.

1. $+\,+\,+\,+\,+\,+\,+\,+\,+$. In any class let $K_{q_1 q_2 m}$ hold for every m if $q_1 = q_2$, and for $m \leqq m_1$ if $q_1 \neq q_2$.

2. $+\,+\,+\,+\,-\,-\,-\,-\,-$. This occurs for the first time in $\mathfrak{Q}^{\mathrm{III}}$. So we need consider it only for $\mathfrak{Q}^{\mathrm{IV}}$. Supposing the elements of $\mathfrak{Q}^{\mathrm{IV}}$ to be q_x, we define our K as follows:

$$K_{q_x q_x m}(m),\ K_{q_{x_1} q_{x_2} m} \text{ hold for } m \leqq 2,\ x_1, x_2 \neq 0 \text{ or } 1;$$

$$K_{q_x q_0 m},\ K_{q_0 q_x m},\ K_{q_1 q_x m},\ K_{q_x q_1 m}, \text{ for } m \leqq m_1, \text{ where } \frac{1}{2^{m_1}} < x.$$

3. $-+++++++$. We suppose that in any class $K_{q_1q_2m}$ holds for every m if $q_1 = q_2$, and for $m \geqq m_1$ if $q_1 \neq q_2$.

4. $-++----$. In any class select three elements, and for these define the K-relations as given in II_3. For all other elements let $K_{q_1q_2m}$ hold for every m if $q_1 = q_2$, and for $m \geqq m_1$ if $q_1 \neq q_2$.

5. $+-++++++$. Let $K_{q_1q_2m}$ hold for every m if $q_1 = q_2$, and for $m \leqq m_1$ for $q_1 \neq q_2$, excepting q' and q'': $K_{q'q''m}$ for $m \leqq m_2 < m_1$.

6. $+-++----$. We need consider only $\mathfrak{Q}^{IV}$. The method of constructing a K-relation from the one holding in $\mathfrak{Q}^{III}$ is similar to that used in 2. This holds also of

7. $+-++----$.

12. $--++++++$. Proceed as in 5, substituting $>$ for $<$.

13. $--++---+$. Let $K_{q'q''m}$ hold for every m if $q' = q''$. Let $K_{\substack{q_i \\ x_1}\ \substack{q_j \\ x_2}\ m}$ hold for m odd if $i > j$ or $x_1 > x_2$, and for m even if $i < j$ or $x_1 < x_2$. We have in this case $\phi^5_m = m$, and $\phi^8_m = m + 1$.

14. $--++----$. Suppose $K_{q'q''m}$ holds for every m if $q' = q''$, and $K_{\substack{q_i \\ x_1}\ \substack{q_j \\ x_2}\ m}$ holds for $m \geqq m_1$ if $i > j$ or $x_1 > x_2$, and for $m < m_1$ if $i < j$ and $x_1 < x_2$. We have $\phi^5_m = m$.

15. $--++----$. Proceed as in 4 above.

We have thus completed the study of the complete existential theory of our eight properties, and have shown that with the exception of the propositions of § 5, there exist no further relations between the properties in question.

While we do not have the entire set of properties completely independent, certain combinations of them are. For instance: (1), (2), (3), (4), (5) form a set of properties, which include all of the rest, and are completely independent.* A further set of completely independent properties is (1), (3), (6), (7). This set of properties is not quite as strong as the above. For if we have K^{12345}, we have also K^{678}, while if we have K^{1367}, we have only in addition K^{58}, and not necessarily K^2 or K^4.† There are a number of other interesting combinations of properties, but they are equivalent to these two, or to the combination K^{1346}.

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* In so far as Fréchet assumes the properties (2), (3), (4), (5) as properties of the δ, and a system $(\mathfrak{Q}; \delta^{2345})$ is a system $(\mathfrak{Q}; K^{23451})$, we have shown incidentally that he has chosen a set of completely independent and consistent properties.

† In the next part we shall show that a K^{1367} is sufficient for the Fréchet δ-theory. We have thus a weaker set of properties than those assumed by Fréchet.

II.

Properties of Subclasses $\mathfrak{R}$ of the Class $\mathfrak{Q}$ of a System $(\mathfrak{Q}; K)$ and of Continuous Functions on $\mathfrak{R}$ to $\mathfrak{A}$.

We have been considering systems $(\mathfrak{Q}; L)$, and $(\mathfrak{Q}; K)$, and have in particular discussed properties of L and the K-relation. We now turn our attention to the properties of the class $\mathfrak{Q}$, considering in particular subclasses* $\mathfrak{R}$ of $\mathfrak{Q}$, and real-valued continuous functions on $\mathfrak{R}$.

We show in § 13 that it is not possible to obtain some theorems of the theory of point sets, relative to $\mathfrak{R}$ of systems $(\mathfrak{Q}; L)$, even though we suppose that L has all of the properties of § 3. Hence the treatment of subclasses $\mathfrak{R}$ of classes $\mathfrak{Q}$ is confined mostly to systems $(\mathfrak{Q}; K)$. However, a theory of sequentially continuous functions is obtainable in a system $(\mathfrak{Q}; L)$. A theory of difference continuous functions can be derived in a system $(\mathfrak{Q}; K)$. These are taken up in §§ 18 ff.

The theorems derived are in the main the theorems of Fréchet. However, instead of permanently conditioning the L and the K in the systems $(\mathfrak{Q}; L)$ and $(\mathfrak{Q}; K)$, we have preferred to indicate in each case the precise properties of L or K, sufficient to carry the argument. In this way it appears that it is not necessary to condition the L for the theorems on continuous functions. Further, that a K-relation having the properties (1), (3), (6), (7), of § 4, is sufficient for all the theorems, and in some cases even weaker conditions on the K will do. It will be noticed that the symmetry property (2) and the property (4) do not occur. The former is really a matter of convenience. It is avoided by the use of properties K^{167}, which combination we have seen is weaker than K^{125}, the combination it replaces. Property (4) serves to avoid the separate consideration of the limit of a sequence which consists of a finite number of elements only. Its presence as a condition restricts the generality of the theorems. We have therefore preferred to gain in generality at the expense of convenience, replacing the property (2) by a weaker combination, and taking up a more detailed discussion, if necessary, instead of using the property (4).

We take up first a consideration of the definitions of properties of the classes $\mathfrak{R}$ in a system $(\mathfrak{Q}; L)$, and the modification of these definitions in case we are operating in a system $(\mathfrak{Q}; K)$, passing thence to the consideration of sub-

* We shall suppose that $\mathfrak{R}$ denotes a subclass of $\mathfrak{Q}$ throughout this part.

classes $\mathfrak{R}$ of $\mathfrak{Q}$ in systems $(\mathfrak{Q}; L)$ and $(\mathfrak{Q}; K)$, and the question of continuous functions of $\mathfrak{R}$ of $(\mathfrak{Q}; L)$ and $(\mathfrak{Q}; K)$.

10. *Definitions.** Suppose $\mathfrak{S}$ a subclass of $\mathfrak{R}$. Then we have:

(1) q is a *limiting element* of $\mathfrak{R}$, if it is the limit of a sequence of distinct elements of the class $\mathfrak{R}$.† We denote limiting elements of $\mathfrak{R}$ by r'.

(2) The *derived class* $\mathfrak{R}'$ of $\mathfrak{R}$ is the class which consists of all of the limiting elements of the class $\mathfrak{R}$, *i. e.*, we have $\mathfrak{R}' = [\text{all } r']$.

(3) $\mathfrak{R}$ is *closed* if it contains its derived class $\mathfrak{R}'$, *i. e.*, if $\mathfrak{R}'^{\mathfrak{R}}$.

(4) $\mathfrak{R}$ is *dense in itself* if its derived class contains it, *i. e.*, if $\mathfrak{R}^{\mathfrak{R}'}$.

(5) $\mathfrak{R}$ is *perfect* if it is identical with its derived class, *i. e.*, if $\mathfrak{R} = \mathfrak{R}'$.

(6) $\mathfrak{R}$ is *compact* if every denumerable infinitude of elements of $\mathfrak{R}$ gives rise to at least one limiting element.‡

(7) $\mathfrak{R}$ is *extremal* if it is compact and closed.

(8) q is an *element of condensation* of $\mathfrak{R}$ if it is a limiting element of any subclass of $\mathfrak{R}$ obtained by removing a denumerable infinitude of elements from $\mathfrak{R}$. We denote an element of condensation by $r^{\wedge}$. Evidently a class $\mathfrak{R}$ has an element of condensation only if it is non-denumerable.

(9) $\mathfrak{R}$ is *condensed* if every non-denumerable set of its elements gives rise to an element of condensation.§

(10) An element q is *interior to* $\mathfrak{S}$ *relative to* $\mathfrak{R}$‖ if q is an element of $\mathfrak{S}$, and every sequence of $\mathfrak{R}$ which has q as a limit ultimately belongs to $\mathfrak{S}$.¶ In symbols:

$$q^{\text{interior } \mathfrak{S}(\mathfrak{R})} \equiv q^{\mathfrak{S}} \cdot \underset{n}{Lr_n} = q \cdot \supset \cdot \exists\, n_1 \,\beta\, n \geqq n_1 \cdot \supset \cdot r_n^{\mathfrak{S}}.$$

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* Cf. Fréchet, *loc. cit.*, p. 6.

† Of course if the limit relation L is such that no sequence with all of its elements distinct has a limit, there will be no limiting elements. A similar statement holds relative to compact classes below.

‡ Accordingly every finite class is compact, for the definitional condition is satisfied vacuously, its hypothesis being incapable of fulfillment in a finite class. Cf. Moore, *Trans. Amer. Math. Soc.*, III, 489.

§ In accordance with the foot-note to (6) above, $\mathfrak{R}$ is condensed if it is denumerable; namely, in the vacuous sense.

‖ Denoted by $q^{\text{interior } \mathfrak{S}(\mathfrak{R})}$. Cf. relativity notation of Moore, *loc. cit.*, pp. 27 ff.

¶ Fréchet speaks of interiority in the strict sense. His definition reads:

$$q^{\text{interior } \mathfrak{S}(\mathfrak{R})} \equiv q^{\mathfrak{S}} \cdot \{r_n\}^{\text{distinct}} \,\beta\, \underset{n}{Lr_n} = q \cdot \supset \cdot r_n^{\mathfrak{S}}.$$

This is evidently not what he means. Judging by the use made of the notion on p. 23, he assumes that he has defined interiority:

$$q^{\text{interior } \mathfrak{S}(\mathfrak{R})} \equiv q^{\mathfrak{S}} \cdot \{r_n\}^{\text{distinct}} \,\beta\, \underset{n}{Lr_n} = q \cdot \supset \cdot \exists\, n_1 \,\beta\, n \geqq n_1 \cdot \supset \cdot r_n^{\mathfrak{S}}.$$

Our definition is, as is easily evident, equivalent to this one if we assume the L to have the properties used by Fréchet, viz., L^{126}.

11. *Two Propositions.* For an L_K defined relative to a K^1, we secure, as follows, a desirable transformation of the definition of interiority:*

(1) $$K^1 \cdot :\supset: s^{\text{interior } \mathfrak{S}(\mathfrak{R})} :\supset: \exists\, m_0 \ni (r \ni K_{rsm_0} \cdot \supset \cdot r^{\mathfrak{S}}).$$

As a converse of (1) we have:

(2) $$K \cdot s^{\mathfrak{S}} \cdot :\supset: \exists\, m_0 \ni (r \ni K_{rsm_0} \cdot \supset \cdot r^{\mathfrak{S}}) :\supset: s^{\text{interior } \mathfrak{S}(\mathfrak{R})}.$$

Joining these two propositions into a single equivalence we have:†

(3) $$K^1 \cdot s^{\mathfrak{S}} \cdot \mathfrak{S}^{\mathfrak{R}} \cdot :\supset: s^{\text{interior } \mathfrak{S}(\mathfrak{R})} :\backsim: \exists\, m_0 \ni (r \ni K_{rsm_0} \cdot \supset \cdot r^{\mathfrak{S}}).$$

We prove (1) first. If possible, suppose the proposition were not holding. Then we have:

$$m \cdot \supset \cdot \exists\, r_m \ni K_{r_m sm} \cdot r_m^{-\mathfrak{S}}.$$

Then evidently by § 7 (4): $\underset{m}{L} r_m = s$, and since $s^{\text{interior } \mathfrak{S}(\mathfrak{R})}$, we have:

$$\exists\, m_1 \ni m \geqq m_1 \cdot \supset \cdot r_m^{\mathfrak{S}}.$$

We have thus reached a contradiction, and our proposition therefore holds.

As for (2) suppose $\underset{n}{L} r_n = s$; *i. e.*,

$$m \cdot \supset \cdot \exists\, n_m \ni n \geqq n_m \cdot \supset \cdot K_{r_n sm}.$$

If, in particular, we take $m = m_0$, we shall have by the hypothesis of (2):

$$n \geqq n_{m_0} \cdot \supset \cdot r_n^{\mathfrak{S}}.$$

Hence s satisfies the conditions of interiority.

If we have K^1, we can define a concept analogous to that of the bounded point set, viz. *limited.* We define:‡

$$\mathfrak{R}^{\text{limited}} \equiv \mathfrak{R} \ni (\exists\, m \ni r_1, r_2 \cdot \supset \cdot K_{r_1 r_2 m}).$$

The property limited is related to the property compact as follows:§

(4) $$K^{17} \cdot \mathfrak{R} :\supset: \mathfrak{R}^{\text{compact}} \cdot \supset \cdot \mathfrak{R}^{\text{limited}}.$$

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* The K-relation having the property (1), if an element s of a subclass $\mathfrak{S}$ of the subclass $\mathfrak{R}$ of $\mathfrak{Q}$ is interior to $\mathfrak{S}$ with respect to $\mathfrak{R}$, then there exists an m_0 such that every element r in the relation K_{rsm_0} belongs to $\mathfrak{S}$. The analogous proposition holds for the weaker interiority defined by Fréchet, if the K-relation has the property (3).

† A similar result holds in a system $(\mathfrak{Q}; \delta)$, δ unconditioned:

$$\delta \cdot \mathfrak{S}^{\mathfrak{R}} \cdot s^{\mathfrak{S}} \cdot :\supset: s^{\text{interior } \mathfrak{S}(\mathfrak{R})} :\backsim: \exists\, a \ni \delta_{rs} \leqq a \cdot \supset \cdot r^{\mathfrak{S}}.$$

In this form we have a conception of interiority which is analogous to the one of the linear point set theory.

‡ $\mathfrak{R}$ is limited, if there exists an m such that for every pair of elements r_1 and r_2 of the class $\mathfrak{R}$, we have $K_{r_1 r_2 m}$.

§ Cf. Fréchet, *loc. cit.*, p. 22.

If possible, suppose it were not so. Then

$$-m \cdot \supset \cdot \exists\, r_{1m} \cdot r_{2m} \ni -K_{r_{1m}r_{2m}(-m)}.$$

There are then four possibilities: (a) The sequences r_{1m} and r_{2m} each contain only a finite number of distinct elements. (b) The sequence r_{1m} contains only a finite number of distinct elements, while the sequence r_{2m} contains an infinitude of distinct elements. (c) The sequence r_{1m} contains an infinitude of distinct elements, while the sequence r_{2m} contains only a finite number of distinct elements. (d) Each sequence contains an infinitude of distinct elements. In case (a), there will be a certain pair of elements r_1, r_2 which will have like subscripts infinitely often. We argue a contradiction by considering that there exists an m_0 such that $K_{r_1 r_2 m_0}$. In case (b), so far as it is not covered by (a), we can show the existence of a set of integers m_n such that r_{2m_n} are all distinct and $\underset{n}{L}\, r_{2m_n} = q$, while $r_{1m_n} = r_1$ for every n. Then, since $(\Omega\,;K)$ is a system,

$$\exists\, m_0 \ni K_{r_1 q m_0}.$$

Further we have:

$$m_0 \cdot \supset \cdot \exists\, n_{m_0} \ni n \geqq n_{m_0} \cdot \supset \cdot K_{r_{2m_n} q m_0}.$$

Then, since K^7,

$$K_{r_1 r_{2m_n} \phi^7 m_0}, \text{ i. e., } K_{r_{1m_n} r_{2m_n} \phi^7 m_0}.$$

By taking $n \geqq n_{m_0}$ and $-m_n < \phi^7_{m_0}$, we obtain a contradiction. Similarly for (c). As for case (d), so far as it is not covered by the preceding cases, we can obtain two sequences $\{r_{1m_n}\}$ and $\{r_{2m_n}\}$, each consisting of distinct elements, and further, since $\mathfrak{R}^{\text{compact}}$, in such a way that:

$$\exists\, (q_1, q_2) \ni \underset{n}{L}\, r_{1m_n} = q_1 \text{ and } \underset{n}{L}\, r_{2m_n} = q_2.$$

Since we have a system $(\Omega\,;K)$, we have:

$$\exists\, m_0 \ni K_{q_1 q_2 m_0}.$$

Also since $\underset{n}{L}\, r_{2m_n} = q_2$:

$$m_0 \cdot \supset \cdot \exists\, n''_{m_0} \ni n \geqq n''_{m_0} \cdot \supset \cdot K_{r_{2m_n} q_2 m_0}.$$

Applying K^7 to these two K-relations there results:

$$K_{r_{2m_n} q_1 \phi^7 m_0}.$$

From $\underset{n}{L} r_{1m_n} = q_1$ it follows:

$$\phi^7_{m_0} \cdot \supset \cdot \exists\, n'_{m_0} \ni n \geqq n'_{m_0} \cdot \supset \cdot K_{r_{1m_n} q_1 \phi^7 m_0}.$$

And hence

$$K_{r_{1m_n} r_{2m_n} \phi^7 \phi^7 m_0}.$$

Hence, taking an integer n exceeding n'_{m_0} and n''_{m_0}, and also such that $-m_n < \phi^7_{\phi^7 m_0}$, we obtain the desired contradiction.

12. *Propositions on Compact Classes in* $(\Omega; L)$. We have as an immediate consequence of the definition of compact classes the propositions:*

(1) *The sum of a finite number of compact classes is compact.*

(2) *Any subclass of a compact class is compact.*

(3) *If every subclass of a class is compact, then the class is compact.* The last two propositions might be joined into one:

(4) $\mathfrak{R}^{\text{compact}} : \backsim : \mathfrak{S}^{\mathfrak{R}} \cdot \supset \cdot \mathfrak{R}^{\text{compact}}$.

(5)† $L^3 \cdot \mathfrak{R}^{\text{compact}} \cdot \mathfrak{R}_n \ni \mathfrak{R}_{n-1}{}^{\mathfrak{R}_n} \cdot \mathfrak{R}_n{}^{\text{closed, non vacuous } (n)} \cdot \supset \cdot \exists\, r \ni r^{\mathfrak{R}_n(n)}$.

The proof is identically the one given by Fréchet, *loc. cit.*, p. 7.

13. *On Derived Classes in a System* $(\Omega; L)$. It is not possible to obtain, even though we suppose that the limit relation has all of the properties of § 3, *i. e.*, L^{123456}, the theorem that the derived set of a given set is closed, or, in class terminology, that the derived class of every subclass of Ω is closed. This may be shown by the following example:‡

We suppose the class Ω to consist of the following elements, all of which are supposed distinct: $\{q_{ln}\}$, $\{q_l\}$, q_0 $(l, n = 1, 2, \ldots, \infty)$. The complete table for the limit L in this class is specified as follows:

(a) A sequence $\{q_{l_0 n}\}$, where l_0 is fixed, taken in any order, or any subsequence of such a sequence, or any sequence obtained by prefixing a finite number of elements of the class, shall have as limit q_{l_0}. (b) A sequence $\{q_l\}$, taken in any order, or any subsequence of such a sequence, or any sequence obtained by prefixing a finite number of elements, shall have as limit q_0. (c) Any identity sequence, or any sequence, which after a certain

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*Cf. Fréchet, *loc. cit.*, p. 7. Notice, however, that there is no condition on L.

† If L has the property (3), and $\mathfrak{R}$ is compact, if, further, we have a sequence of subclasses of $\mathfrak{R}$, each of which is closed, contained in the preceding and containing at least one element, then there is an element common to all of these classes.

‡ The example given is a modification of the example given by Hahn, *Monatshefte* Vol. XIX, p. 248, so as to include the properties 4, 5, of L.

term consists of one element repeated, shall have the repeated element as limit. No other sequences shall have limits. Such a limit is evidently an L^{123456}. Consider now $\mathfrak{R} = [q_{ln}]$. Then $\mathfrak{R}' = [q_l]$, and $\mathfrak{R}'' = q_0$, so that $\mathfrak{R}''$ is not closed. This is due to the fact that the above properties are not sufficient to secure q_0 as the limit of some sequence built up of elements taken from $[q_{ln}]$ only. The connection between the class $\mathfrak{R}'$ and the class $\mathfrak{R}$ as given by these properties of limit is not sufficiently close. We therefore need a weaker system if we desire to have this as a theorem. We obtain this in systems $(\Omega; K)$, the K being suitably conditioned.

14. *Relative to Derived Classes in Systems* $(\Omega; K)$. We have the theorem:*

$$K^{15} :\supset: \mathfrak{R} \cdot \supset \cdot \mathfrak{R}' \text{closed}.$$

Suppose

$$\underset{n}{L} r'_n = r'' \quad \text{and} \quad \underset{l}{L} r_{nl} = r'_n,$$

where the r'_n are all distinct, and the r_{nl} are distinct for every n. Then we have by the definition of limit:

$$m \cdot \supset \cdot \exists\, n_m \ni K_{r'_{n_m} r''_m}$$

and†

$$\exists\, l_m \ni \{r_{n_m l_m}\}^{\text{distinct}} \cdot K_{r_{n_m l_m} r'_{n_m} m}.$$

Then by the fact that K has the property (5) we have

$$K_{r_{n_m l_m} r'' \phi^5 m},$$

and so by § 7 (4): $\underset{m}{L} r_{n_m l_m} = r''$, *i. e.*, $r'' \mathfrak{R}'$ and therefore $\mathfrak{R}'' \mathfrak{R}'$.

15. *On Compact Classes and Their Derivatives in Systems* $(\Omega; K)$. We have:‡

(1) $$K^{16} :\supset: \mathfrak{R}^{\text{compact}} \cdot \supset \cdot \mathfrak{R}' \text{compact}.$$

We need consider only the case in which $\mathfrak{R}'$ contains an infinitude of elements.

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* Cf. Fréchet, *loc. cit.*, p. 18.

† There exists a sequence of l, l_m, such that the elements of the sequence $r_{n_m l_m}$ are all distinct, and $K_{r_{n_m l_m} r'_{n_m} m}$. The distinctness of the elements $r_{n_m l_m}$ can be argued by a step-by-step process from the fact that the elements of the sequences r_{nl} are distinct for every n.

‡ Cf. Fréchet, Rend. di Pal., XXX, p. 4. Also for (6) and (7) below.

Suppose then that $[r'_n]$ is a denumerable infinitude of distinct elements of the class $\mathfrak{R}'$. Then on account of the definition of $\mathfrak{R}'$ we have:

$$n \cdot \supset \cdot \exists \{r_{nl}\}^{\text{distinct}} \ni \underset{l}{L} r_{nl} = r'_n.$$

Then by the definition of limit,

$$\exists l_n \ni \{r_{nl_n}\}^{\text{distinct}} \text{ and } K_{r_{nl_n} r'_n n}.$$

Since $\mathfrak{R}$ is compact, there will be a subsequence of $\{r_{nl_n}\}$, $\{r_{n_k}\}$, and an element r' such that

$$\underset{k}{L} r_{n_k} = r'.$$

Then

$$m \cdot \supset \cdot \exists k_m \ni k \geqq k_m \cdot \supset \cdot K_{r_{n_k} r' m}.$$

But $K_{r_{n_k} r' n_k n_k}$ and so if K^1:

$$m \cdot \supset \cdot \exists k'_m \ni k \geqq k'_m \cdot \supset \cdot K_{r_{n_k} r' n_k m}.$$

Applying the fact that K^6, there results:

$$k \geqq k_m \text{ and } k \geqq k'_m \cdot \supset \cdot K_{r' n_k r' \phi^s m}.$$

Hence by § 7 (4) we have:

$$\underset{k}{L} r'_{n_k} = r'.$$

This shows that there exists a limiting element for $[r'_n]$, *i. e.*, $\mathfrak{R}'$ is compact.

We have as immediate corollaries:

(2) $$K^{156} :\supset: \mathfrak{R}^{\text{compact}} \cdot \supset \cdot \mathfrak{R}'^{\text{extremal}}.$$

(3) $$K^{156} :\supset: \mathfrak{R}^{\text{compact}} \cdot \supset \cdot (\mathfrak{R} + \mathfrak{R}')^{\text{extremal}}.$$

A theorem relating to the class $\mathfrak{R}$ itself is the following:*

(4) $$K^{1367} :\supset: \mathfrak{R}^{\text{compact}} \cdot \supset \cdot [s \ni s^{\mathfrak{R}, -\mathfrak{R}'}]^{\text{denumerable}}.$$

Suppose $s^{\mathfrak{R}, -\mathfrak{R}'}$. Then†

$$m_s = \overline{B}(m \ni K_{r' s m} \mid r' \mathfrak{R}')$$

exists and is finite. For if it were not finite, we should have either: (a) By K^1

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* If K has the properties (1), (3), (6), and (7), and $\mathfrak{R}$ is compact, then the class of elements consisting of elements of $\mathfrak{R}$ which are not limiting elements of $\mathfrak{R}$ is denumerable. Cf. Fréchet, *loc. cit.*, p. 20.

† m_s is the least upper bound of the values of m for which $K_{r' s m}$ holds, r' ranging over the class $\mathfrak{R}'$.

there would exist an r' such that $K_{r'sm}(m)$, in which case $s = r'$, if K^3, which evidently is contradictory to the hypothesis relative to s; or (b)

$$\exists \{r'_m\}^{\text{distinct}} \ni K_{r'_m sm}.$$

But then by § 7 (4) we should have $\underset{m}{L} r'_m = s$. If then K^{15}, we have by § 15 $s^{\Re'}$, which again is contradictory to the hypothesis.

We have further:

$$m \cdot \supset \cdot \exists \text{ finite number of } s \text{ such that } m_s = m.$$

If not, let there be an infinitude for m_0. Since $\Re$ is compact, there exists a limiting element for a sequence of these values. Let $\{s_n\}$ be the sequence, and r'_0 the limiting element. Then

$$m \cdot \supset \cdot \exists n_m \ni n \geqq n_m \cdot \supset \cdot K_{s_n r'_0 m}.$$

Now by § 5 (1), if K^{167}, we have: $K_{r'_0 s_n \phi^9 m}$. To obtain a contradiction, choose m so that $\phi^9_m > m_0$. Since, then, there exist only a finite number of s such that $m_s = m$, the class $[s]$ is evidently denumerable.

As an immediate corollary we have:

$$(5) \qquad K^{1367} :\supset: \Re^{\text{compact}} \cdot \Re'^{\text{denumerable}} \cdot \supset \cdot \Re^{\text{denumerable}}.$$

$$(6) \qquad K^{17} :\supset: \Re^{\text{compact}} \cdot \Re_m = [r \ni (r_1, r_2)^{\Re_m} \cdot \supset \cdot -K_{r_1 r_2 m}] \cdot \supset \cdot \Re_m^{\text{finite}}.$$

Suppose if possible $\Re_{m_1}$ not finite. Then

$$\exists \{r_n\}^{\Re_{m_1}} \ni \underset{n}{L} r_n = q.$$

Then

$$m :\supset: \exists n_m \ni \begin{matrix} n_1 \\ n_2 \end{matrix} \geqq n_m \cdot \supset \cdot K_{r_{n_1} q m} \cdot K_{r_{n_2} q m}.$$

Then

$$K^7 \cdot \supset \cdot K_{r_{n_1} r_{n_2} \phi_m}.$$

If we choose m so that $\phi_m > m_1$, we have a contradiction. Hence

$$m \cdot \supset \cdot \Re_m^{\text{finite}}.$$

$$(7) \qquad K^{17} :\supset: \Re^{\text{compact}} \cdot \supset \cdot \exists \mathfrak{S}^{\text{denumerable}} \ni \Re^{\mathfrak{S}+\mathfrak{S}'}.$$

The proof of this is as in Fréchet: *Rendiconti di Palermo*, XXX, § 5, pp. 3 and 4.

16. *Classes of Elements of Condensation.* Denote the class of elements of condensation of $\mathfrak{R}$, by $\mathfrak{R}^{\backprime}$, *i. e.*, $\mathfrak{R}^{\backprime} = [r^{\backprime}]$. Then we have:

(1) $$K^{15} \cdot \mathfrak{R} \cdot \supset \cdot \mathfrak{R}^{\backprime \text{closed}}.$$

This is an immediate consequence of § 15.*

(2) $$K^{1367} :\supset: \mathfrak{R}^{\text{condensed}} \cdot \supset \cdot \mathfrak{R}^{\backprime \text{dense in itself}}.$$

Let $r^{\backprime}$ be any element of $\mathfrak{R}^{\backprime}$. Then let

$$\mathfrak{R}_m = [\text{all } r \ni K_{rr^{\backprime}m} \cdot -K_{rr^{\backprime}m+1}].$$

Every element of the class $\mathfrak{R}$ excepting $r^{\backprime}$ will be in a uniquely determined class $\mathfrak{R}_m$. For otherwise we should have $K_{rr^{\backprime}m}(m)$, and then by K^3 $r = r^{\backprime}$. There will be an infinitude of non-denumerable classes in the set $\mathfrak{R}$. If not, then:

$$\exists\, m_1 \ni m \geqq m_1 \cdot \supset \cdot \mathfrak{R}_m^{\text{denumerable}}.$$

But, by removing the denumerable class of elements consisting of the elements in the classes $\mathfrak{R}_m$ for $m \geqq m_1$, we obtain a class of which $r^{\backprime}$ is not a limiting element, which would be contrary to the hypothesis, that $r^{\backprime}$ is an element of condensation of the class. Denote by $\mathfrak{R}_{m_n}$ the classes of $[\mathfrak{R}_m]$ which are non-denumerable. Then:

$$\mathfrak{R}^{\text{condensed}} \cdot \supset \cdot \exists\, r^{\backprime}_{m_n} \text{ elements of condensation of } \mathfrak{R}_{m_n}.$$

There are two possibilities: (a) the sequence $\{r^{\backprime}_{m_n}\}$ contains only a finite number of distinct elements, and (b) the sequence $\{r^{\backprime}_{m_n}\}$ contains an infinitude of distinct elements. Suppose if possible (a), and let $r^{\backprime}_0$ be an element infinitely repeated, *i. e.*,

$$r^{\backprime}_0 = r^{\backprime}_{m_{n_k}}(k).$$

We shall have:

$$\exists \{r_{m_{n_k} l}\}^{l} \ni \mathfrak{R}_{m_{n_k}} \ni \lim_{l} r_{m_{n_k} l} = r^{\backprime}_{m_{n_k}} = r^{\backprime}_0.$$

Then

$$m \cdot \supset \cdot \exists l_m \ni l \geqq l_m \cdot \supset \cdot K_{r_{m_{n_k} l} r^{\backprime}_0 m},$$

which holds in particular for $m = m_{n_k}(k)$. Also by hypothesis:

$$K_{r_{m_{n_k} l} r^{\backprime} m_{n_k}}.$$

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* Cf. Fréchet, *loc. cit.*, p. 19. It may be remarked that Fréchet really proves: If $(\Omega; K)$ is such that the derived class of every subclass of Ω is closed, then the class of elements of condensation of any subclass of Ω is also closed.

Hence by K^6 we have:

$$K_{r\check{}_0 r\check{}\phi_{m_{n_k}}}.$$

This will be holding for every k, and hence by § 7 (5) and K^3 we have

$$r\check{}_0 = r\check{}.$$

But by the above we then have:

$$l \geqq l_{(m_{n_k}+1)} \cdot \supset \cdot K_{r_{m_{n_k}l} r\check{}(m_{n_k}+1)},$$

which is contradictory to the definition of $\mathfrak{R}_{m_{n_k}}$. Hence case (a) can not occur. We are thus led to the result that we have (b), *i. e.*, that there is an infinitude of distinct elements in the sequence $\{r\check{}_{m_n}\}$. Let $r\check{}_{m_{n_k}}$ be this infinitude. Then for every k

$$\underset{l}{L} r_{m_{n_k}l} = r\check{}_{m_{n_k}},$$

where $r_{m_{n_k}l}$ is composed of distinct elements of $\mathfrak{R}_{m_{n_k}}$. Then

$$m \cdot \supset \cdot \exists\, l_m \ni l \geqq l_m \cdot \supset \cdot K_{r_{m_{n_k}l} r\check{}_{m_{n_k}} m}.$$

This holds in particular for $m = m_{n_k}(k)$. Moreover, for every k:

$$K_{r_{m_{n_k}l} r\check{}_{m_{n_k}}}.$$

Hence by K^7:

$$K_{r\check{}_{m_{n_k}} r\check{}\phi' m_{n_k}},$$

and by § 7 (4):

$$\underset{k}{L} r\check{}_{m_{n_k}} = r\check{}.$$

Since $r\check{}$ was any element of $\mathfrak{R}\check{}$, this proves that $\mathfrak{R}\check{}$ is dense in itself.

We have the following corollary:

(3) $$K^{1367} :\supset: \mathfrak{R}^{\text{condensed}} \cdot \supset \cdot \mathfrak{R}\check{}^{\text{perfect}}.$$

By a method similar to § 15 (4) it can be shown that:

(4) $$K^{1367} :\supset: \mathfrak{R}^{\text{condensed}} \cdot \supset \cdot [s \ni s^{\mathfrak{R}, -\mathfrak{R}\check{}}]^{\text{denumerable}}.$$

17. *Heine Borel Theorems.*

17a. *Heine Borel Property.* Suppose a class of classes $\mathfrak{S}$, in notation $[\mathfrak{S}]$. We consider a unipartite property P of such classes $[\mathfrak{S}]$, the notation $[\mathfrak{S}]^P$, denoting that the class $[\mathfrak{S}]$ has the property.

Every property P determines a derived property, likewise of classes $[\mathfrak{S}]$, the *Heine Borel Property* with respect to the property P, in notation $H\text{-}B(P)$. A class $[\mathfrak{S}]$ has the property $H\text{-}B(P)$ in case there exists a finite subclass $[\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_n]$ of $[\mathfrak{S}]$, which has the property P, in symbols:

$$[\mathfrak{S}]^{H\text{-}B(P)} \equiv [\mathfrak{S}] \ni ([\mathfrak{S}]^P \cdot \exists\, [\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_n]^{[\mathfrak{S}] \cdot P}).$$

17b. This Heine Borel Property occurs in the theory of linear point sets in the case where the class of classes consists of a set of closed intervals, and where the property P is that every point of a given interval lies within one of the intervals of the set. We shall be concerned with the Heine Borel Property relative to classes $\mathfrak{S}$ of elements s, where the property P is that every element of a given class $\mathfrak{R}$ is interior to some class $\mathfrak{S}$ of the class of classes $[\mathfrak{S}]$ relative to $\mathfrak{R}$. We shall denote this property by $I(\mathfrak{R})$, in symbols:

$$\mathfrak{R} \cdot [\mathfrak{S}]^{I(\mathfrak{R})} \equiv r^{\mathfrak{R}} \cdot \supset \cdot \exists\, \mathfrak{S}^{[\mathfrak{S}]} \ni r^{\text{interior } \mathfrak{S}(\mathfrak{R})}.$$

We then have the following theorem:*

$$(1) \qquad K^{15} \cdot \mathfrak{R}^{\text{extremal}} :\supset: [\mathfrak{S}]^{I(\mathfrak{R}) \cdot \text{denumerable}} = [\mathfrak{S}_n] \cdot \supset \cdot [\mathfrak{S}_n]^{H\text{-}B(I)}.$$

Suppose that the theorem does not hold. Then there will be an r_1, which will not be interior to $\mathfrak{S}_1$ relative to $\mathfrak{R}$. Let $\mathfrak{S}_{n_1}$ be the first class of $[\mathfrak{S}_n]$ to which r_1 is interior. Then there will exist an r_2 not interior to $\mathfrak{S}_1, \ldots, \mathfrak{S}_{n_1}$, but interior to $\mathfrak{S}_{n_2}$. Proceeding in this manner, we obtain in the general case an element r_l not interior to $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_{(n_l-1)}$, but interior to $\mathfrak{S}_{n_l}$. By way of contradiction we shall show that there exists a subsequence $\{r_{l_k}\}$ of $\{r_l\}$, which ultimately consists of elements which are interior to a single class of the set $[\mathfrak{S}_n]$. Since $\mathfrak{R}$ is extremal, we have:

$$\exists\, \{r_{l_k}\} \text{ of } \{r_l\} \ni \underset{k}{L} r_{l_k} = r_0.$$

Now r_0 will be interior to some class $\mathfrak{S}$ of the class $[\mathfrak{S}_n]$, let us say $\mathfrak{S}_i$. We then have by § 11 (1):

$$K^1 \cdot \supset \cdot \exists\, m_0 \ni K_{rr_0m_0} \cdot \supset \cdot r^{\mathfrak{S}_i}.$$

Now since $\underset{k}{L} r_{l_k} = r_0$:

$$m \cdot \supset \cdot \exists\, k_m \ni k \geqq k_m \cdot \supset \cdot K_{r_{l_k} r_0 m}.$$

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*Cf. Fréchet, *loc. cit.*, p. 22. E. R. Hedrick (*Trans. Amer. Math. Soc.*, XII, 285) has recently shown that it is sufficient to replace the hypothesis K^{15} by the hypothesis that the derived class of any subclass of Ω is closed. He supposes a $(\Omega; L)$, the L being the one of Fréchet, an L^{126}. It is easily shown that an L^3 is sufficient. To be sure, the hypothesis that the derived class of any subclass of Ω be closed probably acts in a restrictive way on the L. What this restriction is does not seem to have been as yet determined.

If in particular we take $m = m_0$, we have:

$$k \geqq k_{m_0} \cdot \supset \cdot r_{l_k}^{\mathfrak{S}_i}.$$

To obtain r_{l_k} interior to $\mathfrak{S}_i(\mathfrak{R})$, suppose

$$\{r_{nk}\} \ni \underset{n}{L} r_{nk} = r_{l_k}.$$

Then

$$m \cdot \supset \cdot \exists\, n_{mk} \ni n \geqq n_{mk} \cdot \supset \cdot K_{r_{nk} r_{l_k}^m},$$

and so if K^5:

$$k \geqq k_m \text{ and } n \geqq n_{mk} \cdot \supset \cdot K_{r_{nk} r_{\phi_m}}.$$

If then we take m_1 so that $\phi_{m_1} \geqq m_0$, we shall actually have:

$$k \geqq k_{m_1} \text{ and } n \geqq n_{m_1 k} \cdot \supset \cdot r_{nk}^{\mathfrak{S}_i},$$

that is, if k exceeds k_{m_0} and k_{m_1}, r_{l_k} will be interior to $\mathfrak{S}_i$ relative to $\mathfrak{R}$, which is the desired contradiction.

17c. *Separability.** $\mathfrak{R}$ is *separable relative to* $\mathfrak{Q}$ if there exists a denumerable subclass of $\mathfrak{Q}$ which together with its derivative contains the class $\mathfrak{R}$, symbolically:

$$\mathfrak{R}^{\text{separable }(\mathfrak{Q})} \equiv \exists\, \mathfrak{D}^{\text{denumerable}} \ni \mathfrak{R}^{\mathfrak{D}+\mathfrak{D}'}.$$

We have a special case of this when the class $\mathfrak{R}$ is the class $\mathfrak{Q}$ itself. In so far as the derived class of a subclass of $\mathfrak{Q}$ can not contain any elements not belonging to $\mathfrak{Q}$, we must have, in this case,

$$\mathfrak{Q}^{\text{separable }(\mathfrak{Q})} \equiv \exists\, \mathfrak{D}^{\text{denumerable}} \ni \mathfrak{Q} = \mathfrak{D} + \mathfrak{D}'.$$

The property separable is thus a bipartite property relating to the classes $\mathfrak{R}$ and $\mathfrak{Q}$. From this bipartite property we derive the property of separability of $\mathfrak{Q}$, *i. e.*, we have:

$$\mathfrak{Q}^{\text{separable}} \equiv \mathfrak{Q}^{\text{separable }(\mathfrak{Q})}.$$

We have the following propositions:

(1) $\mathfrak{Q}^{\text{separable}} :\supset: \mathfrak{R}^{\mathfrak{Q}} \cdot \supset \cdot \mathfrak{R}^{\text{separable }(\mathfrak{Q})}.$

(2) $\mathfrak{R}^{\mathfrak{Q}} \cdot \supset \cdot \mathfrak{R}^{\text{separable }(\mathfrak{Q})} :\supset: \mathfrak{Q}^{\text{separable}}.$

(3) $K^{15} :\supset: \mathfrak{R}^{\text{separable }(\mathfrak{Q})} \cdot \supset \cdot \mathfrak{R}'^{\text{separable }(\mathfrak{Q})}.$

(4) $K^{15} :\supset: \mathfrak{R}^{\text{separable }(\mathfrak{Q})} \cdot \supset \cdot (\mathfrak{R} + \mathfrak{R}')^{\text{separable }(\mathfrak{Q})}.$

(5) $\mathfrak{R}^{\text{separable }(\mathfrak{R})} \cdot \supset \cdot \mathfrak{R}^{\text{separable }(\mathfrak{Q})}.$

(6) $K^{17} :\supset: \mathfrak{R}^{\text{compact}} \cdot \supset \cdot \mathfrak{R}^{\text{separable }(\mathfrak{R})} \cdot \supset \cdot \mathfrak{R}^{\text{separable }(\mathfrak{Q})}.$ Cf. § 15 (7).

* Cf. Fréchet, *loc. cit.*, p. 23. The definition of separability given by Fréchet is:

$$\mathfrak{Q}^{\text{separable}} \equiv \exists\, \mathfrak{D}^{\text{denumerable}} \ni \mathfrak{Q} = \mathfrak{D}'.$$

It is easily seen that our definition is somewhat weaker.

17d. *Generalization of the Theorem of Cauchy.** The class $\mathfrak{R}$ has a *generalization of the Theorem of Cauchy relative to the class* $\mathfrak{Q}$ if, for every sequence r_n of $\mathfrak{R}$, which is such that for every n there exists an n_m such that, if n_1 and n_2 exceed n_m, we have $K_{r_{n_1} r_{n_2} m}$, then there exists an element q, which is the limit of the sequence r_n. In notation:

$$\mathfrak{R}^{GC(\mathfrak{Q})} \equiv \{r_n\} \ni m \cdot \supset \cdot \exists\, n_m \ni {}^{n_1}_{n_2} \geqq n_m :\supset: K_{r_{n_1} r_{n_2} m} \because\supset\because \exists\, q \ni Lr_n = q,$$

where $\mathfrak{R}^{GC(\mathfrak{Q})}$ denotes the fact that $\mathfrak{R}$ has a generalization of the theorem of Cauchy relative to $\mathfrak{Q}$. It is possible to replace the class $\mathfrak{R}$ by the class $\mathfrak{Q}$, and thus obtain $\mathfrak{Q}^{GC(\mathfrak{Q})}$, or simply $\mathfrak{Q}^{GC}$. We have at once:

(1) $$\mathfrak{Q}^{GC} \cdot \supset \cdot \mathfrak{R}^{GC(\mathfrak{Q})}.$$

(2) $$\mathfrak{R}^{\text{compact}} \cdot \supset \cdot \mathfrak{R}^{GC(\mathfrak{Q})}.$$

17e. We define finally a property D_m† of a class of classes $[\mathfrak{T}]$, relative to $\mathfrak{R}$. *The classes* $[\mathfrak{T}]$ *have the property* D_m *relative to* $\mathfrak{R}$, if every element r of the class $\mathfrak{R}$ belongs to at least one class of the set $[\mathfrak{T}]$, and if r_1 and r_2 belong to the same $\mathfrak{T}$, then $K_{r_1 r_2 m}$. Symbolically:

$$[\mathfrak{T}]^{D_m(\mathfrak{R})} \equiv (a)\ r \cdot \supset \cdot \exists\, \mathfrak{T}_r^{[\mathfrak{T}]} \ni r^{\mathfrak{T}} r. \quad (b)\ (r_1, r_2)^{\mathfrak{T}} \cdot \supset \cdot K_{r_1 r_2 m}.$$

We have the following lemmas relative to the property D_m:

(1) $$K^{167} :\supset: \mathfrak{R} \cdot m \cdot \supset \cdot \exists\, [\mathfrak{T}_m]^{D_m(\mathfrak{R})}.$$

Suppose m_0 such that $m_0 \geqq m$, $\phi^7_{m_0} \geqq m$, and $\phi^9_{m_0} \geqq m$. Let

$$\mathfrak{T}_{mr_0} = [r \ni K_{rr_0 m_0} \cdot r_0].$$

Evidently every element r will belong to at least one $\mathfrak{T}_{mr}$. Further, if of a pair of elements r_1 and r_2 belonging to the same $\mathfrak{T}_{mr_0}$, one of them is r_0, we shall evidently have $K_{r_1 r_0 m}$ and $K_{r_0 r_2 m}$, since $m_0 \geqq m$, and $\phi^9_{m_0} \geqq m$. In the general case we obtain from K^7 and from $K_{r_1 r_0 m_0}$ and $K_{r_2 r_0 m_0}$: $K_{r_1 r_2 m}$. Hence we have $[\mathfrak{T}_{mr}]^{D_m(\mathfrak{R})}$.

We are, however, interested more especially in the case in which for every m it is possible to find a class $[\mathfrak{T}]^{D_m(\mathfrak{R})}$ which is denumerable or even finite. We have:

(2) $$K^{167} \cdot \mathfrak{R}^{\text{separable}(\mathfrak{Q})} :\supset: m \cdot \supset \cdot \exists\, [\mathfrak{T}_{mn}]^{D_m(\mathfrak{R}) \cdot I(\mathfrak{R})}.$$

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* Cf. Fréchet, *loc. cit.*, p. 23.

† *I. e.*, Development, or division. Cf. Fréchet, p. 25, f.

On account of the separability of $\mathfrak{R}$ relative to $\mathfrak{Q}$, let $\mathfrak{D} = [q_n] \ni \mathfrak{R}^{\mathfrak{D}+\mathfrak{D}'}$. Suppose further

$$m_0 \ni m_0 \geqq m, \text{ and } \phi^{\tau}_{m_0} \geqq m.$$

Then set

$$\mathfrak{T}_{mn} = [r \ni K_{q_n r m_0} \cdot q_n \text{ if } q_n^{\mathfrak{R}}].$$

Evidently

$$r \cdot \supset \cdot \exists\, n_r \ni r^{\mathfrak{T}_{n_r}}.$$

Further, suppose $(r_1, r_2)^{\mathfrak{T}_{mn}}$. Then using K^{τ} we obtain, from $K_{q_n r_1 m_0}$ and $K_{q_n r_2 m_0}$, $K_{r_1 r_2 \phi^{\tau}_{m_0}}$ and so $K_{r_1 r_2 m}$. Hence $[\mathfrak{T}_{mn}]^{D_m(\mathfrak{R})}$.

As for the result $[\mathfrak{T}_{mn}]^{I(\mathfrak{R})}$, there evidently exists

$$m_0' \ni \phi^{\tau}_{m_0'} \geqq m_0 \text{ and } m_0' \geqq m_0.$$

If now $\underset{l}{L} r_l = r$, we have:

$$m_0' :\supset: \exists\, l_{m_0'} \ni l \geqq l_{m_0'} \cdot \supset \cdot K_{r_l r m_0'}.$$

Also, on account of the separability of $\mathfrak{R}$, we have either:

$$r \cdot \supset \cdot \exists\, q_{n_r} \ni K_{q_{n_r} r m_0'};$$

and hence, if we apply K^{τ}, there result $K_{q_{n_r} r_l m_0}$ and $K_{q_{n_r} r m_0}$, *i. e.*, by the definition of $\mathfrak{T}_{mn}$ and the definition of interiority,

$$r^{\text{interior } \mathfrak{T}_{mn_r}(\mathfrak{R})};$$

or $r = q_n$, in which case the interiority is immediate. We therefore have:

$$[\mathfrak{T}_{mn}]^{I(\mathfrak{R})}.$$

As for $[\mathfrak{T}]^{D_m(\mathfrak{R})}$ and finite, we have:

(3) $$K^{167} \cdot \mathfrak{R}^{\text{compact}} :\supset: \exists\, [\mathfrak{T}_{m1}, \mathfrak{T}_{m2}, \ldots, \mathfrak{T}_{mn}]^{D_m(\mathfrak{R}) \cdot I(\mathfrak{R})}.$$

Since the class $\mathfrak{R}$ is compact, we have by § 15 (3) the class $(\mathfrak{R} + \mathfrak{R}')$ extremal. Since further, by § 17c (6), from $(\mathfrak{R} + \mathfrak{R}')^{\text{compact}}$ we obtain $(\mathfrak{R} + \mathfrak{R}')^{\text{separable }(\mathfrak{Q})}$, we can determine by (2) above:

$$[\mathfrak{T}_{mn'}]^{D_m(\mathfrak{R}+\mathfrak{R}') \cdot I(\mathfrak{R}+\mathfrak{R}')}, \quad n' = 1, 2, 3, \ldots.$$

But by § 17b, $(\mathfrak{R} + \mathfrak{R}')^{\text{extremal}} \cdot m :\supset: \exists\, [\mathfrak{T}_{mn'}]^{I(\mathfrak{R}+\mathfrak{R}')} \cdot \supset \cdot [\mathfrak{T}_{mn'}]^{H\text{-}B(I(\mathfrak{R}+\mathfrak{R}'))}$; *i. e.*, there exists a finite set of classes $[\mathfrak{T}_{mn'}]^{I(\mathfrak{R}+\mathfrak{R}')}$. Evidently this set is such that $[\mathfrak{T}_{mn}]^{D_m(\mathfrak{R}) \cdot I(\mathfrak{R})}$.

The following converse of this lemma holds:*

(4) $\quad K^1 \cdot \mathfrak{R}^{GC(\mathfrak{L})} :\supset: m \cdot\supset\cdot \exists\, [\mathfrak{T}_{m1}, \ldots, \mathfrak{T}_{mn}]^{D_m(\mathfrak{R})} ::\supset:: \mathfrak{R}^{\text{compact}}.$

17f. *Heine Borel Theorem in the Non-denumerable Case.* We are now in position to prove the Heine Borel Theorem in the case in which the given $[\mathfrak{S}]^{I(\mathfrak{R})}$ is non-denumerable. We have the theorem:†

(1) $\quad \mathfrak{R} \cdot K^{167} :\supset: \mathfrak{R}^{\text{extremal}} \cdot [\mathfrak{S}]^{I(\mathfrak{R})} \cdot\supset\cdot [\mathfrak{S}]^{H\text{-}B(I)}.$

By §17e (3) we have:

$$\mathfrak{R}^{\text{extremal}} \cdot m \cdot\supset\cdot \exists\, [\mathfrak{T}_m]^{\text{finite}} \cdot D_m(\mathfrak{R}).$$

If the theorem is not true, then for every m it is not true for one of the finite set of classes $[\mathfrak{T}_m]$, *i. e.*:

$$m \cdot\supset\cdot \exists\, \mathfrak{T}_{0m} \ni [\mathfrak{S}]^{-H\text{-}B(I(\mathfrak{T}_{0m}))}.$$

We argue a contradiction by showing that there exists an $\mathfrak{S}_0$ of the class $[\mathfrak{S}]$, such that every element of each class of a subclass of $[\mathfrak{T}_{0m}|m] : [\mathfrak{T}_{0m_n}|n]$, is interior to $\mathfrak{S}_0(\mathfrak{R})$. If r_m be any element of $\mathfrak{T}_{0m}$, we obtain a sequence $\{r_m\}$. There are two possibilities: (a) there is one element repeated infinitely often in the sequence, and (b) no element is repeated infinitely often.

(a) If possible, suppose one element is repeated infinitely often; *i. e.*, let

$$r_0 = r_{m_n}(n).$$

Then

$$\exists\, \mathfrak{S}_0 \ni r_0 \text{ interior } \mathfrak{S}_0(\mathfrak{R}).$$

Then by §11 (1)

$$K^1 \cdot\supset\cdot \exists\, m_0 \ni (r \ni K_{rr_0m_0}) \cdot\supset\cdot r^{\mathfrak{S}_0}.$$

Let $r^{(m_n)}$ be any element of $\mathfrak{T}_{m_n}$. Then:

$$K_{r^{(m_n)}r_{m_n}m_n}; \quad \textit{i. e.,}\quad K_{r^{(m_n)}r_0m_n}.$$

Suppose, further, that $\underset{l}{L} r_l = r^{(m_n)}$. Then

$$m \cdot\supset\cdot \exists\, l_m \ni l \geqq l_m \cdot\supset\cdot K_{r_l r^{(m_n)}m}.$$

This will hold in particular for $m = m_n$, and so we have by K^5 for every n:

$$l \geqq l_{m_n} \cdot\supset\cdot K_{r_l r_0 \phi^5 m_n}.$$

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* For proof see Fréchet, *loc. cit.*, p. 25. Note the weaker hypothesis here.

† Cf. Fréchet, *loc. cit.*, p. 26.

If now we choose n_0 so that

$$n \geqq n_0 \cdot \supset \cdot m_n \geqq m_0 \cdot \phi^5_{m_n} \geqq m_0,$$

we shall have

$$r^{(m_n)} \text{ interior to } \mathfrak{S}_0(\mathfrak{R}).$$

Hence we have the desired contradiction in this case.

(b) No element of the sequence $\{r_m\}$ is repeated infinitely often. Then, on account of the extremality of the class $\mathfrak{R}$,

$$\exists \{r_{m_n}\}, \ r_0 \ni \underset{n}{L} r_{m_n} = r_0.$$

The argument then proceeds very much as in case (a).

We have the following converse of this important theorem:

(2) $$K^{136} \cdot :\supset: \mathfrak{R} \cdot [\mathfrak{S}]^{I(\mathfrak{R})} \cdot \supset \cdot [\mathfrak{S}]^{H\text{-}B(I)} \cdot :\supset: \mathfrak{R}^{\text{extremal}}.$$

(a) $\mathfrak{R}$ *is closed.* Suppose it were not. Then

$$\exists \{r_n\} \ni \underset{n}{L} r_n = q \cdot q^{-\mathfrak{R}}.$$

Let

$$\mathfrak{S}_m = [r \ni -K_{rqm}].$$

Then

$$r \cdot \supset \cdot \exists \ m \ni r^{\text{interior } \mathfrak{S}_m(\mathfrak{R})}.$$

For if not:

$$\exists \{r_l\} \ni (\underset{l}{L} r_l = r \cdot m \cdot \supset \cdot \exists \ l_m \ni l \geqq l_m \cdot \supset \cdot K_{r_l qm}).$$

But then by definition of limit:

$$\underset{l}{L} r_l = q.$$

If now K^{136}, then by § 7 (3), limit is unique, and since $q^{-\mathfrak{R}}$ we have a contradiction. Evidently a finite number of the classes $\mathfrak{S}$ will contain only a finite number of elements of the sequence r_n, and so

$$[\mathfrak{S}_m]^{\text{finite}} \cdot \supset \cdot [\mathfrak{S}_m]^{-I(\mathfrak{R})}.$$

(b) $\mathfrak{R}$ *is compact.* If not, then there exists a sequence of distinct elements $\{r_n\}$ without a limiting element. Then

$$r \neq r_n \cdot \supset \cdot \exists \ \mu_r > \bar{B}(m \ni K_{r_n rm} \mid n)$$

$$r \neq r_{n_0} \cdot \supset \cdot \exists \ \mu_{r_{n_0}} > \bar{B}(m \ni K_{r_n r_{n_0} m} \mid n \neq n_0).$$

If K^{13}, this μ will be finite for every r. Let

$$\mathfrak{S}_{r_0} = [r \ni K_{rr_0\mu_{r_0}} \cdot r_0].$$

Then evidently

$$r_0 \text{ interior } \mathfrak{S}_{r_0}(\mathfrak{R}).$$

But no $\mathfrak{S}_{r_0}$ contains two elements of the sequence r_n, and so

$$[\mathfrak{S}_{r_0}]^{-H\text{-}B(I)}.$$

The proofs of these theorems are sufficient to indicate the method of attack in case we are operating with a K having the properties (1), (3), (6), (7). In the theorems of Fréchet not taken up here, there is little difficulty in showing that such a K is sufficient. Instead of following up the matter of the properties of subclasses $\mathfrak{R}$ of $\mathfrak{Q}$ of systems $(\mathfrak{Q}; K)$ further, we turn our attention briefly to continuous functions on subclasses $\mathfrak{R}$ of systems $(\mathfrak{Q}; L)$ and $(\mathfrak{Q}; K)$.

Continuous Functions on Subclasses $\mathfrak{R}$ *of Systems* $(\mathfrak{Q}; L)$. §§ 18–20.

18. *Functions of Subclasses* $\mathfrak{R}$ *of* $\mathfrak{Q}$ *to the Class of real Numbers* $\mathfrak{A}$. *Sequential Continuity.* By a function on a subclass $\mathfrak{R}$ of $\mathfrak{Q}$ to the class of real numbers $\mathfrak{A}$, we mean a correspondence between elements of the class $\mathfrak{R}$ and real numbers $\mathfrak{A}$ such that to every element of the class $\mathfrak{R}$ there corresponds at least one real number. If this function or correspondence be denoted by μ, our definition might be stated symbolically:

$$r^{\mathfrak{R}} \cdot \supset \cdot \exists a^{\mathfrak{A}} \ni \mu_r = a.$$

We shall suppose, in particular, that we are dealing only with single-valued functions.

If μ is a function on the class $\mathfrak{R}$ to $\mathfrak{A}$, we say that μ *is continuous at the element* r in case

$$\underset{n}{L} r_n = r \cdot \supset \cdot \underset{n}{L} \mu_{r_n} = \mu_r,$$

where the first limit is an L,* the second a real-number limit.

μ is said to be *continuous on* $\mathfrak{R}$ *to* $\mathfrak{A}$, if it is continuous for every element r of $\mathfrak{R}$.

* If L is not unique, *i. e.*, $\underset{n}{L} r_n = r$ and $\underset{n}{L} r_n = r'$, and μ is continuous, then we must of course have $\mu_r = \mu_{r'}$.

19. *Bounded Properties of Continuous Functions.** We have the following theorem:†

$$(1) \qquad \mu^{\text{continuous on } \mathfrak{R}^{\text{extremal}} \text{ to } \mathfrak{A}} \cdot \supset \cdot \exists\, (b, B) \ni b \leqq \mu_r \leqq B\ (r^{\mathfrak{R}}).$$

We show that there exists an upper bound B, the lower bound being shown to exist in a similar manner. Suppose, if possible, that no upper bound exists. Then

$$n \cdot \supset \cdot \exists\, r_n \ni \mu_{r_n} > n.$$

Since μ is defined for every r, the sequence $\{r_n\}$ cannot contain an element infinitely repeated. Then since $\mathfrak{R}$ is extremal we have:

$$\exists\, \{r_{n_l}\}^{\text{distinct}} \cdot r \ni \underset{l}{L} r_{n_l} = r.$$

Now since μ is continuous on $\mathfrak{R}$, it will be continuous at r, and so we have:

$$\underset{l}{L} \mu_{r_{n_l}} = \mu_r.$$

But by the hypothesis on r_{n_l} we would have:

$$\underset{l}{L} \mu_{r_{n_l}} = \infty.$$

We have thus reached a contradiction.

If we denote by b and B the greatest lower and least upper bounds, respectively, of μ on $\mathfrak{R}$, we have:‡

$$(2) \qquad \mu^{\text{continuous on } \mathfrak{R}^{\text{extremal}} \text{ to } \mathfrak{A}} \cdot \supset \cdot \exists\, (r_1, \text{ and } r_2) \ni \mu_{r_1} = b,\ \mu_{r_2} = B.$$

We show that

$$\exists\, r_2 \ni \mu_{r_2} = B.$$

Suppose it were not so. Then

$$n \cdot \supset \cdot \exists\, r_n \ni B > \mu_{r_n} > B - 1/n.$$

The sequence $\{r_n\}$ will again involve an infinitude of distinct elements, and thus using the condition that the class $\mathfrak{R}$ is extremal we obtain:

$$\exists\, \{r_{n_l}\}^{\text{distinct}} \cdot r \ni \underset{l}{L} r_{n_l} = r.$$

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* We consider here a number of the more important theorems of Fréchet (*loc. cit.*, pp. 8–15), to show the method of reasoning when we suppose L unconditioned. The theorems not taken up are also holding in the general $(\Omega; L)$ situation; as a matter of fact, the proofs of Fréchet, when used with some care, will give the desired results.

† If μ is continuous on an extremal subclass $\mathfrak{R}$, then it is bounded on $\mathfrak{R}$. Cf. Fréchet, *loc. cit.*, p. 8.

‡ Cf. Fréchet, *loc. cit.*, p. 8.

Then on account of the continuity of μ:

$$\underset{l}{L}\mu_{r_{n_l}} = \mu_r.$$

But by the construction of $\{r_n\}$:

$$\underset{l}{L}\mu_{r_{n_l}} = B.$$

Hence $\mu_r = B$. Similarly we show that:

$$\exists\, r_1 \ni \mu_{r_1} = b.$$

20. *Sequences of Continuous Functions.* We define uniformity and quasi-uniformity of convergence of sequences first. Suppose a sequence of functions $\{\mu_n\}$ each on $\mathfrak{R}$ to $\mathfrak{A}$, converging to a function μ on $\mathfrak{R}$ to $\mathfrak{A}$, *i. e.*,

$$\underset{n}{L}\mu_{nr} = \mu_r,$$

or
$$r \cdot e \cdot \supset \cdot \exists\, n_{er} \ni n \geqq n_{er} \cdot \supset \cdot |\mu_{nr} - \mu_r| \leqq e.$$

We say that the convergence is *uniform* on $\mathfrak{R}$ if

$$e \cdot \supset \cdot \exists\, n_e \ni n \geqq n_e \cdot \supset \cdot |\mu_{nr} - \mu_r| \leqq e,$$

for every $r^{\mathfrak{R}}$. It evidently differs from the simple convergence in that the n_e does not depend upon the elements r. The convergence is said to be *quasi-uniform*, if

$$e \cdot l :\supset: \exists\, l_{el} \ni r \cdot \supset \cdot \exists\, n_{erl} \ni (l \leqq n_{erl} \leqq l_{el} \cdot |\mu_{n_{erl}r} - \mu_r| \leqq e).$$

Quasi-uniformity of convergence does not insure the convergence of a sequence. The convergence must be assumed separately if desired.

The subject of uniformity and quasi-uniformity is of importance in the convergence of a sequence of continuous functions to a continuous function. We have the following well-known theorem:

(1) *A uniformly convergent sequence of continuous functions converges to a continuous function.*

The hypothesis of the following theorem is, however, less exacting, and hence it covers the former also:*

(2) *A quasi-uniformly convergent sequence of continuous functions converges to a continuous function.*

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* Cf. Fréchet, *loc. cit.*, p. 10.

Suppose a sequence $\{\mu_{nr}\}$ of continuous functions on $\mathfrak{R}$ to $\mathfrak{A}$, which converges to μ_r, *i. e.*,

$$\lim_n \mu_{nr} = \mu_r,$$

the convergence being supposed quasi-uniform. We wish to show that μ_r is continuous, *i. e.*,

$$\lim_i r_i = r_0 \cdot \supset \cdot \lim_i \mu_{r_i} = \mu_{r_0}.$$

Since the sequence converges at r_0, we have:

$$e \cdot \supset \cdot \exists\, n_{er_0} \ni n \geqq n_{er_0} \cdot \supset \cdot |\mu_{nr_0} - \mu_{r_0}| \leqq e.$$

On account of the quasi-uniformity of convergence of our sequence we have:

$$e \cdot n_{er_0} \cdot \supset \cdot \exists\, l_{en_{er_0}} \ni r_i \cdot \supset \cdot \exists\, n_{er_i} n_{er} = n_{ei} \ni n_{er_0} \leqq n_{ei} \leqq l_{en_{er_0}} \text{ and}$$
$$|\mu_{n_{ei}r_i} - \mu_{r_i}| \leqq e \qquad (1)$$

The continuity of the functions $\mu_{n_{ei}}$ gives:

$$e \cdot \supset \cdot \exists\, i_e \ni i \geqq i_e \cdot \supset \cdot |\mu_{n_{ei}r_i} - \mu_{n_{ei}r_0}| \leqq e, \qquad (2)$$

where i_e is the maximum of the i_e corresponding to the values of n between n_{er_0} and $l_{en_{er_0}}$. Finally, since n_{ei} exceeds n_{er_0}, we have:

$$|\mu_{n_{ei}r_0} - \mu_{r_0}| \leqq e. \qquad (3)$$

Adding the inequalities of (1), (2) and (3), we have:

$$|\mu_{r_i} - \mu_{r_0}| \leqq 3e,$$

subject to the condition $i \geqq i_e$, the n being only a subsidiary.

The following converse of this theorem holds also in this general situation:

(3) *If the class $\mathfrak{R}$ is extremal, and a sequence of continuous functions converges to a continuous function, then the convergence is quasi-uniform.*

If the sequence of functions in question be $\{\mu_{nr}\}$ and the limit function μ_r, it is necessary to show:

$$e \cdot l \cdot \supset \cdot \exists\, l_{el} \ni r \cdot \supset \cdot \exists\, n_{erl} \ni (l \leqq n_{erl} \leqq l_{el} \cdot |\mu_{n_{erl}r} - \mu_r| \leqq e).$$

Choose e and l arbitrarily. Let n_{erl} be the minimum of the numbers exceeding l, such that:

$$|\mu_{n_{erl}r} - \mu_r| \leqq e.$$

We wish to show that n_{erl} has a finite upper bound. If this is not the case we have:

$$i \geqq l \cdot \supset \cdot \exists\, r_i \ni n_{er_il} \geqq i.$$

Since the sequence $\{\mu_{nr}\}$ is convergent for every r, no r_i can occur infinitely often, and we obtain from the extremality of $\mathfrak{R}$:

$$\exists \{r_{i_k}\}^{\text{distinct}} \cdot r \ni \underset{k}{L} r_{i_k} = r.$$

Since now μ_r is continuous,

$$e \cdot \supset \cdot \exists k_e \ni k \geqq k_e \cdot \supset \cdot |\mu_{r_{i_k}} - \mu_r| \leqq e.$$

On account of the continuity of μ_{nr},

$$e \cdot \supset \cdot \exists k_{en} \ni k \geqq k_{en} \cdot \supset \cdot |\mu_{nr_{i_k}} - \mu_{nr}| \leqq e.$$

Since further $\underset{n}{L} \mu_{nr} = \mu_r$,

$$e \cdot \supset \cdot \exists n_{er} \ni n \geqq n_{er} \cdot \supset \cdot |\mu_{nr} - \mu_r| \leqq e.$$

Hence,

$$3e \cdot \supset \cdot \exists n_{er} \cdot \exists k_{en} \ni n \geqq n_{er} \cdot k \geqq k_{en} \cdot \supset \cdot |\mu_{nr_{i_k}} - \mu_{r_{i_k}}| \leqq 3e.$$

Let n_0 be the greater of n_{er} and l. Then this last inequality will hold for $k \geqq k_{en_0}$. On account of the convergency of the sequence we have:

$$3e \cdot \supset \cdot \exists n_{ek} \ni n \geqq n_{ek} \cdot \supset \cdot |\mu_{nr_{i_k}} - \mu_{r_{i_k}}| \leqq 3e.$$

There being only a finite number of k less than k_{en_0}, we shall have a finite number of corresponding n_{ek}. Of these and of n_0 we choose the largest. This will serve as an upper bound for the $n_{er_{i_k}l}$. We have thus reached a contradiction in so far as we have demonstrated the existence of a finite bound for $n_{er_{i_k}l}$. Hence the theorem.

Continuous Functions on Subclasses $\mathfrak{R}$ of Systems $(\Omega; K)$.

21. *Difference Continuity.* In case we are operating in a system $(\Omega; K)$, it is possible to define a type of continuity which is analogous to the difference continuity of a function on a real interval. The K-relation serves to replace the absolute value of the difference. In order to distinguish this type from the continuity employed above, we call the former difference continuity, and the latter sequential continuity. We define:

$$\mu^{\text{difference continuous at } r_0} \equiv e \cdot \supset \cdot \exists m_{er_0} \ni K_{rr_0 m_{er_0}} \cdot \supset \cdot |\mu_r - \mu_{r_0}| \leqq e.$$

We say that μ *is difference continuous on the class* $\mathfrak{R}$ if it is continuous at every element of $\mathfrak{R}$.

In order to make the theorems on sequentially continuous functions available here, we must establish some connection. This is contained in the theorem*

$$K^1 \cdot \mathfrak{R} \cdot \mu^{\text{on } \mathfrak{R} \text{ to } \mathfrak{A}} :\supset: \mu^{\text{sequentially continuous on } \mathfrak{R}} \cdot \supset \cdot \mu^{\text{difference continuous on } \mathfrak{R}}.$$

(a) $\mu^{\text{sequentially continuous on } \mathfrak{R}} \cdot \supset \cdot \mu^{\text{difference continuous on } \mathfrak{R}}$. We wish then to show that:

$$r_0^{\mathfrak{R}} \cdot e \cdot \supset \cdot \exists\, m_{er_0} \ni K_{rr_0m_{er_0}} \cdot \supset \cdot |\mu_r - \mu_{r_0}| \leqq e.$$

Suppose this were not so. Then:

$$\exists\, r_0 \cdot \exists\, e_0 \ni m \cdot \supset \cdot \exists\, r_m \ni K_{r_m r_0 m} \cdot |\mu_{r_m} - \mu_{r_0}| > e_0.$$

From § 7 (5) and K^1, it follows that:

$$\lim_m r_m = r_0.$$

Since now μ is sequentially continuous, we have:

$$\lim_m \mu_{r_m} = \mu_{r_0}, \text{ i. e., } e \cdot \supset \cdot \exists\, m_e \ni m \geqq m_e \cdot \supset \cdot |\mu_{r_m} - \mu_{r_0}| \leqq e.$$

Since this will be holding for e_0 also, we have reached a contradiction.

(b) $\mu^{\text{difference continuous on } \mathfrak{R}} \cdot \supset \cdot \mu^{\text{sequentially continuous on } \mathfrak{R}}$. Since μ is difference continuous on $\mathfrak{R}$, we have:

$$r_0 \cdot e \cdot \supset \cdot \exists\, m_e \ni K_{rr_0m_e} \cdot \supset \cdot |\mu_r - \mu_{r_0}| \leqq e.$$

Suppose $\{r_n\}$ is any sequence having r_0 as a limit, *i. e.*:

$$m \cdot \supset \cdot \exists\, n_m \ni n \geqq n_m \cdot \supset \cdot K_{r_n r_0 m}.$$

This will hold in particular when $m = m_e$, and so we have:

$$e \cdot \supset \cdot \exists\, n_e = n_{m_e} \ni n \geqq n_e \cdot \supset \cdot |\mu_{r_n} - \mu_{r_0}| \leqq e,$$

which is the continuity as desired.

22. *Uniform Continuity.* We note that relative to sequential continuity it does not seem possible to define a uniform continuity. However, in the case of difference continuity such a possibility exists. We define:

$$\mu^{\text{uniformly continuous on } \mathfrak{R}} \equiv e \cdot \supset \cdot \exists\, m_e \ni K_{r_1 r_2 m_e} \cdot \supset \cdot |\mu_{r_1} - \mu_{r_2}| \leqq e,$$

the uniformity feature entering in that the m_e is independent of the r. If μ is uniformly continuous it is also continuous. On the other hand we have:*

$$K^{1367} \cdot \mathfrak{R}^{\text{extremal}} :\supset: \mu^{\text{continuous on } \mathfrak{R}} \cdot \supset \cdot \mu^{\text{uniformly continuous on } \mathfrak{R}}.$$

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* Cf. Fréchet, *loc. cit.*, p. 28.

If possible suppose that μ is not uniformly continuous. Then:

$$\exists\, e_0 \ni m \cdot \supset \cdot \exists\, (r_{1m} \cdot r_{2m}) \ni K_{r_{1m}r_{2m}m} \cdot |\mu_{r_{1m}} - \mu_{r_{2m}}| > e_0.$$

We thus obtain two sequences. There are four possibilities: (a) The sequences $\{r_{1m}\}$ and $\{r_{2m}\}$ each contain only a finite number of distinct elements; (b) The sequence $\{r_{1m}\}$ contains only a finite number of distinct elements, while the sequence $\{r_{2m}\}$ contains an infinitude of distinct elements; (c) The sequence $\{r_{1m}\}$ contains an infinitude of distinct elements, while the sequence $\{r_{2m}\}$ contains only a finite number of distinct elements; and (d) Each sequence $\{r_{1m}\}$ and $\{r_{2m}\}$ contain an infinitude of distinct elements.

In case (a) there will be one pair of elements which will have like subscripts infinitely often. These are shown to be equal by K^{13} and § 7 (5). We thus obtain a contradiction at once.

In case (b), so far as it is not covered by (a), it will be possible to select from the sequence $\{r_{2m}\}$ the sequence $\{r_{2m_n}\}$ of distinct elements, and from the sequence $\{r_{1m}\}$ the set $\{r_{1m_n}\}$ consisting of one element repeated, *i. e.*:

$$r_{1m_n} = r_1\,(n).$$

Then by our hypothesis:

$$K_{r_{1m_n}r_{2m_n}m_n}, \quad i.\ e., \quad K_{r_1r_{2m_n}m_n}.$$

From K^{167} by § 5 (1) it follows that:

$$K_{r_{2m_n}r_1\phi^9 m_n}.$$

Since this holds for every n and $L_m \phi^9_m = \infty$, we have by § 7 (4):

$$L_n r_{2m_n} = r_1.$$

But μ is continuous. Hence:

$$e_0 \cdot \supset \cdot \exists\, n_{e_0} \ni n \geqq n_{e_0} \cdot \supset \cdot |\mu_{r_1} - \mu_{r_{2m_n}}| \leqq e_0 \text{ or } |\mu_{r_{1m_n}} - \mu_{r_{2m_n}}| \leqq e_0,$$

which is contrary to the definition of r_{1m} and r_{2m}. Hence we do not have (b).

We show similarly that case (c) can not occur.

In case (d), in so far as it is not covered by the preceding, it will be possible to select the sequences $\{r_{1m_n}\}$ and $\{r_{2m_n}\}$ each of distinct elements, and on account of the extremality of $\mathfrak{R}$ in such a way that

$$L_n r_{1m_n} = r_1.$$

Then $$m \cdot \supset \cdot \exists\, n'_m \ni n \geqq n'_m \cdot \supset \cdot K_{r_{1m_n}r_1m}.$$

Also by K^1 and the hypothesis on r_{1m} and r_{2m}:

$$m \cdot \supset \cdot \exists\, n''_m \ni n \geqq n''_m \cdot \supset \cdot K_{r_{1m_n} r_{2m_n} m}.$$

Then using K^6 we have:

$$n \geqq n'_m \cdot n \geqq n''_m \cdot \supset \cdot K_{r_{2m_n} r_1 \phi^6 m}.$$

Hence, by K^1 and § 7 (4):

$$\underset{n}{L}\, r_{2m_n} = r_1.$$

But on account of the continuity of μ we shall have:

$$e/2 \cdot \supset \cdot \exists\, n_e \ni n \geqq n_e \cdot \supset \cdot |\mu_{r_{1m_n}} - \mu_{r_1}| \leqq e/2 \cdot |\mu_{r_{2m_n}} - \mu_{r_1}| \leqq e/2,$$

i. e.,

$$n \geqq n_e \cdot \supset \cdot |\mu_{r_{1m_n}} - \mu_{r_{2m_n}}| \leqq e.$$

But this is contradictory to our hypothesis on r_{1m_n} and r_{2m_n} if $e = e_0$.

We have thus shown that none of the cases (a), (b), (c), (d) can occur; that is, we have established the uniform continuity of μ.

We could proceed to consider finally the theorem of Fréchet, *loc. cit.*, p. 31, as extended by Hahn,* and show that this is also holding in case K^{1367}, *i. e.*, in case δ^{0367}. With some care in the use of the proof given by Hahn, there is little difficulty in proving the existence of the non-constant continuous function in a δ^{0367}, and deriving the theorem in question, *i. e*:

$$K^{1367} \cdot \mathfrak{R} :\supset: \mu^{\text{continuous on } \mathfrak{R} \text{ and bounded on } \mathfrak{R}} \cdot \supset \cdot \mathfrak{R}^{\text{extremal}}.$$

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* *Monatshefte für Mathematik und Physik*, XIX, 251 ff. We might remark in this connection that, on account of what seems to be an oversight, the limit L, defined by Hahn on p. 250 of this note in the construction of a system $(\mathfrak{Q}; L)$ on which every continuous function is constant, is not unique. The uniqueness is secured if the series of inequalities in l. 8: $i_1 < i_2 < \ldots < i_n$ is replaced by $k_1 < k_2 < \ldots < k_n$.

On the Perspective Jonquières Involutions Associated with the (2, 1) Ternary Correspondence.

By Paul Prentice Boyd.

Introduction.

In 1871 Clebsch* developed a correspondence between two planes such that two points of one plane are made to correspond to one point of another plane, while to a point of the first corresponds but one on the second. The correspondence was made to depend upon the bisection of Abelian functions connected with a curve of the fourth order, lying in a so-called double plane, which is so related to a surface that to a point on the surface there corresponds one point in the plane, but to a point on the plane there correspond two points on the surface. Inspired by this article, de Paolis,† in 1877, worked out the general theory of such a (2, 1) correspondence between two planes entirely independent of any surface depiction. This theory, as he remarked, naturally presents itself immediately as a first generalization of the rational transformations of Cremona. In a second paper de Paolis carried the discussion over into non-euclidian geometry, and in a third paper he worked out the analytic details of that (2, 1) transformation with which the Geiser involution‡ is associated. Noether,§ in 1878, published a note concerning the (2, 1) transformation associated with the involution of order 17 discovered by Bertini,‖ and again in 1889 published two extended articles involving the same transformation.

* Clebsch: "Ueber den Zusammenhang einer Classe von Flächenabbildungen mit der Zweitheilung der Abel'schen Functionen," *Mathematische Annalen*, Vol. III (1871).

† Riccardo de Paolis: 1. "Le trasformazioni piane doppia." 2. "La trasformazione piana doppia di secondo ordine, e la sua applicazione alla geometria non euclidea." 3. "La tras. p. dop. di terzo ord. prim. gen. e la sua app. a curve del quarto ord." *Atti della R. Accademia dei Lincei*, Serie 3, T. 1, 2, 1877.

‡ Geiser: "Ueber zwei geometrische Probleme," *Journal für Mathematik*, Vol. LXVII (1867), pp. 78–89. Milinowsky: "Bemerkung zu der Geiser'schen... ," *Journal für Mathematik*, Vol. LXXVII (1874), pp. 263–268.

§ M. Noether: "Ueber die einzweideutigen Ebenentransformationen," *Sitzungsber. der physik.-medic. Soc. zu Erlangen*, Jan. 14, 1878. "Ueber eine Classe von auf die einfache Ebene abbildbaren Doppelebenen," *Math. Annalen*, Vol. XXXIII (1889), pp. 524–545. "Ueber die rationalen Flächen vierter Ordnung," *Math. Annalen*, Vol. XXXIII (1889), pp. 546–571.

‖ Bertini: "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Mat.*, Serie 2, T. 8, 1877, pp. 11–23; 146; 244–286.

In the first of these two articles Noether classified the (2, 1) transformations according to three of the four irreducible types of involutions found by Bertini in 1877–1880 — the perspective Jonquières,* the Geiser and that of order 17. Clebsch's work came under the second class, and, as before mentioned, de Paolis worked out the same in great detail in his third paper. de Paolis gives, at the end of his first paper, a table stating the possible fundamental configurations in the two planes for orders of transformation from 1 to 10, and for nets of curves of genus 0 and 1. He also states the facts for n (the order of the transformation) $=4$, p (genus of net) $=2$; $n=5$, $p=2$; $n=5$, $p=3$; $n=6$, $p=3$. Other writers have treated special quadratic transformations.† Schottky‡ has treated the subject in connection with Abelian functions.§

Aside from the work of Noether, nothing has been done with the third class of (2, 1) correspondences — that involving the involution of order 17.‖ No complete analytic treatment has yet appeared. Aside from the cases mentioned by de Paolis of $n=2$, $p=0$; and $n=4$, $p=2$: $n=5$, $p=3$; $n=5$, $p=2$, all of which involve involutions that are perspective Jonquières or that reduce to such, no work has been done upon the first class. It is the purpose of this paper to give a general synthetic and analytic treatment of the (2, 1) correspondence associated with perspective Jonquières involutions and to connect with this the depiction of certain surfaces upon the two planes.

I. *The* (2, 1) *Ternary Correspondence.*

Suppose we have a correspondence established between the points of two planes, Σ and Σ', of such a character that to one point X of Σ there corresponds a single point Y of Σ', but that to a point Y of Σ' correspond two points X and $\bar{X}$ in Σ. This correspondence will be denoted by the symbol (2, 1) or simply T_{21}. The plane Σ is called the simple plane, and Σ' the double plane. If, further,

* de Jonquières: "Memoire sur les figures isographiques....," produced in 1859, presented to Paris Academy, 1860, published in *Batt. G.*, Vol. XXIII (1885), pp. 48–73. "De la transformation geometrique des figures planes," *Nouv. Ann.*, Ser. 2, T. 3 (1864), pp. 97–111.

† V. Retali: "Memorie dell' Instituto de Bologna," Series 4, Vol. X.

H. Liebmann: "Die einzweideutigen projektiven Punktverwandtschaften der Ebene," *Dissertation*, Jena, 1895.

‡ Schottky: "Zur Theorie der Abel'schen Functionen von vier Variabeln," *Crelle*, Vol. CII. "Ueber specielle Abel'sche Functionen vierten Ranges," *Crelle*, Vol. CIII.

§ R. Sturm ("Die Lehre von den geometrischen Verwandtschaften," Vol. IV, Chap. IX) gives a general treatment of the [2, 1] correspondence and works out in detail the cases of $n=2$; $n=3$, $p=0$; $n=3$, $p=1$.

‖ For the derivation of the equations of this involution see V. Snyder: "The Involutorial Birational Transformation of the Plane of Order 17," AM. JOUR. OF MATH., Vol. XXXIII, pp. 327–336.

the correspondence is such that the image in Σ' of a line in Σ is a curve C_n' of order n, and the image in Σ of a line in Σ' is a curve C_n also of order n, the correspondence or transformation is said to be of the n-th order. The rational character of the transformation from Σ to Σ' insures that the curves of order n in Σ', which are images of lines in Σ, shall be unicursal, while in the transformation from Σ' to Σ the genus is not preserved. If the genus of the curves C_n corresponding to lines in Σ' is p, then p is said to be the genus of the transformation.

Furthermore, the fact that two points, X and $\bar{X}$, in Σ correspond to the same point Y in Σ' establishes an involutorial correspondence between these two points. So that, connected with every (2, 1) correspondence between the two planes Σ and Σ', there is an involution T_{11} in the plane Σ. We shall speak of the two points X and $\bar{X}$, which are the images in Σ of the same point Y in Σ', as conjugate points. The image in Σ of a curve in Σ' may consist of a single curve, self-conjugate, that is, bearing both sets of points X and $\bar{X}$; or may consist of two curves conjugate to each other, one being the locus of X and the other the locus of $\bar{X}$.

In every (2, 1) correspondence there will be a curve Ω in Σ which is the curve of coincident points; it is the locus of points wherein X and $\bar{X}$ coincide. The corresponding points Y in Σ' will also lie on a curve Ω', the image of Ω. We shall call it the limit curve. The points of Ω and Ω' are in a one-to-one correspondence, and the curves are birationally equivalent and, therefore, of the same genus. It may happen that besides this curve Ω there are isolated coincident points, but no such points will appear in the present work.

It will be useful in studying the curves Ω and Ω' to think of the point pairs X and $\bar{X}$ on Ω as approaching coincidence. As the conjugate pairs approach coincidence on Ω, every pair does so in a definite direction, which is called the principal direction for the given point on Ω. A self-conjugate curve will cut Ω in the principal direction, and so corresponding to such an intersection the image in Σ' would make a simple passage across Ω'; but if a curve in Σ, not self-conjugate, should cross Ω in a direction not the principal direction, then its image in Σ' would touch Ω' at the corresponding point. In particular, a random line in Σ, which cuts Ω ω times (ω being the order of Ω), will have an image in Σ' which touches Ω' ω times. This can be seen as follows: Such a line or curve cuts the curve Ω regarded as two curves Ω and $\bar{\Omega}$, just becoming coincident, in two points that are not conjugate, because of which the image in Σ'

must contain the two points on Ω' corresponding to these two non-conjugate points in Σ. That is, the image in Σ' must be tangent to Ω'. On the other hand, the image in Σ of a curve C' in Σ' which touches Ω' at a certain point, must have a double point at the corresponding point on Ω. For, by considering the part of the curve C' near the point of tangency as cutting Ω' in two points which are approaching coincidence, we have the image of C' in Σ cutting Ω in two directions at two points that are approaching coincidence. So that in the limit we have C' tangent to Ω' while its image goes through the corresponding point on Ω twice. This double point may either be an ordinary double point on a self-conjugate curve in Σ or be an intersection of the two conjugate curves that compose the image of C'. If a non-self-conjugate curve in Σ should cut Ω in the principal direction, its image in Σ' would have a cusp at the corresponding point on Σ', while if such a curve has contact of any order with Ω at a given point, the image in Σ' must have contact of twice that order at the corresponding point on Ω'. Again, if a curve has three-fold contact with the limit curve Ω', the corresponding curve in Σ has a cusp on Ω with the tangent to Ω as the cuspidal tangent.*

This correspondence between the simple plane and the double plane requires that the system of curves C_n in Σ corresponding to the doubly infinite system of lines in Σ' shall form a net. Two points in Σ' determine a line; then two pairs of conjugate points in Σ must determine the corresponding C_n. But since, in general, the C_n will be a self-conjugate curve, two points, one of each pair of conjugate points, will insure that the C_n goes through all four and so fix the curve. Also, there must be only two variable intersections for two curves C_n, the two conjugate points corresponding to the intersection of the two lines in Σ'. So that the intersections of the two curves C_n at the fundamental points must total $n^2 - 2$, and we have the formula

$$\sum_{i=1}^{n-1} \alpha_i i^2 = n^2 - 2, \tag{1}$$

in which α indicates the number of i-fold fundamental points and the summation includes all possible orders of multiplicity.

It would be impossible for every curve of the net to have a double point outside of the fundamental points, for, if so, all the curves corresponding to a certain pencil of lines in Σ' would have the same double point and we would get

* For these and other theorems, see reference to de Paolis on p. 291.

four variable intersections, in contradiction to formula (1). Hence the maximum number of double points possessed by all curves of the net will be that to which the multiplicities of the fundamental points are equivalent. We have, therefore, the following formula for the genus of the transformation:

$$p = \frac{(n-1)(n-2)}{2} - \sum_{i=1}^{n-1} \frac{i(i-1)}{2} \alpha_i. \tag{2}$$

Let us see how these formulas (1) and (2) affect the determination of a curve of the system. By subtracting (2) from (1), multiplying the result by 2 and adding to (2), we get the result

$$\sum_{i=1}^{n-1} \alpha_i \frac{i(i+1)}{2} = \frac{n(n+3)}{2} - 3 + p. \tag{3}$$

From this we see that when $p=0$, the fundamental system determines a web of curves to which the net considered belongs; that when $p=1$, the fundamental points are exactly snfficient to fix the net; but that when $p>1$ a net is not established by the fundamental system, and can be gotten only by imposing extra conditions upon these points. Very little has been done in (2, 1) correspondences for genus greater than one.

A number of other formulas will be used in the work to follow. Two straight lines in Σ will each have an image in the involutorial transformation T_{11} of order N, which is connected with the given (2, 1) transformation. Each line cuts the image of the other line in N points, which $2N$ points must make up N pairs of conjugate points. The intersections, then, of the images of these two lines in Σ' in the transformation T_{21} must be made up of the intersections at the fundamental points in Σ', of N intersections at points corresponding to the N pairs mentioned above, and of one intersection corresponding to the intersection of the two lines in Σ. That is,

$$n^2 = \sum_{j=1}^{n-1} \alpha'_j j^2 + N + 1, \tag{4}$$

α'_j being the number of j-fold fundamental points in Σ'.

Again, denoting by δ the class* of the involution T_{11}, that is, the number of pairs of conjugate points on an arbitrary line in Σ, we know that every curve C'_n, the image in Σ' of a line in Σ, must have δ double points outside of the

* Caporali: "Sulle trasformazioni univoche piane involutorie," *Rendiconti dell' accademia di Napoli*, 1879, pp. 212–219. Also: "Memorie di Geometria," Napoli, 1888, pp. 116–125.

fundamental points. Therefore, since the curves C_n' are rational, we have for class of T_{11} the formula

$$\delta = \frac{(n-1)(n-2)}{2} - \sum_{j=1}^{n-1} \frac{j(j-1)}{2} \alpha_j'. \tag{5}$$

Further, a line in Σ must cut its image in the involution T_{11} in N points. Evidently these N points must be made up of coincident points; that is, intersections with the curve of coincident points Ω and the points of the δ pairs of conjugate points lying on the given line. Whence, denoting the order of Ω by ω, we have the relation

$$N = \omega + 2\delta. \tag{6}$$

By means of these formulas it is seen that there are just enough conditions on the curves C_n' in Σ' to establish the net. The fundamental system imposes $\sum_{j=1}^{n-1} \alpha_j' \frac{j(j+1)}{2}$ conditions, the ω tangencies of C_n' with Ω' impose ω conditions, and the δ double points impose δ conditions. Thus in all there are $\sum \alpha_j' \frac{j(j+1)}{2} + \omega + \delta$ conditions. By means of (4), (5) and (6) this expression reduces to

$$\frac{n(n+3)}{2} - 2.$$

It is also to be noticed that the curve of coincident points Ω is a part of the Jacobian of the net of curves C_n. The curves C_n corresponding to lines of a pencil with center P' on Ω', being self-conjugate in general, will pass through Ω at the point P corresponding to P' on Ω' in the principal direction; in particular, the curve C_n corresponding to the tangent to Ω' will have a double point on Ω. The fact that Ω is a part of the Jacobian of the net of C_n's, as well as the character of the rest of the Jacobian, may be seen in another way. Eliminating N from (4) and (6), and eliminating $\Sigma j^2 \alpha_j'$ and δ from the resulting equation and (5),

$$\sum_{j=1}^{n-1} j\alpha_j' + \omega = 3(n-1).$$

But $\Sigma j \alpha_j'$ represents the totality of fundamental curves in Σ corresponding to fundamental points in Σ', and ω is the order of Ω and $3(n-1)$ is the order of the Jacobian of the net of C_n's. Therefore, the Jacobian of the net of C_n's in Σ is made up of the curve of coincident points and the system of fundamental curves in T_{21}.

The order of the limit curve Ω' in Σ' may now be derived. The Jacobian has a $(3i-1)$-fold point at an i-fold fundamental point.* Hence, a C_n will cut the Jacobian outside the fundamental points in the following number of points:

$$3(n-1)n - \sum_{i=1}^{n-1}(3i-1)i.\alpha_i.$$

If now we solve equations (1) and (2) for $\Sigma\alpha_i i$ and substitute its value, and also the value of $\Sigma\alpha_i i^2$ from (1) in the expression just obtained, we find the number of intersections of C_n with the Jacobian outside of the fundamental points to be

$$2(p+1).$$

But we know here, as in the case of the (1, 1) correspondence, that the curve C_n can not cut the fundamental curves outside of the fundamental points; whence, it follows that C_n must cut Ω in these $2(p+1)$ points. This means that in the involution of conjugate points on a self-conjugate C_n there are $2(p+1)$ double points. But, further, since to every one of these $2(p+1)$ intersections there corresponds in Σ' an intersection of Ω' and the line corresponding to C_n, and since there can evidently be no other intersections of the line and Ω', the order of Ω' must be $2(p+1)$.

II. *Synthetic Procedure.*

Take a curve of the n-th order, C_n, with an $(n-2)$-fold point at the point 1_{n-2}; a curve of the $(n-1)$-th order, C_{n-1}, with an $(n-3)$-fold point at 1_{n-2}; and two lines, l_1 and l_2, through the same point, 1_{n-2}. Then the equation

$$\lambda C_n + \mu l_1 C_{n-1} + \nu l_2 C_{n-1} = 0 \qquad (1)$$

represents a system of curves of order n, having an $(n-2)$-fold point in common, and passing through all the other intersections of C_n and C_{n-1}. The basis points then consist of one $(n-2)$-fold point, and of simple points to the number

$$n(n-1) - (n-2)(n-3) = 4n-6.$$

The conditions imposed on a curve C_n of the system at the fundamental points amount to

$$\tfrac{1}{2}(n-2)(n-1) + 4n - 6 = \tfrac{1}{2}(n^2+5n-10),$$

and the difference between this number and the total number of conditions completely determining a curve of order n is

$$\tfrac{1}{2}(n^2+5n-10) - \tfrac{1}{2}n(n+3) = n-5.$$

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* Nichols: "On Certain Jacobians," *Math. Rev.*, Vol. I, No. 1, p. 67.

In particular, if $n=2$, equation (1) will represent a web of conics; if $n=3$, a net of cubics. But, in general, when $n>3$, in order to have the degrees of freedom represented by the parameters in the equation, $n-3$ of the $4n-6$ simple fundamental points must be determined by the others. That is to say, only $3n-3$ of the simple fundamental points can be taken arbitrarily. This agrees with the statements made previously that when, in the system of curves in the simple plane of the (2, 1) correspondence, $p>1$, extra conditions must be imposed upon the points of the fundamental system; for, assuming that the curves of order n have no multiple points other than the $(n-2)$-fold point, their genus is

$$\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}(n-2)(n-3)=n-2,$$

and when $n>3$, $p>1$.

The intersections of two C_n's of the special net (1) at the fundamental points amount to $(n-2)^2+4n-6=n^2-2$ simple intersections. It is therefore a non-homoloidal net, each pencil contained in it having two variable basis points. This can be seen by substituting the coördinates of some point in (1) and eliminating one of the parameters λ, μ, or ν, by means of the resulting equation. The result is a pencil of curves of order n through n^2-1 points. But such a pencil has n^2 points in common; hence, the choice of the first point of the pencil determines a second point. There is, then, connected with the net (1) an involution of points throughout the plane. These pairs of points may be considered the images of single points in another plane.

The net of curves of order n, (1), *generates an involution of points in the plane and establishes a* (2,1) *correspondence between the points of its own and another plane.*

To find the order N of the involution T_{11} consider two arbitrary lines of the plane and two pencils of curves C_n. Through any point X of the first line pass a curve of the first pencil. This curve cuts the second line in n points. Through each one of these n points pass a curve of the second pencil. Each of these n curves cuts the first line in n points. Associate these n^2 intersections with the original point X. In this way an (n^2, n^2) correspondence is established between the points of the first line, and in this correspondence there are $n^2+n^2=2n^2$ coincidences. Of these $2n^2$ coincidences, n^2 are the n intersections of the C_n common to the two pencils, each point counted n times; 1 is the intersection of the two given lines; $(n-1)^2$ are the $n-1$ intersections of the C_{n-1} of the net, each counted $n-1$ times, since any curve of the net having a point

on C_{n-1} must have C_{n-1} as a part. There remain

$$2n^2 - [n^2 + 1 + (n-1)^2] = 2n - 2$$

coincidences. There are therefore $2n-2$ points on each of the two lines such that two curves, one from each of the two pencils, intersecting at one of these points on the first line will intersect again at one of the points on the second line. That is, there are $2n-2$ points on the first line whose conjugate points in the involution lie on the second line; which amounts to saying that the second line cuts the image of the first line in $2n-2$ points. *Therefore the order of the involution* T_{11} *is*

$$N = 2n - 2.$$

The fundamental system of the involution T_{11} may be investigated by the same methods. In order to find the multiplicity of the point 1_{n-2} on the C_{2n-2} which is the image of a line in T_{11}, take again two pencils of the net. Any curve of the first pencil will cut the line in n points. A curve of the second pencil through any one of these points will have $n-2$ tangents at 1_{n-2}. The n curves of the second pencil each passing through one of the intersections of the curve of the first pencil with the given line will together have $n(n-2)$ tangents at 1_{n-2}. By associating one of the tangents at 1_{n-2} of the curve of the first pencil with these n^2-2n tangents to curves of the second pencil a (n^2-2n, n^2-2n) correspondence is formed among the rays of the pencil on 1_{n-2}. This correspondence has $2n^2-4n$ coincidences. The C_n common to the two pencils will furnish $n(n-2)$ of these coincidences, and the C_{n-1} of the net will furnish $(n-1)(n-3)$. There are left

$$2n^2 - 4n - [n(n-2) + (n-1)(n-3)] = 2n - 3$$

coincidences. This means that there are $2n-3$ points on the given arbitrary line each having the property that the curves of the two pencils passing through it have a common tangent at 1_{n-2}. That is, these two curves have an additional intersection at 1_{n-2} and this additional point of intersection is the conjugate point in T_{11} to the point on the given line common to the two curves. Therefore the image of the line in T_{11} has a $(2n-3)$-fold point at 1_{n-2} and the fundamental curve corresponding to the point 1_{n-2} is a curve of order $2n-3$.

The multiplicity of the $4n-6$ simple fundamental points may be found in the same way. At one of these points the one tangent to a curve of the first pencil may be associated with the n tangents of the n curves of the second pencil and thus a (n, n) correspondence be established in the pencil of rays at the given

point. This correspondence will have $2n$ coincidences, of which n are due to the C_n common to the two pencils and $n-1$ to the C_{n-1} of the net. There is one other coincidence, whence it follows that there is only one point on the line whose conjugate point is at the simple fundamental point. The $4n-6$ simple fundamental points are therefore simple points on the C_{2n-2} corresponding in T_{11} to an arbitrary line, and the fundamental curves corresponding to these fundamental points are straight lines.

The system for fundamental curves for T_{11} consists of a C_{2n-3} and $4n-6$ lines. That this is the whole system is shown by the fact that the Jacobian, which comprises the totality of fundamental curves, is of order

$$3(2n-2-1) = 6n-9 = (2n-3)+(4n-6).$$

The multiplicity of the point 1_{n-2} on its fundamental curve C_{2n-3} may be determined by taking a line through 1_{n-2}. Then a C_n of the first pencil cuts the line in two points outside of 1_{n-2}. Through these two points there go two curves of the second pencil, having $2n-4$ tangents at 1_{n-2}. A $(2n-4, 2n-4)$ correspondence is thus set up in the pencil of rays at 1_{n-2} having $4n-8$ coincidences. Of these, $2n-4$ are due to the C_n common to the two pencils and $2n-6$ to the C_{n-1} of the net; and 1 to the given line, since the C_n's of the two pencils tangent to the given line at 1_{n-2} furnish a coincidence. There remains

$$4n-8-[(2n-4)+(2n-6)+1] = 1.$$

That is, there is one point on the given line outside of 1_{n-2} conjugate to 1_{n-2}. 1_{n-2} must therefore be a $(2n-4)$-fold point on its fundamental curve C_{2n-3}.

If the line is taken through one of the simple fundamental points, it will determine the multiplicity of these points on the fundamental curve C_{2n-3}. Take a line through 1_1. A C_n of the first pencil cuts this line in $n-1$ points outside of 1_1. Through every one of these $n-1$ points can be passed a C_n of the other pencil, and these $n-1$ C_n's will have altogether $(n-1)(n-2)$ tangents at 1_{n-2}. In this way a $[(n-1)(n-2), (n-1)(n-2)]$ correspondence is formed in the pencil of rays at 1_{n-2}, with $2(n-1)(n-2)$ coincidences. Subtracting $(n-1)(n-2)$ for the C_n common to the two pencils, and $(n-2)(n-3)$ for the C_{n-1} of the net, there are left

$$2(n-1)(n-2)-(n-1)(n-2)-(n-2)(n-3) = 2n-4.$$

Therefore, there are $2n-4$ points on the given line outside of 1_1 whose conjugate points are at 1_{n-2}, these points being intersections of the line with the fundamental curve C_{2n-3}. Hence, 1_1 and all the other $4n-6$ simple fundamental

points are simple points on C_{2n-3}. This result was to be expected, since a line through 1_{n-2} can not cut C_{2n-3} in more than one point apart from 1_{n-2}.

In order to find whether the fundamental lines corresponding to the $4n-6$ simple fundamental points pass through 1_{n-2}, pass a line through 1_{n-2}. A C_n of the first pencil will cut this line in two points outside of 1_{n-2}, whence a (2, 2) correspondence will be established in the pencil of rays at one of the simple fundamental points. There are then four coincidences, two of which are due to the C_n common to the two pencils, and two to the C_{n-1} of the net. There are no others, and hence there is no point on the line through 1_{n-2} outside of 1_{n-2} whose conjugate is at the simple fundamental point in question. Therefore, the $4n-6$ simple fundamental lines corresponding to the $4n-6$ simple points all pass through 1_{n-2}.

By taking the line through one of the simple fundamental points, it is seen that a fundamental line passes through its corresponding point. The line so taken through 1_1 is cut by a C_n of the first pencil in $n-1$ points outside of the fundamental point 1_1. There is then a $[(n-1), (n-1)]$ correspondence in the pencil of rays at 1_1, which has $2n-2$ coincidences. These are completely accounted for by the $n-1$ due to the C_n common to the two pencils, the $n-2$ due to the C_{n-1} of the net and 1 due to the given line itself. Therefore, the point 1_1 lies on its fundamental line. The $4n-6$ fundamental lines are therefore the $4n-6$ lines joining 1_{n-2} to the corresponding $4n-6$ fundamental points.

The configuration of the fundamental system of the involution T_{11} is now completely determined, and the results are consistent with the requirements of the Jacobian of the C_{2n-2}'s. The point 1_{n-2} is of order

$$3(2n-3)-1=(2n-4)+(4n-6)$$

on the Jacobian; and each of the $4n-6$ simple fundamental points is of order

$$3-1=1+1.$$

This involution of order $2n-2$ having one fundamental point of multiplicity $2n-3$ and $4n-6$ simple fundamental points is the perspective Jonquières.* It is to be noticed that the order of the involution here established for any value of n is even. In the case of $n=3$ the Jonquières of order 4 is a special case of the Geiser in which six of the seven fundamental points lie on a conic.

The Jacobian of the net of C_n's, (1), contains the C_{n-1} as a factor, for any ray of the pencil at 1_{n-2} forms with the C_{n-1} a curve of the net, having double

* See reference on p. 292.

points at the intersection of the line with C_{n-1}. It was shown also in Part I that the curve of coincident points of T_{11} is a factor of the Jacobian of the net in T_{21}. This Jacobian is of order $3(n-1)$, whence the curve of coincident points Ω is of order

$$3(n-1)-(n-1)=2n-2.$$

This result may also be established from the fact that the order of Ω in a perspective Jonquières is always equal to the order of the involution. Or it may be used to prove that the T_{11} is a perspective Jonquières, since in general, if the order of Ω equals the order of the involution, the involution is the persective Jonquières.*

The multiplicity of 1_{n-2} on Ω is equal to the multiplicity of 1_{n-2} on its fundamental curve, since every passage of C_{2n-3} through 1_{n-2} makes a coincidence of conjugate points. Also, since the Jacobian in T_{21} has a $(3(n-2)-1)$-fold point at 1_{n-2} and C_{n-1} has a $(n-3)$-fold point there, it follows that Ω has a $3(n-2)-1-(n-3)=(2n-4)$-fold point. The same considerations prove that Ω passes once through each of the $4n-6$ simple fundamental points. And, further, since a line through 1_{n-2} cuts Ω twice outside of 1_{n-2}, a fundamental line must be tangent to Ω at its corresponding fundamental point.

The class of Ω is found from Plücker's numbers to be

$$n=(2n-2)^2-(2n-2)-2\frac{(2n-4)(2n-5)}{2}=8n-14.$$

Subtracting $2(2n-4)$ for the lines tangent at 1_{n-2}, there remain just $4n-6$ tangents that can be drawn from 1_{n-2} to Ω at outside points. The $4n-6$ fundamental lines therefore make up this set of tangent lines.

Ω has no multiple points other than the one at 1_{n-2}; it is hyperelliptic and is of genus $2n-4$.

The fundamental system in the double plane Σ' may be derived in the following way: Substituting in equation (4), Part I,

$$\Sigma a_j' \cdot j^2=(n-1)^2.$$

Substituting also in equation (6), Part I,

$$\delta=0,$$

which of course follows from the fact that involution T_{11} is a perspective Jonquières and therefore of class zero. Also from equation (5), Part I, and the

* Doehlemann: "Geometrische Transformationen," II. Teil, p. 168.

above,

$$\Sigma j(j-1)\alpha_j' = (n-1)(n-2).$$

Combining these results,

$$\Sigma \alpha_j' \cdot j = n-1.$$

There is therefore in Σ' one fundamental point $1'_{n-1}$ of order $(n-1)$, and this is evidently the point corresponding to the fundamental curve C_{n-1} in Σ.

The limit curve Ω' in Σ' corresponding to Ω is by Part I of order

$$2(p+1) = 2n-2.$$

In the case, then, of a (2, 1) correspondence with an associated Jonquières involution, the orders of the involution, of Ω and Ω' are all the same. Ω' must have a $(2n-4)$-fold point at $1'_{n-1}$, for Ω cuts C_{n-1} at the fundamental points in $(2n-4)(n-3)+(4n-6) = 2n^2-6n+6$ points and $(2n-2)(n-1)-2n^2+6n-6 = 2n-4$. Ω and Ω' then have the same sort of a multiple point, and since neither can have other multiple points, their genus is the same, namely, $(2n-4)$; and their class is the same, $8n-14$. It was shown in Part I that the genus of Ω is always the same as that of Ω', since Ω and Ω' are birationally equivalent.

In order that a line in Σ' may have as image in Σ a curve C_n with an $(n-2)$-fold point at 1_{n-2}, there must be a fundamental curve C'_{n-2} in Σ' corresponding to 1_{n-2}. This C'_{n-2} must have an $(n-3)$-fold point at $1'_{n-1}$, since C_{n-1} has an $(n-3)$-fold point at 1_{n-2}. Again, C'_{n-2} cuts Ω' outside of $1'_{n-1}$ in $(2n-2)(n-2)-(2n-4)(n-3) = 4n-8$ points, and since all these must be points of tangencies, C'_{n-2} touches Ω' outside of $1'_{n-1}$ $2n-4$ times. The curve C_n also passes through $4n-6$ simple fundamental points in Σ, and this requires that there be $4n-6$ fundamental lines in Σ' to correspond. Each of these lines must go through $1'_{n-1}$, since C_{n-1} passes through each of the corresponding points in Σ. Each of these lines must touch Ω' once outside of $1'_{n-1}$, and since, as in the case of Ω, there can be only $4n-6$ tangents from $1'_{n-1}$ to Ω', the fundamental lines make up this set of tangents.

In this transformation the net, a curve of which is generated by a pencil of lines on an arbitrary point in Σ and their image curves in T_{11}, and which then is the locus of conjugate points on the lines of the pencil exclusive of the coincident points on Ω, does not appear, since the class of the involution T_{11} is zero. Also the net, a curve of which is the envelope of the lines joining conjugate points on a self-conjugate C_n, reduces to the point 1_{n-2}. Again, the curve (E)

which is the locus of pairs of conjugate points separated harmonically by two given lines, and which de Paolis finds to be of order $N+1$ and to have an image in Σ' of order n with an $(n-1)$-fold point at $1'_{n-1}$, is here of order $2n-1$ with an image in Σ' of order n, having an $(n-1)$-fold point at $1'_{n-1}$.

An arbitrary line l in Σ is transformed by T_{21} into a curve C'_n of order n with an $(n-1)$-fold point at $1'_{n-1}$, and touching Ω' $2n-2$ times. These $2n-2$ points are the images of the points of intersection of l and Ω, or the points in which l cuts its image in T_{11}, C_{2n-2}. The conditions on such a C'_n are: an $(n-2)$-fold point at a given point and touching a given curve Ω' in $n-2$ points, the points of tangency not being assigned. Since

$$\tfrac{1}{2}(n-1)n+2n-2=\tfrac{1}{2}n(n+3)-2,$$

it follows that there is a double infinity of C'_n's corresponding to the double infinity of lines in Σ. Of the n^2 points of intersection of two C'_n's there are $(n-1)^2$ at $1'_{n-1}$. The remaining $2n-1$ intersections are the images of the $2n-2$ point pairs in Σ which lie, one on the one line and one on the other, and the point of intersection of the two lines.

A line l through 1_{n-2} goes into a C'_n in Σ' which is composed of the fundamental curve C'_{n-2} and a C'_2 made up of a line l' through $1'_{n-1}$ counted twice. The C'_{n-2} touches Ω' $2n-4$ times, and since the line through 1_{n-2} cuts Ω in the principal direction, the image line l' in Σ through $1'_{n-1}$ cuts Ω' twice.

The fundamental line l_1, through 1_{n-2} and its corresponding fundamental point 1_1, goes into the fundamental C'_{n-2} and the corresponding fundamental line l'_1 counted twice. l_1 is tangent to Ω at 1_1, hence l'_1 touches Ω'. There are $4n-6$ such lines.

A line l through the fundamental point 1_1, but not through 1_{n-2}, goes into a C'_n composed of the fundamental line l'_1 and a C'_{n-1} having an $(n-2)$-fold point at $1'_{n-1}$. This C'_{n-1} touches Ω' in $2n-3$ points. When l goes through 1_{n-2} also, we have the former case, the C'_n degenerating into the fundamental C'_{n-2} and the fundamental line l'_1 counted again.

A line l through two simple fundamental points 1_1 and 2_1 goes into a C'_n composed of the fundamental lines l'_1 and l'_2 and a C'_{n-2} with an $(n-3)$-fold point at $1'_{n-1}$ and touching Ω' $2n-4$ times. There are $(2n-3)(4n-7)$ such lines.

The image of Ω_{2n-2} in Σ' is a C'_{2n^2-2n} made up of the fundamental C'_{n-2} counted $2n-4$ times, the $4n-6$ fundamental lines and Ω' of order $(2n-2)$.

An arbitrary line in Σ' has as image in Σ a self-conjugate C_n of the original net. These C_n's cut Ω in $2(p+1) = 2n-2$ points outside of the fundamental points; hence the involution on one of these self-conjugate curves has $2n-2$ self-corresponding points.

But by Plücker's numbers there can be drawn from the point 1_{n-2} to C_n

$$n^2 - n - (n-2)(n-3) - 2(n-2) = 2n-2$$

tangents. These self-corresponding points on C_n are the $2n-2$ points of tangency of the set of tangents from 1_{n-2} to C_n, as well as the intersection of C_n and Ω.

A line l' through the point $1'_{n-1}$ has as image in Σ the fundamental C_{n-1} and a line l through 1_{n-2}. This line l then must be a self-conjugate line, as was already known from the character of the involution, and the two self-corresponding points of the involution on l are the two intersections of l with Ω outside of 1_{n-2}. There is therefore a projective correspondence between the rays of the two pencils on 1_{n-2} and $1'_{n-1}$ in which, as has already been found, the $4n-6$ tangents to Ω correspond to the $4n-6$ tangents to Ω'.

A line l' tangent to Ω' at one point has as image in Σ a C_n with a double point on Ω corresponding to the point of tangency on Ω'. In the case of the $4n-6$ fundamental lines in Σ corresponding to the tangent lines through $1'_{n-1}$, the intersection of the fundamental C_{n-1} and the fundamental line at its associated fundamental point counts as the node.

The number of bitangents to Ω' is $16(n-2)(2n-5)$. The images of these bitangents in Σ have two double points on Ω.

There are also $18(n-2)$ stationary tangents to Ω' each of which has as image in Σ a C_n with a cusp on Ω, the cuspidal tangent being also tangent to Ω.

If there should be a contact line, that is, a line touching Ω'_{2n-2} at all their common points, an $(n-1)$-fold tangent, the corresponding C_n would have $n-1$ double points on Ω. But the genus of C_n is $n-2$; hence the image of a contact line must break up. Suppose its factors are C_s and C_{n-s}, where $n-1 \geqq s \geqq \frac{n}{2}$, n even, or $n-1 \geqq s \geqq \frac{n+1}{2}$, n odd. The conditions necessary to fix these two parts, C_s and C_{n-s}, are

$$\frac{s(s+3)}{2} + \frac{(n-s)(n-s+3)}{2}.$$

This is less than $\frac{1}{2}n(n+3)$, the number of conditions necessary to fix the C_n

in general. But on p. 297 it was found that the conditions imposed upon a self-conjugate C_n of the net are

$$\gtreqless \tfrac{1}{2}n(n+3),$$

according as $n \lesseqgtr 5$. Therefore, when $n \geqq 5$, if the C_n is to break up into two conjugate curves C_s and C_{n-s}, the fundamental points must lie on one or both of these curves to a number greater than that sufficient to fix the part. That is, the possibility of getting $(n-1)$-fold tangents to Ω' depends on the configuration of the fundamental points in Σ.

If a C_n of the net breaks up into a C_{n-1} with an $(n-2)$-fold point at 1_{n-2} and passing through $3n-5$ simple fundamental points, and a C_1 containing the remaining $n-1$ simple fundamental points, then by taking the images of these two curves in Σ' it is found that each has as image besides fundamental curves a straight line, and that further it is the same line for both, and that it is an $(n-1)$-fold tangent to Ω'. Again, if the C_n breaks up into a C_{n-2} with an $(n-3)$-fold point at 1_{n-2} and passing through $3n-7$ simple fundamental points, and a C_2 passing through 1_{n-2} and $n+1$ simple fundamental points, by taking the images of these two curves in Σ' it is found that each has besides fundamental curves a line which is an $(n-1)$-fold tangent to Ω'. These two curves, then, the C_{n-2} and the C_2, make up the image of an $(n-1)$-fold tangent line to Ω'. In the same way it is found that a C_{n-3} with an $(n-4)$-fold point at 1_{n-2} and through $3n-9$ simple fundamental points, and a C_3 with a double point at 1_{n-2} and through $n+3$ simple fundamental points, would be the image of an $(n-1)$-fold tangent line to Ω'. If n is even, the last possibility is when the C_n breaks up into two curves of order $\frac{n}{2}$. Two such curves, each having an $\left(\frac{n}{2}-1\right)$-fold point at 1_{n-2} and each passing through $3n-\left[5+2\left(\frac{n}{2}-1\right)\right]=2n-3$ simple fundamental points, make up the image of an $(n-1)$-fold tangent to Ω'. If n is odd, the last possibility is when the C_n breaks up into a $C_{\frac{n+1}{2}}$ with an $\left(\frac{n+1}{2}-1\right)$-fold point at 1_{n-2}, through $3n-\left[5+2\left(\frac{n-1}{2}-1\right)\right]=2n-2$ simple fundamental points, and a $C_{\frac{n-1}{2}}$ with an $\left(\frac{n-1}{2}-1\right)$-fold point and through $2n-4$ simple fundamental points. These compose the image of an $(n-1)$-fold tangent to Ω'. These results may all be checked by computing the number of intersections with Ω outside of the fundamental points. For

example, in the last possibility, n even, $C_{\frac{n}{2}}$ cuts Ω_{2n-2} outside of the fundamental points in

$$\frac{n}{2}\left(2n-2\right)-\left(\frac{n}{2}-1\right)\left(2n-4\right)-\left(2n-3\right)=n-1$$

points.

Similarly, in every set of possible curves obtained in the above way to make up a degenerate C_n each part will be found to have $n-1$ intersections with Ω outside of the fundamental points. The two parts of the C_n must have these $n-1$ points on Ω in common, since each curve bears one of the conjugate points, and the corresponding line in Σ' must touch Ω' at each of the $n-1$ corresponding points.

Since the C_n of the net is the image of a line in Σ', the image in Σ of a conic in Σ' will be a C_{2n} having a $(2n-4)$-fold point at 1_{n-2} and double points at the $4n-6$ simple fundamental points. If the conic C_2' in Σ' is a contact conic, that is, if the conic touches Ω'_{2n-2} $2n-2$ times, the corresponding C_{2n} must have $2n-2$ double points on Ω. The C_{2n} then has at the fundamental points and on Ω, in all,

$$\tfrac{1}{2}(2n-4)(2n-5)+4n-6+2n-2=\tfrac{1}{2}(2n-1)(2n-2)+1$$

double points, which is one more than a proper C_{2n} can have. A C_{2n}, therefore, which is the image of a contact conic C_2', must break up into two parts, each part bearing one of the conjugate points.

If one of the parts of the degenerate C_{2n} is a line C_1, this line can not pass through 1_{n-2}, for in that case it would be self-conjugate. Suppose that C_1 goes through x simple fundamental points. For the intersections of C_1 with Ω_{2n-2}

$$(2n-2)\,.\,1=x+2n-2;$$

whence $x=0$. The other factor, therefore, of the C_{2n} must be a C_{2n-1} with a $(2n-4)$-fold point at 1_{n-2} and with double points at the $4n-6$ simple fundamental points. But then the C_{2n-1} would have

$$\tfrac{1}{2}(2n-4)(2n-5)+(4n-6)=\tfrac{1}{2}(2n-2)(2n-3)+1$$

double points, or one more than a proper C_{2n-1} can have. It follows that a contact conic can not have an image in Σ one part of which is a straight line.

Next, suppose that the C_{2n} is composed of a C_2 and a C_{2n-2}. If the C_2 does not pass through 1_{n-2} but goes through x simple fundamental points, computing intersections of C_2 and Ω_{2n-2} as above,

$$x=2(2n-2)-(2n-2)=2n-2.$$

The C_{2n-2} must then have a $(2n-4)$-fold point at 1_{n-2}, pass simply through $2n-2$ simple fundamental points and have double points at the remaining $2n-4$. On computing the intersections of C_{2n-2} and Ω_{2n-2} it is found that C_{2n-2} cuts Ω_{2n-2} in

$$(2n-2)^2-[(2n-4)^2+2(2n-4)+(2n-2)]=2n-2$$

points outside of the fundamental points. These $2n-2$ points must be the same as the $2n-2$ intersections of C_2 and Ω and correspond to the $2n-2$ points of tangency of C_2' and Ω'. Counting the double points of C_{2n-2},

$$\tfrac{1}{2}(2n-4)(2n-5)+2n-4=\tfrac{1}{2}(2n-3)(2n-4),$$

which is the maximum number possible for a unicursal C_{2n-2}. These facts are therefore consistent and it is possible for a contact conic in Σ' to be so represented in Σ. This will be found to be the only possibility when one factor of the C_{2n} is a conic.

In general, if one factor of the image of a contact conic is a C_s with an $(s-2)$-fold point at 1_{n-2}, where $2\leqq s\leqq n$, and having simple points at x of the $4n-6$ simple fundamental points, and double points at y of these simple fundamental points, from the intersection of C_s and Ω_{2n-2} it is found that

$$x+2y=s(2n-2)-(s-2)(2n-4)-(2n-2)=2n+2s-6.$$

Since $y\leqq s-2$, $s-2$ being the number of double points which C_s may have outside of 1_{n-2}, $x\leqq 2n-2$. The other factor of the image of the given contact conic must be a C_{2n-s} with a $[(2n-4)-(s-2)]=(2n-s-2)$-fold point at 1_{n-2}, passing through x of the simple fundamental points simply, and through $4n-6-x-y$ doubly. The number of intersections of C_{2n-s} and Ω_{2n-2} outside of the fundamental points is

$$(2n-s)(2n-2)-(2n-s-2)(2n-4)-x-2(4n-6-x-y)$$
$$=x+2y-2s+4=2n-2.$$

The difference between the maximum number of double points possible for C_{2n-s} and the number required at the fundamental points is

$$\tfrac{1}{2}(2n-s-1)(2n-s-2)-\tfrac{1}{2}(2n-s-2)(2n-s-3)-(4n-6-x-y)$$
$$=x+y+4-2n-s=2n+2s-6+4-2n-s-y=s-2-y.$$

But $y\leqq s-2$; therefore, $s-2-y\geqq 0$ and the requirements placed upon the C_{2n-s} in regard to double points are possible. Two curves, therefore, C_s and C_{2n-s}, meeting the above conditions will compose the image of a conic C_2' touching Ω' $2n-2$ times.

When the contact curve in Σ' is unicursal, it can be stated in general that its image in Σ must be a degenerate curve of two factors. Such a curve C_k' has $\frac{1}{2}(k-1)(k-2)$ double points, each one of which has two double points on the image C_{nk} to correspond. The total number of double points on C_{nk} is therefore

$$\tfrac{1}{2}(nk-2k)(nk-2k-1)+\tfrac{1}{2}(k^2-k)(4n-6)+(k-1)(k-2)+k(n-1)$$
$$=\tfrac{1}{2}(nk-1)(nk-2)+1.$$

Hence the C_{nk}, having one more double point than the maximum number possible for a proper C_{nk}, must break up.

III. *Analytic Procedure.*

The curve C_n of the net in Σ having an $(n-2)$-fold point at (0, 1, 0) will have an equation*

$$\Phi_n \equiv x_2^2 \phi_{n-2}(x) + x_2 \phi_{n-1}(x) + \phi_n(x) = 0.$$

The fundamental curve C_{n-1} with an $(n-3)$-fold point at (0, 1, 0) will have as its equation

$$\Psi_{n-1} \equiv x_2^2 \psi_{n-3}(x) + x_2 \psi_{n-2}(x) + \psi_{n-1}(x) = 0.$$

In these equations the functions ϕ and ψ are homogeneous expressions in x_1 and x_3 of order indicated by the subscripts. The two lines through (0, 1, 0) may be taken as

$$x_1 = 0, \qquad x_3 = 0.$$

The net of C_n's may now be represented by

$$\lambda \Phi_n + \mu x_1 \Psi_{n-1} + \nu x_3 \Psi_{n-1} = 0.$$

The Jacobian of this net has the equation

$$\begin{vmatrix} \Phi_1, & x_1\Psi_1 + \Psi, & x_3\Psi_1 \\ \Phi_2, & x_1\Psi_2, & x_3\Psi_2 \\ \Phi_3, & x_1\Psi_3, & x_3\Psi_3 + \Psi \end{vmatrix} = 0,$$

where now the subscripts indicate partial differentiation with respect to the corresponding coordinate. Simplifying,

$$x_1 x_3 \,|\, \Phi_1, \Psi_2, \Psi_3 \,| + x_1 \Psi \,|\, \Phi_1, \Psi_2 \,| + x_3 \Psi \,|\, \Phi_2, \Psi_3 \,| + \Phi_2 \Psi^2$$
$$= \Psi [x_1 \,|\, \Phi_1, \Psi_2 \,| + x_3 \,|\, \Phi_2, \Psi_3 \,| + \Phi_2 \Psi] = 0.$$

The locus of double points for the net of C_n's is therefore made up of Ψ_{n-1} and a curve of order $(2n-2)$, which from Part II is known to be the curve of coin-

* In the work to follow any function $F_k(u_1, u_3)$ will be abbreviated to $F_k(u)$.

cident points, Ω_{2n-2}, for the associated perspective Jonquières involution. The expression for Ω above seems to show that the curve has a $(2n-5)$-fold point at $(0, 1, 0)$, but, on expanding, it is found that the terms containing x_2^3 disappear, whence the curve has a $(2n-4)$-fold point at $(0, 1, 0)$, as is known (from Part II) to be the case for Ω.* Since the Jacobian has a 2-fold point at each of the $4n-6$ simple fundamental points and Ψ_{n-1} has simple points there, it follows that Ω_{2n-2} must go through each of them once.

(a) *Equations of* T_{11}.

Since the character of the part of the Jacobian Ω_{2n-2} is known, it may now be written in the simpler form

$$x_2^2\,\theta_{2n-4}(x) + x_2\,\theta_{2n-3}(x) + \theta_{2n-2}(x) = 0, \tag{1}$$

and from this the equations of transformation for the involution T_{11} may be derived. The equation of any line through the point $O(0, 1, 0)$ may be written $x_1 + k x_3 = 0$. Substituting the coordinates of a point on this line, $P(a, b, c)$, and eliminating k, the equation becomes

$$x_1 = x_3 \frac{a}{c}. \tag{2}$$

Now the image of P, $\overline{P}(\bar{a}, \bar{b}, \bar{c})$, in T_{11} is the harmonic conjugate of P with respect to the two intersections of OP and Ω, these two intersections being the self-corresponding points in the involution on the line OP. To find these intersections substitute the value of x_1 from (2) in (1):

$$x_2^2\,\theta_{2n-4}\left(\frac{a}{c}x_3, x_3\right) + x_2\,\theta_{2n-3}\left(\frac{a}{c}x_3, x_3\right) + \theta_{2n-2}\left(\frac{a}{c}x_3, x_3\right) = 0.$$

Simplifying,

$$x_2^2 c^2\,\theta_{2n-4}(a, c) + x_2 x_3 c\,\theta_{2n-3}(a, c) + x_3^2\,\theta_{2n-2}(a, c) = 0. \tag{3}$$

Let the two roots of this quadratic equation in $\frac{x_2}{x_3}$ be r_1 and r_2 and call the two points common to OP and Ω determined by r_1 and r_2, R and S. Join the points P, R, S and $\overline{P}(\bar{a}, \bar{b}, \bar{c})$, the conjugate point of P, to $B(1, 0, 0)$. Then the equations BF, $B\overline{P}$, BR, BS are respectively

$$c x_2 - b x_3 = 0, \qquad \bar{c} x_2 - \bar{b} x_3 = 0,$$
$$x_2 - r_1 x_3 = 0, \qquad x_2 - r_2 x_3 = 0.$$

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* See reference to Nichols, p. 297.

In order to find the condition that these four lines shall be harmonic, represent the equation of BR as $u = 0$, and the equation of BS as $s = 0$. Then

$$x_2 - r_1 x_3 = u, \qquad x_2 - r_2 x_3 = s,$$

and subtracting,

$$x_3 = \frac{u - s}{r_2 - r_1},$$

and solving for x_2,

$$x_2 = \frac{r_2 u - r_1 s}{r_2 - r_1}.$$

Substituting these values for x_2 and x_3 in the equations of BP and $B\overline{P}$,

$$u = \frac{b - r_1 c}{b - r_2 c} \cdot s, \qquad u = \frac{\bar{b} - r_1 \bar{c}}{\bar{b} - r_2 \bar{c}} \cdot s.$$

These are the equations of BP and $B\overline{P}$ referred to the lines BR and BS. If BP and $B\overline{P}$ are harmonically separated by BR and BS,

$$\frac{b - r_1 c}{b - r_2 c} \cdot \frac{\bar{b} - r_1 \bar{c}}{\bar{b} - r_2 \bar{c}} = -1,$$

whence

$$2 r_1 r_2 c \bar{c} + 2 b \bar{b} - (r_1 + r_2)(b \bar{c} + \bar{b} c) = 0. \tag{4}$$

But from (3),

$$r_1 + r_2 = -\frac{\theta_{2n-3}(a, c)}{c\, \theta_{2n-4}(a, c)}; \qquad r_1 r_2 = \frac{\theta_{2n-2}(a, c)}{c^2\, \theta_{2n-4}(a, c)}.$$

Substituting in (4) and clearing of fractions,

$$\bar{b}\,[2 b c\, \theta_{2n-4}(a, c) + c\, \theta_{2n-3}(a, c)] + \bar{c}\,[2\, \theta_{2n-2}(a, c) + b\, \theta_{2n-3}(a, c)] = 0.$$

This is the condition that $\overline{P}$ be the harmonic conjugate of P with respect to S and R. For the equations of transformation, then, connecting any two conjugate points of the plane, $(a, b, c) \equiv (x_1, x_2, x_3)$ and $(\bar{a}, \bar{b}, \bar{c}) \equiv (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, this last relation gives

$$\bar{x}_2 = 2\, \theta_{2n-2}(x) + x_2\, \theta_{2n-3}(x),$$
$$\bar{x}_3 = -x_3\,[2 x_2\, \theta_{2n-4}(x) + \theta_{2n-3}(x)],$$

the second of which gives, since $\bar{x}_1 = \bar{x}_3 \frac{x_1}{x_3}$,

$$\bar{x}_1 = -x_1\,[2 x_2\, \theta_{2n-4}(x) + \theta_{2n-3}(x)].$$

The image in T_{11} of the line

$$a_1 \bar{x}_1 + a_2 \bar{x}_2 + a_3 \bar{x}_3 = 0$$

is therefore the curve whose equation is

$$-a_1 x_1 [2 x_2 \theta_{2n-4}(x) + \theta_{2n-3}(x)] + 2 a_2 \theta_{2n-2}(x) + a_2 x_2 \theta_{2n-3}(x) - a_3 x_3 [2 x_2 \theta_{2n-4}(x) + \theta_{2n-3}(x)] = 0;$$

or

$$2 x_2 [a_1 x_1 + a_3 x_3] \theta_{2n-4}(x) + [a_1 x_1 - a_2 x_2 + a_3 x_3] \theta_{2n-3}(x) - 2 a_2 \theta_{2n-2}(x) = 0.$$

This is seen to be a curve of order $(2n-2)$ having a $(2n-3)$-fold point at $(0, 1, 0)$.

Arranging the terms in this last equation according to the parameters a_1, a_2 and a_3, the net of C_{2n-2}'s may be written

$$a_1 x_1 \Theta - a_2 \mathrm{X} + a_3 x_3 \Theta = 0,$$

where

$$\Theta \equiv 2 x_2 \theta_{2n-4}(x) + \theta_{2n-3}(x),$$

and

$$\mathrm{X} \equiv 2 \theta_{2n-2}(x) + x_2 \theta_{2n-3}(x).$$

The Jacobian of this net is

$$\begin{vmatrix} x_1 \Theta_1 + \Theta, & -\mathrm{X}_1, & x_3 \Theta_1 \\ x_1 \Theta_2, & -\mathrm{X}_2, & x_3 \Theta_2 \\ x_1 \Theta_3, & -\mathrm{X}_3, & x_3 \Theta_3 + \Theta \end{vmatrix} = 0.$$

Simplifying,

$$\Theta [x_3 \mid \mathrm{X}_2, \Theta_3 \mid + x_1 \mid \Theta_1, \mathrm{X}_2 \mid + \Theta \mathrm{X}_2] = 0.$$

Expanding the expression in brackets,

$$x_3 [\mathrm{X}_2 \Theta_3 - \mathrm{X}_3 \Theta_2] + x_1 [\Theta_1 \mathrm{X}_2 - \Theta_2 \mathrm{X}_1] + \Theta \mathrm{X}_2 = 0.$$

Rearranging,

$$\mathrm{X}_2 [x_1 \Theta_1 + x_3 \Theta_3] - \Theta_2 [x_1 \mathrm{X}_1 + x_3 \mathrm{X}_3] + \Theta \mathrm{X}_2 = 0.$$

From this, by Euler's Theorem,

$$(2n-3) \mathrm{X}_2 \Theta - (2n-2) \Theta_2 \mathrm{X} + \Theta \mathrm{X}_2 = 0,$$

or

$$\Theta \mathrm{X}_2 - \Theta_2 \mathrm{X} = 0.$$

Substituting the values in this last equation,

$$[2 x_2 \theta_{2n-4}(x) + \theta_{2n-3}(x)] \theta_{2n-3}(x) - 2 \theta_{2n-4}(x) [2 \theta_{2n-2}(x) + x_2 \theta_{2n-3}(x)] = 0,$$

whence

$$\theta^2_{2n-3}(x) - 4 \theta_{2n-4}(x) \theta_{2n-2}(x) = 0.$$

This equation is of degree $(4n-6)$ and contains no x_2. It therefore represents $4n-6$ straight lines through the point $(0, 1, 0)$. The Jacobian of the net of

C_{2n-2}'s is therefore a curve of order $(2n-3)$ with the equation

$$\Theta \equiv 2x_2\theta_{2n-4}(x) + \theta_{2n-3}(x) = 0,$$

and $4n-6$ lines through (0, 1, 0) whose equation is

$$\theta^2_{2n-3}(x) - 4\theta_{2n-4}(x)\theta_{2n-2}(x) = 0.$$

$\Theta = 0$ is the fundamental C_{2n-3} corresponding to the point (0, 1, 0), and the $4n-6$ lines are the fundamental lines corresponding to the $4n-6$ simple fundamental points.

To show that the equation

$$\theta^2_{2n-3}(x) - 4\theta_{2n-4}(x)\theta_{2n-2}(x) = 0$$

represents the $4n-6$ tangents from (0, 1, 0) to Ω, proceed by Joachimsthal's method. Substitute

$$x_1 = \lambda . 0 + \mu x_1', \quad x_2 = \lambda . 1 + \mu x_2', \quad x_3 = \lambda . 0 + \mu x_3'$$

in the equation of Ω, (1):

$$(\lambda + \mu x_2')^2\theta_{2n-4}(\mu x_1', \mu x_3') + (\lambda + \mu x_2')\theta_{2n-3}(\mu x_1', \mu x_3') + \theta_{2n-2}(\mu x_1', \mu x_3') = 0.$$

From this,

$$(\lambda + \mu x_2')^2\theta_{2n-4}(x') + (\lambda + \mu x_2')\mu\theta_{2n-3}(x') + \mu^2\theta_{2n-2}(x') = 0,$$

whence

$$\lambda^2\theta_{2n-4}(x') + \lambda\mu[2x_2'\theta_{2n-4}(x') + \theta_{2n-3}(x')] + \mu^2[\theta_{2n-2}(x') + x_1'^2\theta_{2n-4}(x') + x_2'\theta_{2n-3}(x')] = 0.$$

This is a quadratic expression in $\frac{\lambda}{\mu}$, the discriminant of which, placed equal to zero, represents the system of tangent lines from (0, 1, 0) to Ω.

This system of tangent lines has then this equation,

$$[2x_2'\theta_{2n-4}(x') + \theta_{2n-3}(x')]^2 - 4\theta_{2n-4}(x')[\theta_{2n-2}(x') + x_1'^2\theta_{2n-4}(x') + x_2'\theta_{2n-3}(x')] = 0,$$

or

$$\theta^2_{2n-3}(x) - 4\theta_{2n-4}(x)\theta_{2n-2}(x) = 0. \qquad \text{Q. E. D.}$$

(b) *Equations of* T_{21}.

The equations of transformation between the simple plane Σ and the double plane Σ' may be derived from the equation of the net,

$$\lambda\Phi + \mu x_1\Psi + \nu x_3\Psi = 0,$$

by placing

$$y_1 = x_1\Psi, \quad y_2 = \Phi, \quad y_3 = x_3\Psi, \qquad (1')$$

where (y_1, y_2, y_3) represent the coordinates of a point in Σ'. Expanding,

$$y_1 = x_1 x_2^2 \psi_{n-3}(x) + x_1 x_2 \psi_{n-2}(x) + x_1 \psi_{n-1}(x), \tag{1}$$
$$y_2 = x_2^2 \phi_{n-2}(x) + x_2 \phi_{n-1}(x) + \phi_n(x), \tag{2}$$
$$y_3 = x_2^2 x_3 \psi_{n-3}(x) + x_2 x_3 \psi_{n-2}(x) + x_3 \psi_{n-1}(x). \tag{3}$$

Dividing equation (1) by (3) and (2) by (3),

$$\frac{y_1}{y_3} = \frac{x_1}{x_3}, \tag{4}$$

$$\frac{y_2}{y_3} = \frac{x_2^2 \phi_{n-2}(x) + x_2 \phi_{n-1}(x) + \phi_n(x)}{x_3 [x_2^2 \psi_{n-3}(x) + x_2 \psi_{n-2}(x) + \psi_{n-1}(x)]}. \tag{5}$$

Substituting in (5) the value of $\frac{x_1}{x_3}$ from (4),

$$\frac{y_2}{y_3} = \frac{x_2^2 x_3^{n-2} \phi_{n-2}\left(\frac{y_1}{y_3}, 1\right) + x_2 x_3^{n-1} \phi_{n-1}\left(\frac{y_1}{y_3}, 1\right) + x_3^n \phi_n\left(\frac{y_1}{y_3}, 1\right)}{x_3\left[x_2^2 x_3^{n-3} \psi_{n-3}\left(\frac{y_1}{y_3}, 1\right) + x_2 x_3^{n-2} \psi_{n-2}\left(\frac{y_1}{y_3}, 1\right) + x_3^{n-1} \psi_{n-1}\left(\frac{y_1}{y_3}, 1\right)\right]}$$

$$= \frac{x_2^2 y_3^2 \phi_{n-2}(y) + x_2 x_3 y_3 \phi_{n-1}(y) + x_3^2 \phi_n(y)}{y_3 [x_2^2 y_3^2 \psi_{n-3}(y) + x_2 x_3 y_3 \psi_{n-2}(y) + x_3^2 \psi_{n-1}(y)]}.$$

Clearing of fractions,

$$\left.\begin{aligned} x_2^2 [y_3^2 \phi_{n-2}(y) - y_2 y_3^2 \psi_{n-3}(y)] + x_2 x_3 [y_3 \phi_{n-1}(y) - y_2 y_3 \psi_{n-2}(y)] \\ + x_3^2 [\phi_n(y) - y_2 \psi_{n-1}(y)] = 0. \end{aligned}\right\} \tag{6}$$

Solving,

$$\frac{x_2}{x_3} = \frac{-[\phi_{n-1}(y) - y_2 \psi_{n-2}(y)] \pm \sqrt{[\phi_{n-1}(y) - y_2 \psi_{n-2}(y)]^2 - 4[\phi_{n-2}(y) - y_2 \psi_{n-3}(y)][\phi_n(y) - y_2 \psi_{n-1}(y)]}}{2 y_3 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)]}.$$

From (4),

$$\frac{x_1}{x_3} = \frac{y_1}{y_3} = \frac{2 y_1 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)]}{2 y_3 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)]}.$$

The equations of transformation become, therefore,

$$\left.\begin{aligned} x_1 &= 2 y_1 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)], \\ x_2 &= -[\phi_{n-1}(y) - y_2 \psi_{n-2}(y)] \\ &\quad \pm \sqrt{[\phi_{n-1}(y) - y_2 \psi_{n-2}(y)]^2 - 4[\phi_{n-2}(y) - y_2 \psi_{n-3}(y)][\phi_n(y) - y_2 \psi_{n-1}(y)]}, \\ x_3 &= 2 y_3 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)]. \end{aligned}\right\} \tag{7}$$

The image in Σ', then, of an arbitrary line in Σ,

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0,$$

is

$$\begin{aligned} &2 a_1 y_1 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)] - a_2 [\phi_{n-1}(y) - y_2 \psi_{n-2}(y)] \\ &\pm a_2 \sqrt{[\phi_{n-1}(y) - y_2 \psi_{n-2}(y)]^2 - 4 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)][\phi_n(y) - y_2 \psi_{n-1}(y)]} \\ &+ 2 a_3 y_3 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)] = 0. \end{aligned}$$

Rationalizing,

$$\begin{aligned} a_2^2 [\phi_{n-1}(y) - y_2 \psi_{n-2}(y)]^2 &- 4 a_2^2 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)] [\phi_n(y) - y_2 \psi_{n-1}(y)] \\ = 4 a_1^2 y_1^2 [\phi_{n-2}(y) &- y_2 \psi_{n-3}(y)]^2 + a_2^2 [\phi_{n-1}(y) - y_2 \psi_{n-2}(y)]^2 \\ + 4 a_3^2 y_3^2 [\phi_{n-2}(y) &- y_2 \psi_{n-3}(y)]^2 - 4 a_1 a_2 y_1 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)] \\ [\phi_{n-1}(y) &- y_2 \psi_{n-2}(y)] + 8 a_1 a_3 y_1 y_3 [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)]^2 \\ - 4 a_2 a_3 y_3 [\phi_{n-1}(y) &- y_2 \psi_{n-2}(y)] [\phi_{n-2}(y) - y_2 \psi_{n-3}(y)]. \end{aligned}$$

Simplifying,

$$\left.\begin{aligned} &[\phi_{n-2}(y) - y_2 \psi_{n-3}(y)] [(a_1 y_1 + a_3 y_3)^2 (\phi_{n-2}(y) - y_2 \psi_{n-3}(y)) \\ &- a_2 (a_1 y_1 + a_3 y_3)(\phi_{n-1}(y) - y_2 \psi_{n-2}(y)) + a_2^2 (\phi_n(y) - y_2 \psi_{n-1}(y))] = 0. \end{aligned}\right\} \quad (8)$$

But equations (7) show that the first factor,

$$\phi_{n-2}(y) - y_2 \psi_{n-3}(y) = 0,$$

is the fundamental curve C'_{n-2} in Σ' corresponding to the point (0, 1, 0) in Σ. It is of order $(n-2)$ and has an $(n-3)$-fold point at (0, 1, 0).

The image in Σ' of an arbitrary line in Σ,

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0,$$

may be obtained in the following way, in which the fundamental C'_{n-2} does not appear as a factor. Dividing by x_3,

$$a_1 \frac{x_1}{x_3} + a_2 \frac{x_2}{x_3} + a_3 = 0.$$

Substituting $\frac{x_1}{x_3} = \frac{y_1}{y_3}$,

$$a_1 \frac{y_1}{y_3} + a_2 \frac{x_2}{x_3} + a_3 = 0.$$

From this,

$$\frac{x_2}{x_3} = -\frac{a_1 y_1 + a_3 y_3}{a_2 y_3}.$$

Substituting in (6),

$$\frac{(a_1y_1+a_3y_3)^2}{a_2^2y_3^2}\Big[y_3^2\Phi_{n-2}(y)-y_2y_3^2\Psi_{n-3}(y)\Big]-\frac{a_1y_1+a_3y_3}{a_2y_3}\Big[y_3\Phi_{n-1}(y)-y_2y_3\Psi_{n-2}(y)\Big]$$
$$+\Phi_n(y)-y_2\Psi_{n-1}(y)=0.$$

Whence,

$$\left.\begin{aligned}(a_1y_1+a_3y_3)^2[\Phi_{n-2}(y)-y_2\Psi_{n-3}(y)]-a_2(a_1y_1+a_3y_3)[\Phi_{n-1}(y)-y_2\Psi_{n-2}(y)]\\+a_2^2[\Phi_n(y)-y_2\Psi_{n-1}(y)]=0.\end{aligned}\right\}\quad(9)$$

This is the proper image C_n' of an arbitrary line in Σ. It is seen to be a curve of order n, with an $(n-1)$-fold point at (0, 1, 0).

Equations (1′) show that the coordinates of all points in Σ on the fundamental curve Ψ render $y_1=y_3=0$. Hence the image in Σ' of Ψ is the point (0, 1, 0). A line through (0, 1, 0),

$$y_1+ky_3=0,$$

is transformed by formulas (1′) into $x_1\Psi+kx_3\Psi=0$, or into the fundamental Ψ and a line through (0, 1, 0),

$$x_1+kx_3=0.$$

On the other hand, this line through (0, 1, 0) in Σ is transformed by formulas (7) into

$$2y_1[\Phi_{n-2}(y)-y_2\Psi_{n-3}(y)]+2ky_3[\Phi_{n-2}(y)-y_2\Psi_{n-3}(y)]=0;$$

that is, into the fundamental curve and a line through (0, 1, 0),

$$y_1+ky_3=0.$$

This shows the projective relation existing between the two pencils of lines in Σ and Σ'.

The equations of transformation (7) show that in general to a single point in Σ' there correspond two points in Σ; but that when the coordinates of the point in Σ' are such as to reduce the expression under the radical sign in the formula for x_2 to zero, there is only one corresponding point in Σ. The equation of the limit curve Ω' is therefore

$$[\Phi_{n-1}(y)-y_2\Psi_{n-2}(y)]^2-4[\Phi_{n-2}(y)-y_2\Psi_{n-3}(y)][\Phi_n(y)-y_2\Psi_{n-1}(y)]=0.\quad(10)$$

This shows that Ω' is of order $(2n-2)$ and has a $(2n-4)$-fold point at (0, 1, 0).

Comparing equations (9) and (10), it is seen that the left side of the equation of C_n' is a quadratic expression in $(a_1 y_1 + a_3 y_3)$ the discriminant of which is the left side of the equation of Ω'. Regarding (9) and (10) as simultaneous equations, it appears that at each of the points common to C_n' and Ω' there are two equal values for $\frac{y_1}{y_3}$. Therefore, the curves C_n' and Ω' are tangent at all their common points. This same result may be gotten by regarding a_1, a_2, a_3, in (9), as parameters and finding that the envelope of the C_n''s is the curve Ω'.

Joachimsthal's method may again be used to find the equation of the $4n-6$ tangents from (0, 1, 0) to Ω' which, by Part II, are known to be the fundamental lines in Σ' corresponding to the $4n-6$ simple points in Σ. Substitute in (10):

$$y_1 = \lambda \,.\, 0 + \mu y_1'; \quad y_2 = \lambda \,.\, 1 + \mu y_2'; \quad y_3 = \lambda \,.\, 0 + \mu y_3':$$

$$[\Phi_{n-1}(\mu y_1', \mu y_3') - (\lambda + \mu y_2')\Psi_{n-2}(\mu y_1', \mu y_3')]^2 - 4[\Phi_{n-2}(\mu y_1', \mu y_3') - (\lambda + \mu y_2')\Psi_{n-3}(\mu y_1', \mu y_3')]\,[\Phi_n(\mu y_1', \mu y_3') - (\lambda + \mu y_2')\Psi_{n-1}(\mu y_1', \mu y_3')] = 0.$$

Expanding,

$$\begin{aligned}\lambda^2 [\Psi_{n-2}^2(y') - 4\Psi_{n-1}(y')\Psi_{n-3}(y')] + \lambda\mu [&-2\Phi_{n-1}(y')\Psi_{n-2}(y') \\ &+ 4\Phi_{n-2}(y')\Psi_{n-1}(y') + 4\Phi_n(y')\Psi_{n-3}(y') - 8 y_2'\Psi_{n-1}(y')\Psi_{n-3}(y') \\ &+ 2 y_2'\Psi_{n-2}^2(y')] + \mu^2 [\Phi_{n-1}^2(y') - 2 y_2'\Phi_{n-1}(y')\Psi_{n-2}(y') + y_2'\Psi_{n-2}^2(y') \\ &- 4\Phi_n(y')\Phi_{n-2}(y') + 4 y_2'\Phi_{n-2}(y')\Psi_{n-1}(y') + 4 y_2'\Phi_n(y')\Psi_{n-3}(y') \\ &- 4 y_2'^2\Psi_{n-1}(y')\Psi_{n-3}(y')] = 0.\end{aligned}$$

Equating the discriminant of this quadratic expression in $\frac{\lambda}{\mu}$ to zero, we obtain the equation of the $4n-6$ tangent lines from (0, 1, 0) to Ω', which compose the fundamental lines in Σ' corresponding to the simple fundamental points in Σ.

IV. *Surface Depiction.*

Consider the surface F^n whose equation is

$$z_4^2[\Phi_{n-2}(z) - z_2\Psi_{n-3}(z)] + z_4[\Phi_{n-1}(z) - z_2\Psi_{n-2}(z)] + [\Phi_n(z) - z_2\Psi_{n-1}(z)] = 0.$$

This surface has an $(n-1)$-fold point at (0, 1, 0, 0) and an $(n-2)$-fold point at (0, 0, 0, 1); while the line $z_1 = 0$, $z_3 = 0$ is an $(n-3)$-fold line. If the surface

is projected from the point $(0, 1, 0, 0)$ upon the plane $z_2 = 0$, there is a one-to-one correspondence between points of surface and plane.

In order to get the equations of transformation, represent any point on a line through $(0, 1, 0, 0)$ and (z_1, z_2, z_3, z_4) on F^n by the coordinates

$$x_1 = \lambda . 0 + \mu z_1; \quad x_2 = \lambda . 1 + \mu z_2; \quad x_3 = \lambda . 0 + \mu z_3; \quad x_4 = \lambda . 0 + \mu z_4.$$

Let the point in question lie in the plane $x_2 = 0$. Then

$$\lambda + \mu z_2 = 0.$$

The coordinates of the point on the surface F^n therefore, in terms of the coordinates of the point on the plane $x_2 = 0$, are

$$z_1 = \frac{x_1}{\mu}; \quad z_2 = -\frac{\lambda}{\mu}; \quad z_3 = \frac{x_3}{\mu}; \quad z_4 = \frac{x_4}{\mu}. \tag{1}$$

Substituting in the equation of F^n,

$$\frac{x_4^2}{\mu^2}\left[\phi_{n-2}\left(\frac{x_1}{\mu}, \frac{x_3}{\mu}\right) + \frac{\lambda}{\mu}\psi_{n-3}\left(\frac{x_1}{\mu}, \frac{x_3}{\mu}\right)\right] + \frac{x_4}{\mu}\left[\phi_{n-1}\left(\frac{x_1}{\mu}, \frac{x_3}{\mu}\right) + \frac{\lambda}{\mu}\psi_{n-2}\left(\frac{x_1}{\mu}, \frac{x_3}{\mu}\right)\right]$$
$$+ \phi_n\left(\frac{x_1}{\mu}, \frac{x_3}{\mu}\right) + \frac{\lambda}{\mu}\psi_{n-1}\left(\frac{x_1}{\mu}, \frac{x_3}{\mu}\right) = 0,$$

or

$$\frac{x_4^2}{\mu^2}\left[\frac{1}{\mu^{n-2}}\phi_{n-2}(x) + \frac{\lambda}{\mu^{n-2}}\psi_{n-3}(x)\right] + \frac{x_4}{\mu}\left[\frac{1}{\mu^{n-1}}\phi_{n-1}(x) + \frac{\lambda}{\mu^{n-1}}\psi_{n-2}(x)\right]$$
$$+ \frac{1}{\mu^n}\phi_n(x) + \frac{\lambda}{\mu^n}\psi_{n-1}(x) = 0,$$

or

$$x_4^2[\phi_{n-2}(x) + \lambda\psi_{n-3}(x)] + x_4[\phi_{n-1}(x) + \lambda\psi_{n-2}(x)] + \phi_n(x) + \lambda\psi_{n-1}(x) = 0;$$

whence

$$-\lambda = \frac{x_4^2\phi_{n-2}(x) + x_4\phi_{n-1}(x) + \phi_n(x)}{x_4^2\psi_{n-3}(x) + x_4\psi_{n-2}(x) + \psi_{n-1}(x)}.$$

Now, by equations (1),

$$z_1 : z_2 : z_3 : z_4 = \frac{x_1}{\mu} : \frac{-\lambda}{\mu} : \frac{x_3}{\mu} : \frac{x_4}{\mu} = x_1 : -\lambda : x_3 : x_4.$$

Substituting the value of $-\lambda$,

$$z_1 : z_2 : z_3 : z_4 = x_1 : \frac{x_4^2 \phi_{n-2}(x) + x_4 \phi_{n-1}(x) + \phi_n(x)}{x_4^2 \psi_{n-3}(x) + x_4 \psi_{n-2}(x) + \psi_{n-1}(x)} : x_3 : x_4$$

$$= x_1 [x_4^2 \psi_{n-3}(x) + x_4 \psi_{n-2}(x) + \psi_{n-1}(x)] : x_4^2 \phi_{n-2}(x) + x_4 \phi_{n-1}(x) + \phi_n(x)$$
$$: x_3 [x_4^2 \psi_{n-3}(x) + x_4 \psi_{n-2}(x) + \psi_{n-1}(x)]$$
$$: x_4 [x_4^2 \psi_{n-3}(x) + x_4 \psi_{n-2}(x) + \psi_{n-1}(x)].$$

This gives the following formulas, by means of which the image on F^n of a point on the plane $x_2 = 0$ may be obtained:

$$\left.\begin{aligned} z_1 &= x_1 [x_4^2 \psi_{n-3}(x) + x_4 \psi_{n-2}(x) + \psi_{n-1}(x)], \\ z_2 &= x_4^2 \phi_{n-2}(x) + x_4 \phi_{n-1}(x) + \phi_n(x), \\ z_3 &= x_3 [x_4^2 \psi_{n-3}(x) + x_4 \psi_{n-2}(x) + \psi_{n-1}(x)], \\ z_4 &= x_4 [x_4^2 \psi_{n-3}(x) + x_4 \psi_{n-2}(x) + \psi_{n-1}(x)]. \end{aligned}\right\} \quad (2)$$

From these equations or from equations (1) it is seen that

$$z_1 : z_3 : z_4 = x_1 : x_3 : x_4.$$

Hence the formulas fixing on the plane $x_2 = 0$ the image of a point on the surface F^n are

$$x_1 = z_1; \quad x_3 = z_3; \quad x_4 = z_4. \quad (3)$$

Now project the surface F^n from the point $(0, 0, 0, 1)$ upon the plane $z_4 = 0$. Since the center of projection is an $(n-2)$-fold point, there are in general two points on the surface corresponding to one point on the plane. Proceeding in the same way, represent a point on the line through $(0, 0, 0, 1)$ and (z_1, z_2, z_3, z_4) on F^n by

$$y_1 = \lambda . 0 + \mu z_1; \quad y_2 = \lambda . 0 + \mu z_2; \quad y_3 = \lambda . 0 + \mu z_3; \quad y_4 = \lambda . 1 + \mu z_4. \quad (4)$$

Then, if the point (y_1, y_2, y_3, y_4) lies in the plane $y_4 = 0$,

$$\lambda + \mu z_4 = 0. \quad (5)$$

Solving now for z_1, z_2, z_3, z_4 and substituting in the equation of F^n,

$$\frac{\lambda^2}{\mu^2}\left[\frac{1}{\mu^{n-2}} \phi_{n-2}(y) - \frac{y_2}{\mu^{n-2}} \psi_{n-3}(y)\right] - \frac{\lambda}{\mu}\left[\frac{1}{\mu^{n-1}} \phi_{n-1}(y) - \frac{y_2}{\mu^{n-1}} \psi_{n-2}(y)\right]$$
$$+ \frac{1}{\mu^n} \phi_n(y) - \frac{y_2}{\mu^n} \psi_{n-1}(y) = 0.$$

Solving for $\frac{\lambda}{\mu}$, $\frac{\lambda}{\mu} =$

$$\frac{\frac{1}{\mu^{n-1}}\phi_{n-1}(y) - \frac{y_2}{\mu^{n-1}}\psi_{n-2}(y) \pm \sqrt{\left[\frac{1}{\mu^{n-1}}\varphi_{n-1}(y) - \frac{y_2}{\mu^{n-1}}\psi_{n-2}(y)\right]^2 - 4\left[\frac{1}{\mu^{n-2}}\varphi_{n-2}(y) - \frac{y_2}{\mu^{n-2}}\psi_{n-3}(y)\right]\left[\frac{1}{\mu^{n}}\varphi_{n}(y) - \frac{y_2}{\mu^{n}}\psi_{n-1}(y)\right]}}{2\left[\frac{1}{\mu^{n-2}}\phi_{n-2}(y) - \frac{y_2}{\mu^{n-2}}\psi_{n-3}(y)\right]},$$

or

$$\frac{\lambda}{\mu} = \frac{\phi_{n-1}(y) - y_2\psi_{n-2}(y) \pm \sqrt{[\phi_{n-1}(y) - y_2\psi_{n-2}(y)]^2 - 4[\phi_{n-2}(y) - y_2\psi_{n-3}(y)][\phi_n(y) - y_2\psi_{n-1}(y)]}}{2\mu[\phi_{n-2}(y) - y_2\psi_{n-3}(y)]}.$$

This, in connection with (4) and (5), gives the ratios

$$z_1 : z_2 : z_3 : z_4 = \frac{y_1}{\mu} : \frac{y_2}{\mu} : \frac{y_3}{\mu}$$

$$: - \frac{\phi_{n-1}(y) - y_2\psi_{n-2}(y) \pm \sqrt{[\phi_{n-1}(y) - y_2\psi_{n-2}(y)]^2 - 4[\phi_{n-2}(y) - y_2\psi_{n-3}(y)][\phi_n(y) - y_2\psi_{n-1}(y)]}}{2\mu[\phi_{n-2}(y) - y_2\psi_{n-3}(y)]};$$

whence

$$z_1 : z_2 : z_3 : z_4 = 2y_1[\phi_{n-2}(y) - y_2\psi_{n-3}(y)] : 2y_2[\phi_{n-2}(y) - y_2\psi_{n-3}(y)]$$
$$: 2y_3[\phi_{n-2}(y) - y_2\psi_{n-3}(y)] : -[\phi_{n-1}(y) - y_2\psi_{n-2}(y)]$$
$$\pm \sqrt{[\phi_{n-1}(y) - y_2\psi_{n-2}(y)]^2 - 4[\phi_{n-2}(y) - y_2\psi_{n-3}(y)][\phi_n(y) - y_2\psi_{n-1}(y)]}.$$

These ratios give the following formulas for finding the two points on F^n corresponding to a single point on the plane $x_4 = 0$:

$$\left.\begin{aligned} z_1 &= 2y_1[\phi_{n-2}(y) - y_2\psi_{n-3}(y)], \\ z_2 &= 2y_2[\phi_{n-2}(y) - y_2\psi_{n-3}(y)], \\ z_3 &= 2y_3[\phi_{n-2}(y) - y_2\psi_{n-3}(y)], \\ z_4 &= -[\phi_{n-1}(y) - y_2\psi_{n-2}(y)] \\ &\quad \pm \sqrt{[\phi_{n-1}(y) - y_2\psi_{n-2}(y)]^2 - 4[\phi_{n-2}(y) - y_2\psi_{n-3}(y)][\phi_n(y) - y_2\psi_{n-1}(y)]}. \end{aligned}\right\} \quad (6)$$

From these equations, or from equations (4), it is seen that

$$z_1 : z_2 : z_3 = y_1 : y_2 : y_3;$$

from which are derived the equations for passing from a point on F^n to its image on the plane $z_4 = 0$,

$$y_1 = z_1; \quad y_2 = z_2; \quad y_3 = z_3.$$

This surface depiction upon the two planes $z_2 = 0$ and $z_4 = 0$ is now seen to connect the planes in a (2, 1) correspondence, for a point on the plane $z_4 = 0$ or Σ' is projected from (0, 0, 0, 1) into two points on F^n and these two points are projected from (0, 1, 0, 0) into two points on the plane $z_2 = 0$ or Σ. Proceeding in the opposite direction, a point in Σ is projected into a single point on F^n and this point is projected into a single point on Σ' which, however, is at the same time the image of another point on F^n and another point on Σ. By a linear transformation any two planes may be related in the (2, 1) correspondence by means of F^n. An examination of the equations obtained will show that the correspondence established between the two planes is precisely the one under consideration.

The section of the surface F^n made by the simple plane, $z_2 = 0$, is

$$z_4^2 \phi_{n-2}(z) + z_4 \phi_{n-1}(z) + \phi_n(z) = 0,$$

or

$$x_4^2 \phi_{n-2}(x) + x_4 \phi_{n-1}(x) + \phi_n(x) = 0.$$

This is the curve of the net Φ_n, the coordinates in the simple plane being now (x_1, x_3, x_4) and the $(n-2)$-fold point being at (0, 0, 1) or (0, 0, 0, 1). The fundamental curve Ψ_{n-1} would now have the equation

$$x_4^2 \psi_{n-3}(x) + x_4 \psi_{n-2}(x) + \psi_{n-1}(x) = 0.$$

But from (2) points satisfying this equation require that $z_1 = z_3 = z_4 = 0$ on F^n. Hence the image on F^n of Ψ_{n-1} in Σ is the center of projection (0, 1, 0, 0), which is to say that Ψ_{n-1} is the section made by Σ of the tangent cone at the point (0, 1, 0, 0).

The section of F^n by the double plane, $z_4 = 0$, is

$$\phi_n(z) - z_2 \psi_{n-1}(z) = 0,$$

or

$$\phi_n(y) - y_2 \psi_{n-1}(y) = 0.$$

This is one of the curves C_n' of the net in Σ' having an $(n-1)$-fold point at (0, 1, 0) or (0, 1, 0, 0). Equations (6) show that if

$$\phi_{n-2}(y) - y_2 \psi_{n-3}(y) = 0,$$

$z_1 = z_2 = z_3 = 0$. Therefore, the image of this curve, which is the fundamental

C'_{n-2} having an $(n-3)$-fold point at $(0, 1, 0, 0)$, is the point $(0, 0, 0, 1)$; that is, the fundamental C'_{n-2} is the section made by the plane Σ of the cone tangent to F^n at the point $(0, 0, 0, 1)$. Again, from equation (6) it is seen that when the coordinates of a point in Σ' satisfy the equation of Ω',

$$[\phi_{n-1}(y) - y_2\psi_{n-2}(y)]^2 - 4[\phi_{n-2}(y) - y_2\psi_{n-3}(y)][\phi_n(y) - y_2\psi_{n-1}(y)] = 0,$$

then there is only one point corresponding to it on F^n; in which case the projecting line from $(0, 0, 0, 1)$ must be tangent to F^n. Therefore, the image on F^n of the limit curve Ω' in Σ' is the curve of contact of the tangent cone to F^n from the point $(0, 0, 0, 1)$.

An arbitrary line l in Σ is projected into a plane section K_1 of F^n having an $(n-1)$-fold point at $(0, 1, 0, 0)$. The corresponding curve in Σ', C'_n, is projected from $(0, 0, 0, 1)$ into this plane section K_1 and another curve K_2 which is the image on F^n of the curve in Σ, C_{2n-2}, conjugate to l in T_{11}. The $2n-2$ intersections of l and C_{2n-2} on Ω correspond to the $2n-2$ intersections of K_1 and K_2 on the contact curve of the tangent cone from $(0, 0, 0, 1)$, which points, in turn, correspond to the $2n-2$ points of tangency of C'_n with Ω'.

The pencil of planes on the line $z_1 = 0$, $z_3 = 0$ cuts out the two projective pencils of lines in Σ and Σ', these projective pencils of lines being here in perspective position. Since the sets of conjugate points in Σ are collinear with $(0, 0, 0, 1)$, the pairs of conjugate points on F^n are co-planar with the line $z_1 = 0$, $z_3 = 0$, and collinear with $(0, 0, 0, 1)$. The line $z_1 = 0$, $z_3 = 0$ is an $(n-3)$-fold line on F^n; hence the pencil of planes cuts out of F^n a system of rational cubics, each having a double point at $(0, 1, 0, 0)$ and a simple point at $(0, 0, 0, 1)$ and each being self-conjugate, with the pairs of conjugate points collinear with $(0, 0, 0, 1)$.

To obtain the equation of a cubic of this system take a plane of the pencil

$$z_3 = kz_1.$$

Substitute in the equation of F^n:

$$z_4^2[\phi_{n-2}(z_1, kz_1) - z_2\psi_{n-3}(z_1, kz_1)] + z_4[\phi_{n-1}(z_1, kz_1) - z_2\psi_{n-2}(z_1, kz_1)] + \phi_n(z_1, kz_1) - z_2\psi_{n-1}(z_1, kz_1) = 0;$$

whence

$$z_1^{n-3}z_4^2[z_1\phi_{n-2}(k) - z_2\psi_{n-3}(k)] + z_1^{n-2}z_4[z_1\phi_{n-1}(k) - z_2\psi_{n-2}(k)] + z_1^n\phi_n(k) - z_1^{n-1}z_2\psi_{n-1}(k) = 0.$$

Discarding the part of the intersection composed of the $n-3$ straight lines, the residue is

$$\phi_n(k)\,z_1^3 - \psi_{n-1}(k)\,z_1^2 z_2 + \phi_{n-1}(k)\,z_1^2 z_4 - \psi_{n-2}(k)\,z_1 z_2 z_4 + \phi_{n-2}(k)\,z_1 z_4^2 - \psi_{n-3}(k)\,z_2 z_4^2 = 0.$$

This cubic, cut out of F^n by the plane of the pencil determined by k, is seen to have a simple point at $(0, 0, 0, 1)$ and a double point at $(0, 1, 0, 0)$.

The $4n-6$ planes of the pencil containing the $4n-6$ simple fundamental lines in Σ and their images in Σ' are the planes through the line $z_1 = 0$, $z_3 = 0$ tangent to the cone which is tangent to F^n from $(0, 0, 0, 1)$.

If the equations of the contact cone from $(0, 0, 0, 1)$ and F^n are treated simultaneously and the coordinate z_2 is eliminated, the equation of the cone through the contact curve and with vertex at $(0, 1, 0, 0)$ is obtained. From the equation of F^n,

$$z_2 = \frac{z_4^2\phi_{n-2}(z) + z_4\phi_{n-1}(z) + \phi_n(z)}{z_4^2\psi_{n-3}(z) + z_4\psi_{n-2}(z) + \psi_{n-1}(z)} = \frac{\Phi(z_1, z_4, z_3)}{\Psi(z_1, z_4, z_3)}.$$

Substituting in the equation of the contact cone, whose equation is

$$[\phi_{n-1}(z) - z_2\psi_{n-2}(z)]^2 - 4[\phi_{n-2}(z) - z_2\psi_{n-3}(z)][\phi_n(z) - z_2\psi_{n-1}(z)] = 0,$$

the result is

$$\left.\begin{aligned}&[\phi_{n-1}(z)\,\Psi(z_1, z_4, z_3) - \Phi(z_1, z_4, z_3)\,\psi_{n-2}(z)]^2 \\ &\quad - 4[\phi_{n-2}(z)\,\Psi(z_1, z_4, z_3) - \Phi(z_1, z_4, z_3)\,\psi_{n-3}(z)] \\ &\qquad [\phi_n(z)\,\Psi(z_1, z_4, z_3) - \Phi(z_1, z_4, z_3)\,\psi_{n-1}(z)] = 0.\end{aligned}\right\} \quad (7)$$

Expanding and collecting terms,

$$[z_4^2\phi_{n-1}(z)\,\psi_{n-3}(z) - z_4^2\phi_{n-2}(z)\,\psi_{n-2}(z) + 2z_4\phi_n(z)\,\psi_{n-3}(z) - 2z_4\phi_{n-2}(z)\,\psi_{n-1}(z) + \phi_n(z)\,\psi_{n-2}(z) - \phi_{n-1}(z)\,\psi_{n-1}(z)]^2 = 0.$$

This is the equation of the cone standing on the contact curve of the cone from $(0, 0, 0, 1)$ and having its vertex at $(0, 1, 0, 0)$. As is to be expected for the contact curve the equation of the projecting cone appears squared. Substituting x's for z's according to formulas (3), the equation of Ω squared is obtained. The equation of Ω is therefore:

$$\left.\begin{aligned}&x_4^2[\phi_{n-1}(x)\,\psi_{n-3}(x) - \phi_{n-2}(x)\,\psi_{n-2}(x)] \\ &\quad + 2x_4[\phi_n(x)\,\psi_{n-3}(x) - \phi_{n-2}(x)\,\psi_{n-1}(x)] \\ &\qquad + [\phi_n(x)\,\psi_{n-2}(x) - \phi_{n-1}(x)\,\psi_{n-1}(x)] = 0.\end{aligned}\right\} \quad (8)$$

This is seen to be a curve of order $2n-2$, with a $(2n-4)$-fold point at $(0, 0, 0, 1)$. This form of the equation for Ω gives the values of the functions θ_{2n-4}, θ_{2n-3}, θ_{2n-2} in terms of the functions ϕ and ψ of the original net of C_n's, thus making it possible to state the formulas of transformation for T_{11} in terms of the same functions appearing in the formulas for T_{21}. That this last form for Ω is what would be gotten by transforming directly from Σ' to Σ appears at once when the formulas

$$y_1 = x_1\Psi, \qquad y_2 = \Phi, \qquad y_3 = x_3\Psi$$

are substituted in Ω'. The image consists of the fundamental Ψ taken $2n-4$ times and equation (7) in terms of x_1, x_2, x_3. This again reduces to equation (8).

Some Geometrical Theorems Connected with Laplace's Equation and the Equation of Wave Motion.

By H. Bateman.

§ 1.

If A_1, A_2 are two points whose coordinates referred to rectangular axes (OX, OY) are $(a+\rho\cos\theta,\ b+\rho\sin\theta)$, $(a-\rho\cos\theta,\ b-\rho\sin\theta)$ respectively, then the points B_1, B_2 whose coordinates are $(a+i\rho\sin\theta,\ b-i\rho\cos\theta)$, $(a-i\rho\sin\theta,\ b+i\rho\cos\theta)$ respectively are called the *anti-points* of A_1 and A_2.* The lines A_1B_2, A_1B_1, A_2B_1, A_2B_2 are isotropic lines and the four points A_1, A_2, B_1, B_2 may be regarded as the four foci of a conic; if A_1 and A_2 are the real foci, then B_1 and B_2 are the imaginary foci. The lines A_1A_2, B_1B_2 are perpendicular and are bisected at their point of intersection $C(a, b)$.

In the following theorems the quantities a, b, ρ, θ need not be real, but the interpretation of the theorems is facilitated when this is the case.

Theorem I. *If V is a uniform solution of Laplace's equation*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0,$$

the sum of the values of V at A_1 and A_2 is equal to the sum of the values of V at B_1 and B_2.

Let

$$V=f(x+iy)+g(x-iy),$$

then

$$\begin{aligned}
V(A_1)&=f(a+ib+\rho e^{i\theta})+g(a-ib+\rho e^{-i\theta}),\\
V(A_2)&=f(a+ib-\rho e^{i\theta})+g(a-ib-\rho e^{-i\theta}),\\
V(B_1)&=f(a+ib+\rho e^{i\theta})+g(a-ib-\rho e^{-i\theta}),\\
V(B_2)&=f(a+ib-\rho e^{i\theta})+g(a-ib+\rho e^{-i\theta}).
\end{aligned}$$

* These points are also called associated points. See Darboux: "Sur une classe remarquable de courbes et de surfaces algebriques," 2nd ed., p. 61. The term used here is due to Cayley.

Therefore,

$$V(A_1) + V(A_2) = V(B_1) + V(B_2),$$

the symbol $V(A_1)$ being used to denote the value of V at the point A_1.

If in particular we take

$$V = \log \sqrt{(x-\xi)^2 + (y-\eta)^2} \text{ or } V = \tan^{-1} \frac{y-\eta}{x-\xi},$$

where (ξ, η) are the coordinates of an arbitrary point P, we obtain the well-known theorems

$$MA_1 \,.\, MA_2 = MB_1 \,.\, MB_2,$$

$$A_1 \hat{M} X + A_2 \hat{M} X = B_1 \hat{M} X + B_2 \hat{M} X.$$

The last theorem indicates that the angles $A_1 MA_2$ and $B_1 MB_2$ have the same bisectors.

THEOREM II. *If* $u + iv = f(x+iy)$, *so that*

$$2u = f(x+iy) + f(x-iy),$$
$$2iv = f(x+iy) - f(x-iy),$$

then the difference of the two values of u *at the points* A_1 *and* A_2 *is equal to the difference of the two values of* $2iv$ *at the points* B_1 *and* B_2.

We have

$$\begin{aligned}
2u(A_1) &= f(a+ib+\rho e^{i\theta}) + f(a-ib+\rho e^{-i\theta}),\\
2u(A_2) &= f(a+ib-\rho e^{i\theta}) + f(a-ib-\rho e^{-i\theta}),\\
2iv(B_1) &= f(a+ib+\rho e^{i\theta}) - f(a-ib-\rho e^{-i\theta}),\\
2iv(B_2) &= f(a+ib-\rho e^{i\theta}) - f(a-ib+\rho e^{-i\theta});
\end{aligned}$$

hence

$$u(A_1) - u(A_2) = iv(B_1) - iv(B_2).$$

If we take

$$u = \log \sqrt{(x-\xi)^2 + (y-\eta)^2},$$
$$v = \tan^{-1} \frac{y-\eta}{x-\xi},$$

we obtain the well-known theorem*

$$\log \frac{MA_1}{MA_2} = iB_1 \hat{M} B_2.$$

This second theorem is important because, if A_1 and A_2 lie on a curve $u = \text{const.}$, we have $u(A_1) = u(A_2)$, and so it follows that B_1 and B_2 lie on a curve $v = \text{const.}$

* Cf. Darboux, *loc. cit.*, p. 63; Laguerre, *Nouvelles Annales de Mathématique* (1853), p. 7.

This is a theorem of very wide application, and so it may be worth while to consider a few particular cases.

If $x + iy = \cos(u + iv)$, the curves $u = \text{const.}$, $v = \text{const.}$ are confocal conics, and we have the theorem that, if A_1 and A_2 are any two points on a conic, the two anti-points B_1 and B_2 lie on a confocal. If we take a system of parallel chords $A_1 A_2$, it is easy to see that the anti-points B_1, B_2 trace out a confocal, and by varying the direction of the chords different confocals are obtained.

If $x + iy = \text{sn}^2 \frac{1}{2}(u + iv)$, the curves $u = \text{const.}$, $v = \text{const.}$ are confocal Cartesian ovals. Hence, if A_1, A_2 are any two points on a Cartesian oval, the anti-points B_1, B_2 lie on a confocal Cartesian oval.

This theorem may be used to obtain a property of the lines of curvature of a quadric surface in the following way:

It is known that, if the points of a quadric are projected from an umbilic on to a plane parallel to the tangent plane at the umbilic, then the lines of curvature project into confocal Cartesian ovals,* and the other three umbilici, which lie on the same focal conic as the vertex of projection, project into three collinear foci of the ovals. Now a quadrilateral $a_1 b_1 a_2 b_2$ formed of generators of the quadric will project into a quadrilateral formed of isotropic lines, for any plane section of the quadric projects into a circle, and so a section consisting of two generators will project into two isotropic lines. The points A_1, A_2, which are the projections of a_1, a_2, will thus be the anti-points of the points B_1, B_2, which are the projections of b_1, b_2. The theorem relating to the points A_1, A_2, B_1, B_2 and the two confocal Cartesian ovals tells us that, if a_1 and a_2 lie on a line of curvature $u = \text{const.}$, the points b_1 and b_2 will lie on a line of curvature $v = \text{const.}$ This theorem may also be stated as follows:

If one pair of opposite corners of a quadrilateral formed of generators of a hyperboloid of one sheet lie on a confocal ellipsoid, the other pair of opposite corners will lie on a confocal hyperboloid of two sheets.

This is really a particular case of a well-known theorem relating to the geodesics on a quadric.

§ 2.

We shall now consider some extensions of Theorems I and II to spaces of three and four dimensions.

* W. K. Clifford, *Educational Times* (1873), Reprint, Vol. XIX, pp. 73, 74. *Math. Papers*, p. 177.

If, in a space of three dimensions, S_3, we take the anti-points of $A_1 A_2$ for all the planes through $A_1 A_2$, the locus of these anti-points B_1, B_2 is a circle, for the locus is obtained by rotating $A_1 B_1 A_2$ about the line $A_1 A_2$. The points A_1, A_2 are called the *foci* of this circle,* for they are the centres of isotropic cones passing through the circle.

Conversely, if we are given a circle c, we can determine the vertices of the two isotropic cones which pass through the circle, and call these the foci of the circle. If the circle is of radius ρ and its centre is at a, b, c, and the direction cosines of its plane are (l, m, n), then the coordinates of the two foci are

$$(a \pm i\rho l, \quad b \pm i\rho m, \quad c \pm i\rho n).$$

It is clear that, if the circle is real, the coordinates of the foci are complex quantities. This fact has been used by Chasles† and Laguerre‡ to obtain a geometrical representation of points with complex coordinates, a point being represented by a real circle. To make the correspondence unique Laguerre introduces the idea of a cycle; *i. e.*, a circle described in a definite direction; one focus is then represented by one cycle and the other focus by the opposite cycle.

In a space of four dimensions, S_4, the anti-points to $A_1 A_2$, which lie in planes through $A_1 A_2$, all lie on a sphere centre C; the linear space S_3 containing this sphere is perpendicular to the line $A_1 A_2$; and the points A_1, A_2 are the vertices of the isotropic cones containing the sphere. The points A_1, A_2 are called the *foci* of the sphere.

If we use (x_1, x_2, x_3, x_4) to denote the coordinates of a point in S_4 when rectangular axes are used, then the coordinates of the foci of a sphere of radius ρ having its centre at the point (c_1, c_2, c_3, c_4) are

$$c_1 \pm i\rho\lambda_1, \quad c_2 \pm i\rho\lambda_2, \quad c_3 + i\rho\lambda_3, \quad c_4 \pm i\rho\lambda_4,$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the direction cosines of the line joining the two foci. These quantities are connected by the equation

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1,$$

and the equation of the space containing the sphere is

$$\lambda_1 (x_1 - c_1) + \lambda_2 (x_2 - c_2) + \lambda_3 (x_3 - c_3) + \lambda_4 (x_4 - c_4) = 0.$$

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* Darboux, *Annales de l'École Normale*, t. 1 (1872), 2nd series, p. 323.

† "Géometrie supérieure," pp. 546–566; "Théorie des coniques sphériques homofocales," *Liouville's Journal*, t. 5 (1860), 2nd series, p. 426. See also Cayley, *Annali di Matematica*, t. 1.

‡ *Bulletin de la Société Philomatique* (1870); *Nouvelles Annales de Mathématiques* (1872).

The line whose equations are

$$\frac{x_1 - a_1}{l_1} = \frac{x_2 - a_2}{l_2} = \frac{x_3 - a_3}{l_3} = \frac{x_4 - a_4}{l_4}$$

is said to be *isotropic* when $l_1^2 + l_2^2 + l_3^2 + l_4^2 = 0$. The isotropic lines which pass through the point (a_1, a_2, a_3, a_4) all lie on the *isotropic cone* or point-hypersphere, whose equation is

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 + (x_4 - a_4)^2 = 0.$$

The two isotropic cones whose equations are

$$(x_1 - c_1 - i\rho\lambda_1)^2 + (x_2 - c_2 - i\rho\lambda_2)^2 + (x_3 - c_3 - i\rho\lambda_3)^2 + (x_4 - c_4 - i\rho\lambda_4)^2 = 0,$$
$$(x_1 - c_1 + i\rho\lambda_1)^2 + (x_2 - c_2 + i\rho\lambda_2)^2 + (x_3 - c_3 + i\rho\lambda_3)^2 + (x_4 - c_4 + i\rho\lambda_4)^2 = 0$$

intersect in the sphere whose equations are

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \rho^2,$$
$$\lambda_1(x_1 - c_1) + \lambda_2(x_2 - c_2) + \lambda_3(x_3 - c_3) + \lambda_4(x_4 - c_4) = 0;$$

and this is the sphere considered previously.

The sphere is evidently real when $c_1, c_2, c_3, c_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and ρ are all real, and in this case the coordinates of the two foci are conjugate complex quantities.

A real sphere may be used to represent a point in S_4 with complex coordinates, just as a real circle was used to represent points in S_3. This representation is of special interest when the sphere lies in the space $x_4 = 0$ and will be discussed later.

If a sphere K passes through a fixed circle γ, its foci will always lie on a second fixed circle $\bar{\gamma}$, which is the locus of the foci of γ for different spaces containing γ. The circles γ and $\bar{\gamma}$ are evidently concentric and lie in perpendicular planes. The relation between the two circles is a mutual one and each is the locus of the vertex of an isotropic cone passing through the other. We shall say that each circle is the focal circle of the other.

If we project two such circles γ and $\bar{\gamma}$ orthgonally onto a space S_3, we obtain two conics, Γ and $\bar{\Gamma}$, which are focal conics of a system of confocal quadrics.

The focal properties of quadrics may in fact be developed very conveniently in connection with the focal properties of circles; thus we have the following theorem:

The foci of a set of circular sections of a quadric lie on a focal conic. Conversely, if we take the circles whose foci are the extremities of a system of parallel chords of a conic, these circles will generate a quadric having the conic as a focal conic. By varying the direction of the chords a system of confocal quadrics is obtained.

If we call the other focal conics of a quadric having Γ as focal conic, the focals of Γ, we have the following theorem:*

If Γ lies on a quadric Q and $\overline{\Gamma}$ is a focal of Γ, then $\overline{\Gamma}$ lies on a quadric confocal to Q

These theorems are probably well-known,† but they are not given in the usual text-books; so we shall proceed to prove them.

Let $Q=0$, $\Omega=0$ be the tangential equations of a quadric and of the circle at infinity respectively. Let $C=0$ be the tangential equation of the conic cut out on Q by a plane t, and let $T=0$ be the tangential equation of the pole of t; then there is an identity of the form

$$Q \equiv C - \lambda T^2.$$

Now let $\Gamma=0$ be the tangential equation of a focal conic of C; then we have an identity of the form

$$\Gamma = C + \mu \Omega.$$

These two equations give

$$Q + \mu \Omega = \Gamma - \lambda T^2.$$

This identity shows that Γ is a plane section of a quadric $Q + \mu \Omega = 0$, which is confocal with Q. If C is a circle, Γ may consist of the two foci of the circle and of the line joining them, and this line will lie in one of the principal planes of Q on account of the position of a circular section of Q. Also the point T lies in this principal plane; hence, the equation

$$Q + \mu \Omega = \Gamma - \lambda T^2$$

represents a conic confocal with Q, and so the two foci of C lie on a focal conic of Q.

Now let the point T move along a diameter of Q; then the conics C will be cut out by a family of parallel planes. If C' is a conic cut out by the polar

* Some analogous theorems have been given by Chasles, *Liouville's Journal*, t. 5 (1860), pp. 444–454, and Darboux, "Sur les théorèms d'Ivory, relatifs aux surfaces homofocales du second degré," Paris, Gauthier-Villars (1872).

† Most of them are given by Humbert, *Bull. de la Soc. Math. de France*, t. 13 (1885), p. 95.

plane of T' and

$$Q = C' - \lambda' T'^2,$$
$$\Gamma' = C' + \mu' \Omega,$$

we shall have

$$\Gamma' - \Gamma = C' - C + (\mu' - \mu)\Omega = \lambda' T'^2 - \lambda T^2 + (\mu' - \mu)\Omega.$$

Now, since C' is similar and similarly situated to C, Γ' will also be similar and similarly situated to Γ. Also the centres of Γ and Γ' lie on $T T'$, and the planes of the two conics are parallel; hence, the cones containing Γ and Γ' will have their vertices on the line $T T'$ and will touch the quadric $\Gamma' - \Gamma = 0$.

On the other hand, the equation

$$\lambda' T'^2 - \lambda T^2 + (\mu' - \mu)\Omega = 0$$

either represents a surface of revolution having the points T, T' as foci and the line $T T'$ as axis, or, if $\mu' = \mu$, it represents a pair of points.

In the first case the tangent cone from a point on $T T'$ would be a cone of revolution, and so the two cones containing the conics Γ and Γ' would be cones of revolution. This, however, is only the case when these conics are circles having their planes perpendicular to $T T'$, and this is not true in the general case under consideration. Consequently, we must have $\mu' = \mu$, and this implies that the conic Γ' lies on the same confocal as Γ. Hence, we have the theorem that when the conics C are parallel sections of Q, the conics Γ are parallel sections of a confocal quadric $Q + \mu\Omega$. In the particular case when the conics C are circular sections of Q, the conics Γ will be pairs of points which are extremities of a system of parallel chords of a focal conic of Q.

We have paid special attention to these theorems because in a sense they provide us with an analogue of Theorem II, when we take into account the fact that a system of confocal ellipsoids are equipotential surfaces and the confocal hyperboloids their orthogonal trajectories. The ellipsoids may thus be supposed to correspond to the loci $u = \text{const.}$ and the confocal hyperboloids to the loci $v = \text{const.}$ The pair of points on $u = \text{const.}$ and the pair of anti-points on $v = \text{const.}$ are now replaced by a conic on $u = \text{const.}$ and a focal conic on $v = \text{const.}$

It looks as if there should be analogous theorems for other equipotential surfaces, but we shall not investigate the matter here. We shall now pass on to some analogues of the first theorem.

THEOREM III. *If V is a solution of the partial differential equation*

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial^2 V}{\partial x_4^2}=0,$$

the mean value of V over a sphere in S_4 is equal to the mean value of V along the line joining the foci.

This theorem is practically proved in the usual derivation of Poisson's formula for a solution of the equation of wave motion.*

To prove the theorem we observe in the first place that the differential equation is unaltered by a change of rectangular axes; hence we may choose the axes in such a way that the equations of the sphere are

$$(X_1-x_1)^2+(X_2-x_2)^2+(X_3-x_3)^2=\rho^2,\quad X_4=x_4,$$

where X_1, X_2, X_3, X_4 are current coordinates. We have then to establish the identity of the two integrals

$$W_1=\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} F(x_1+\rho\sin\theta\cos\phi,\ x_2+\rho\sin\theta\sin\phi,\ x_3+\rho\cos\theta,\ x_4)\sin\theta\,d\theta\,d\phi,$$

$$W_2=\frac{1}{2\rho}\int_{-\rho}^{+\rho} F(x_1,\ x_2,\ x_3,\ x_4+iu)\,du,$$

where $V=F(x_1, x_2, x_3, x_4)$ is a solution of the partial differential equation.

We shall assume that the function F possesses second derivatives which are finite and continuous over the domains of integration, when the variables x_1, x_2, x_3, x_4 are limited to a certain domain. The integrals W_1 and W_2 may then be differentiated by the rule of Leibnitz, and it is easy to verify that they satisfy the partial differential equations

$$\frac{\partial^2 W}{\partial x_1^2}+\frac{\partial^2 W}{\partial x_2^2}+\frac{\partial^2 W}{\partial x_3^2}+\frac{\partial^2 W}{\partial x_4^2}=0,$$

$$\frac{\partial^2 W}{\partial x_1^2}+\frac{\partial^2 W}{\partial x_2^2}+\frac{\partial^2 W}{\partial x_3^2}=\frac{\partial^2 W}{\partial \rho^2}-\frac{2}{\rho}\frac{\partial W}{\partial \rho},$$

$$\frac{\partial^2 W}{\partial x_4^2}+\frac{\partial^2 W}{\partial \rho^2}-\frac{2}{\rho}\frac{\partial W}{\partial \rho}=0.$$

Now let $F(x_1, x_2, x_3, x_4)$ be a regular function of x_4 within some circle of radius $R>\rho$ surrounding the point x_4; then it can be proved that there is only one solution of the last equation which reduces to $F(x_1, x_2, x_3, x_4)$ when $\rho=0$, and can be expanded in a convergent power-series of the form

$$W=A_0+A_1\rho+A_2\rho^2+A_3\rho^3+\cdots\cdots.$$

* See for instance Rayleigh's *Sound*, Vol. II, p. 99.

For, when this power-series is substituted in the differential equation, we obtain the relations

$$0 = A_1 = A_3 = A_5 = \ldots.$$

and a recurrence formula by which the coefficients $A_2, A_4, \ldots.$ may be calculated in a unique manner from the first coefficient A_0, which is equal to $F(x_1, x_2, x_3, x_4)$.

Now, both W_1 and W_2 are equal to $F(x_1, x_2, x_3, x_4)$ when $\rho = 0$; hence, since they both satisfy the differential equation, we must have $W_1 = W_2$. This gives the formula

$$\int_{-\rho}^{+\rho} F(x_1, x_2, x_3, x_4 + iu)\,du = \frac{\rho}{2\pi}\int_0^{\pi}\int_0^{2\pi} F(x_1 + \rho\sin\theta\cos\phi,$$
$$x_2 + \rho\sin\theta\sin\phi,\ x_3 + \rho\cos\theta,\ x_4)\sin\theta\,d\theta\,d\phi.$$

To derive Poisson's integral from this formula, we express the integral from $-\rho$ to $\rho + 2\varepsilon$ by means of an integral taken over a consecutive sphere, thus:

$$\int_{-\rho}^{\rho+2\varepsilon} F(x_1, x_2, x_3, x_4 + iu)\,du = \frac{\rho+\varepsilon}{2\pi}\int_0^{\pi}\int_0^{2\pi} F(x_1 + \overline{\rho+\varepsilon}\sin\theta\cos\phi,$$
$$x_2 + \overline{\rho+\varepsilon}\sin\theta\sin\phi,\ x_3 + \overline{\rho+\varepsilon}\cos\theta,\ x_4 + i\varepsilon)\sin\theta\,d\theta\,d\phi.$$

Subtracting, we get

$$F(x_1, x_2, x_3, x_4 + i\rho) = \frac{1}{4\pi}\frac{d}{d\rho}\rho\int_0^{\pi}\int_0 \quad F(x_1 + \rho\sin\theta\cos\phi,$$
$$x_2 + \rho\sin\theta\sin\phi,\ x_3 + \rho\cos\theta,\ x_4)\sin\theta\,d\theta\,d\phi$$
$$+\frac{i\rho}{4\pi}\frac{\partial}{\partial x_4}\int_0^{\pi}\int_0^{2\pi} F(x_1 + \rho\sin\theta\cos\phi,\ x_2 + \rho\sin\theta\sin\phi,$$
$$x_3 + \rho\cos\theta,\ x_4)\sin\theta\,d\theta\,d\phi.$$

Putting $\quad x_1 = x,\quad x_2 = y,\quad x_3 = z,\quad x_4 = 0,\quad \rho = t,$

$$F(x, y, z, 0) = f(x, y, z),$$
$$\left[\frac{\partial}{\partial t}F(x, y, z, it)\right]_{t=0} = g(x, y, z),$$

we obtain the equation

$$F(x, y, z, t) = \frac{1}{4\pi}\frac{d}{dt}t\int_0^{\pi}\int_0^{2\pi} f(x_1 + t\sin\theta\cos\phi,$$
$$x_2 + t\sin\theta\sin\phi,\ x_3 + t\cos\theta)\sin\theta\,d\theta\,d\phi$$
$$+\frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi} g(x_1 + t\sin\theta\cos\phi,\ x_2 + t\sin\theta\sin\phi,$$
$$x_3 + t\cos\theta)\sin\theta\,d\theta\,d\phi.$$

This is Poisson's formula.

THEOREM IV. *If* $V = F(x_1, x_2, x_3, x_4)$ *is a solution of*

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} + \frac{\partial^2 V}{\partial x_4^2} = 0,$$

the mean value of V *round a circle in* S_4 *is equal to* i *times the mean value of* V *round the focal circle.*

As before, we can transform the axes so that the equations of the two circles take a simple form such as

$$x_1^2 + x_2^2 = \rho^2, \qquad x_3 = x_4 = 0,$$
$$x_3^2 + x_4^2 = -\rho^2, \qquad x_1 = x_2 = 0.$$

We now have to establish the identity of the two integrals

$$U_1 = \frac{1}{2\pi}\int_0^{2\pi} F[x_1 + \rho\cos\alpha,\ x_2 + \rho\sin\alpha,\ x_3,\ x_4]\, d\alpha,$$

$$U_2 = \frac{1}{2\pi}\int_0^{2\pi} F[x_1,\ x_2,\ x_3 + i\rho\cos\theta,\ x_4 + i\rho\sin\theta]\, d\theta.$$

These integrals both satisfy the differential equations

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} = 0,$$
$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho},$$
$$\frac{\partial^2 U}{\partial x_3^2} + \frac{\partial^2 U}{\partial x_4^2} + \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} = 0,$$

and reduce to $F(x_1, x_2, x_3, x_4)$ when $\rho = 0$.

Now, if F is a regular function of x_3 and x_4 in circles of radii $R > \rho$ surrounding the points x_3 and x_4, in their complex planes, it may be shown as before that there is only one solution of the last differential equation which reduces to $F(x_1, x_2, x_3, x_4)$ when $\rho = 0$. Hence, we must have $U_1 = U_2$ and so

$$\int_0^{2\pi} F[x_1 + \rho\cos\alpha,\ x_2 + \rho\sin\alpha,\ x_3,\ x_4]\, d\alpha$$
$$= \int_0^{2\pi} F[x_1,\ x_2,\ x_3 + i\rho\cos\theta,\ x_4 + i\rho\sin\theta]\, d\theta.$$

We now proceed to some applications of this result.

(1) Let F have the same value at all points of a circle

$$(X_1 - x_1)^2 + (X_2 - x_2)^2 = \rho^2, \quad X_3 = 0, \quad X_4 = 0;$$

then this value $F\{x_1, x_2, x_3, x_4, \rho\}$ is given by the formula

$$F\{0, 0, x_3, x_4, \rho\} = \frac{1}{2\pi}\int_0^{2\pi} F[0, 0, x_3 + i\rho\cos\theta,\ x_4 + i\rho\sin\theta]\, d\theta.$$

If, then, we know the value of F at points of the plane $X_1 = x_1$, $X_2 = x_2$, its value at other points of space can be derived from the formula. The formula may be written in the more convenient form

$$V = F\{x_3, x_4, \rho\} = \frac{1}{2\pi}\int_0^{2\pi} f[x_3 + i\rho\cos\theta,\ x_4 + i\rho\sin\theta]\, d\theta,$$

where $f(x_3, x_4)$ is the value of V when $\rho = 0$. If V is independent of x_4, and therefore a solution of Laplace's equation

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} = 0,$$

the general formula becomes

$$\int_0^{2\pi} F[x_1 + \rho\cos\alpha,\ x_2 + \rho\sin\alpha,\ x_3]\, d\alpha = \int_0^{2\pi} F[x_1, x_2, x_3 + i\rho\cos\theta]\, d\theta.$$

This equation expresses that the mean value of V round a circle is equal to the integral of V multiplied by a certain function along the line joining the foci of the circle. The formula may be used to obtain the well-known formula

$$V = \frac{1}{2\pi}\int_0^{2\pi} F[x_3 + i\rho\cos\theta]\, d\theta$$

for a solution which is symmetrical about the axis of x_3. A more general result may be obtained by using the integrals along a pair of focal circles situated arbitrarily in S_4. Thus we have the relation

$$\int_0^{2\pi} F(\xi_1 + \rho\cos\alpha\cos\omega,\ \xi_2 + \rho\sin\alpha,\ \xi_3 + \rho\cos\alpha\sin\omega,\ \xi_4)\, d\alpha$$
$$= \int_0^{2\pi} F(\xi_1 - i\rho\cos\theta\sin\omega,\ \xi_2,\ \xi_3 + i\rho\cos\theta\cos\omega,\ \xi_4 + i\rho\sin\theta)\, d\theta,$$

when $V = F(\xi_1, \xi_2, \xi_3, \xi_4)$ is a suitable solution of

$$\frac{\partial^2 V}{\partial \xi_1^2} + \frac{\partial^2 V}{\partial \xi_2^2} + \frac{\partial^2 V}{\partial \xi_3^2} + \frac{\partial^2 V}{\partial \xi_4^2} = 0.$$

When V is independent of ξ_1 and consequently a solution of

$$\frac{\partial^2 V}{\partial \xi_2^2} + \frac{\partial^2 V}{\partial \xi_3^2} + \frac{\partial^2 V}{\partial \xi_4^2} = 0,$$

this equation becomes

$$\int_0^{2\pi} F(x+\rho\sin\alpha,\ y+\rho\cos\alpha\sin\omega,\ z)\,d\alpha$$
$$=\int_0^{2\pi} F(x,\ y+i\rho\cos\theta\cos\omega,\ z+i\rho\sin\theta)\,d\theta,$$

where x, y, z are written in place of ξ_2, ξ_3, x_4.

This equation indicates that an integral of a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

taken round the ellipse

$$\frac{(X-x)^2}{\rho^2}+\frac{(Y-y)^2}{\rho^2\sin^2\omega}=1,\quad Z=z,$$

is equal to an integral of V taken over a portion of the focal conic

$$\frac{(Y-y)^2}{\rho^2\cos^2\omega}+\frac{(Z-z)^2}{\rho^2}+1=0,\quad X=x.$$

If we take an imaginary value of ω so that $\sin^2\omega>1$, $\cos^2\omega<0$, the focal conic is the focal hyperbola of the ellipse.

The theorem may be used in potential theory in the following way: If a solution of Laplace's equation is known to be constant at all real and imaginary points of an ellipse or hyperbola, then its value on this conic may be found when the values of the function at points on a focal conic are known. The theorem may be regarded as an analogue of Theorem I.

§ 3.

We have already remarked that a sphere in the space $x_4=0$ may be used to represent a point in S_4 whose coordinate x_4 is imaginary. This representation has been used with great success by Darboux* in investigating properties of systems of spheres. We shall use it here in connection with a second representation, in which a particle which is at the point (x, y, z) at time t is represented by a point in S_4 whose coordinates are (x_1, x_2, x_3, x_4), where

$$x_1=x,\quad x_2=y,\quad x_3=z,\quad x_4=ict,$$

and c is the velocity of light.

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* *Annales de l'École Normale*, t. 1 (1872), 2nd ser., p. 323. The points on a hypersphere in S_4 correspond to spheres which cut a fixed sphere at a constant angle, the points on a sphere in S_4 to a system of spheres touching two fixed spheres and the points on a circle in S_4 to spheres touching three fixed spheres.

This transformation arises naturally in the reduction of the equation of wave motion,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},$$

to the symmetrical form

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} + \frac{\partial^2 V}{\partial x_4^2} = 0.$$

It has been used recently by Minkowski* in connection with the equations of electrodynamics and the transformations which form the basis of the theory of relativity. The transformation reduces the equations of electrodynamics to a symmetrical form, and so enables us to use mathematical symmetry as a guide to further investigations.

When we combine the representations of Darboux and Minkowski, we are led to a representation of a space-time point (*i. e.*, the instantaneous position of a particle at a given time) by means of a sphere with a definite sign attached to its radius. It is convenient to regard the sphere as a spherical wave; the sign of its radius is then indicated by the direction in which the wave is moving. A space-time point (x, y, z, t) is represented by a spherical wave whose centre is at the point (x, y, z) and whose radius is equal to ct, the wave being a convergent wave, if t is positive, and a divergent wave when t is negative. The wave may be regarded as the position at time $t = 0$ of a disturbance which will be absorbed by (or has been emitted by) a particle which is at (x, y, z) at time t.

We shall call the wave the representative wave of the particle at time t. It is clear that the career of a particle may be studied very conveniently with the aid of its representative waves at different times. If the particle is moving with a velocity less than that of light, no two of the convergent representative waves can intersect and similarly no two of the divergent representative waves can intersect.

It follows from this result and the principle of Huyghens that there is only one position of the particle in which it can create a disturbance which will arrive at the point (x, y, z) at time t. For, if a disturbance issuing from a point (x_0, y_0, z_0) at time t_0 is propagated in the form of a spherical wave with this point as centre, then the wave will only reach the point (x, y, z) at time t, if

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = c^2 (t - t_0)^2.$$

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* *Göttinger Nachrichten* (1908).

The geometrical interpretation of this equation is that the representative wave of (x, y, z, t) is touched internally by the representative wave of (x_0, y_0, z_0, t_0), if both waves are convergent (or divergent), and touched externally by it, if one wave is convergent and the other one divergent.

Now, since no two convergent waves associated with the particle intersect, there is only one wave of the system which has the proper type of contact with the representative wave of the point (x, y, z, t).

This theorem, which is proved analytically by Prof. A. W. Conway,* is of considerable importance in the mathematical theory of the motion of an electron. If the electron be treated as a simple singularity of an electromagnetic field, Liénard and Wiechert have shown that an appropriate electromagnetic field for an electron moving in an arbitrary manner may be obtained in the following way:

Let the motion of the electron or singularity be specified by the equations

$$\xi = \xi(\tau), \quad \eta = \eta(\tau), \quad \zeta = \zeta(\tau);$$

then there is one value of $\tau < t$ which satisfies the equation

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 = c^2(t - \tau)^2.$$

If, with this value of τ,

$$M = 2c^2(t - \tau) - 2\xi'(\tau)[x - \xi(\tau)] - 2\eta'(\tau)[y - \eta(\tau)] - 2\zeta'(\tau)[z - \zeta(\tau)],$$

the components (E_x, E_y, E_z), (H_x, H_y, H_z) of the electric and magnetic forces at the point (x, y, z) at time t are given by the equations

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}, \qquad H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \text{ etc.,}$$

where

$$A_x = e\frac{\xi'(\tau)}{M}, \quad A_y = e\frac{\eta'(\tau)}{M}, \quad A_z = e\frac{\zeta'(\tau)}{M}, \quad \Phi = \frac{ec}{M}.$$

These potentials A_x, A_y, A_z, Φ are all solutions of the equation of wave motion,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},$$

and in fact this equation is satisfied by any function of the form

$$V = \frac{1}{M} f(\tau).$$

These expressions for the components of the electric and magnetic forces show that the field at (x, y, z, t) depends only on the position τ of the moving

* *Proc. London Math. Soc.*, Vol. I (1903), Ser. 2, p. 154.

singularity. Lorentz* calls this position of the electron the *effective position.* Minkowski† calls it the "light point" of (x, y, z, t). It is convenient, however, to have a term which will apply to an electron irrespective of its path, and so we shall say that an electron (or particle) $P(\xi, \eta, \zeta, \tau)$ is *active* relative to $Q(x, y, z, t)$ when the conditions

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=c^2(t-\tau)^2, \quad t>\tau \tag{1}$$

are satisfied. Thus P is active relative to Q whenever a convergent representative wave of Q is touched internally by a convergent representative wave of P, or in fact, whenever the two representative waves touch in the proper way.

If we represent P, Q by points $(\xi_1, \xi_2, \xi_3, \xi_4)$, (x_1, x_2, x_3, x_4) in S_4, where

$$\xi_1=\xi, \quad \xi_2=\eta, \quad \xi_3=\zeta, \quad \xi_4=ic\tau,$$
$$x_1=x, \quad x_2=y, \quad x_3=z, \quad x_4=ict,$$

then the condition (1) takes the form

$$(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2+(x_4-\xi_4)^2=0,$$

and this implies that the line joining the two representative points in S_4 is an *isotropic line.*

Hence, if P is active relative to Q (or vice-versa), the representative points in S_4 lie on an isotropic line. In studying different configurations of electrons we may either consider the positions of the different electrons at a given time, or we may consider the positions in which they are active relative to one another. The second method is preferable for some purposes, for the relation between the electrons is unaltered by changing the axes from a system at rest to a system in uniform motion. We shall therefore proceed to investigate some properties of isotropic lines in S_4, for the properties of systems of electrons which are actively related to one another can be expressed in terms of these.

§ 4.

Since the equation of an isotropic line is of the form

$$(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2+(x_4-\xi_4)^2=0,$$

it is clear that an isotropic line is transformed into an isotropic line by any transformation which leaves this condition unaltered in form. The group of conformal transformations of the space S_4 satisfies this requirement and will be

* "The Theory of Electrons," Leipzig (1909), p. 251.

† *Physikalische Zeitschrift* (1909).

referred to as the group G_{15}.* The equivalent group of transformations in the variables (x, y, z, t) leaves the equation

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=c^2(t-\tau)^2$$

unaltered in form and so transforms a spherical wave into a spherical wave.† The transformations of the group have on this account been called *spherical wave transformations*; they are characterized by a relation of the form

$$dx'^2+dy'^2+dz'^2-c^2dt'^2=\lambda^2[dx^2+dy^2+dz^2-c^2dt^2].$$

It has been proved elsewhere‡ that the fundamental equations of the theory of electrons are covariant for transformations of this type. The equation of

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* It is a fifteen-parameter group. The group of Lorentzian transformations, considered by Poincaré, *Rend. Palermo* (1906), and Minkowski, *Gött. Nachr.* (1908), is a subgroup of G_{15}; it is only a six-parameter group and is included in the ten-parameter group of transformations of rectangular axes in S_4.

† Einstein's postulate of the constancy of the velocity of light (*Ann. d. Physik*, Vol. XVII, 1905) is really equivalent to the assumption that there is a group of transformations for which the equations

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=c^2(t-\tau)^2 \qquad t>\tau \tag{1}$$

are covariant. If we regard these as the conditions of observation or for the action of one particle on another, then the corresponding group of transformations, for which these conditions are covariant may be regarded as the *principal group* underlying all physical phenomena. According to the general principles of group-theory, the quantities and relations which are invariant with regard to the principal group should represent true physical quantities and relations. Some of these invariants for the group G_{15} have been found by Einstein, Poincaré, Minkowski, Planck, Cunningham and the author. It is desirable that all the principal invariants and covariants for the group should be found, for then we shall perhaps be able to decide whether Einstein's conditions of observation are the right ones or not.

At first sight it would look as if these conditions of observation were only valid for the free æther, since light and other disturbances travel with different velocities in different material media. If, however, we adopt Lorentz's hypothesis that the fundamental equations of electromagnetism for ponderable bodies can be derived from the electron equations by a process of averaging, then we may assume that the condition of observation holds for two neighboring particles, when there is only free æther in between. The conditions of action of two *neighboring* particles are then

$$dx^2+dy^2+dz^2=c^2dt^2 \qquad dt>0. \tag{2}$$

Now, it is a remarkable fact that the group of transformations for which these conditions are covariant is exactly the same as that for which the more general conditions

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=c^2(t-\tau)^2 \qquad t>\tau \tag{1}$$

are covariant. Hence, if we may assume that (2) is the condition for contact action and define the principal group by means of this condition, then we may conclude that the condition for action at a distance is covariant for the principal group whenever the condition can be expressed in the form (1). This result may be expressed in another way by saying that any transformation of the group G transforms free æther into free æther.

The conditions of contact action expressed by equations (2) are the most suitable for the discussion of wave theory in isotropic media, but it is useful to know what difference it would make in the general equations of the electromagnetic field if the condition of contact action were of a more general character. This question has been considered by the author, *Proc. London Math. Soc.*, Vol. VIII (1910), Ser. 2, p. 255.

‡ *Proc. London Math. Soc.*, Vol. VIII (1910), Ser. 2, p. 223.

wave motion is also covariant in the sense that if V is a solution of

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0$$

and

$$V ds = V' ds',$$

where

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$
$$ds'^2 = dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 + dx_4'^2,$$

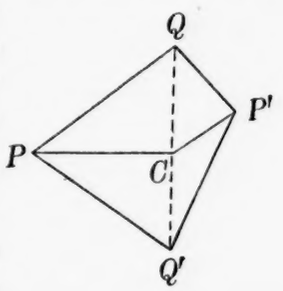
then V' is a solution of

$$\frac{\partial^2 V'}{\partial x'^2} + \frac{\partial^2 V'}{\partial y'^2} + \frac{\partial^2 V'}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2 V'}{\partial t'^2} = 0.$$

The last theorem* shows that $\int V ds$ taken along any curve in S_4 is an invariant for transformations of the group G. On this account there is reason to suppose that, when V is properly chosen, the integral represents an important physical quantity. The integral evidently vanishes along an isotropic line, since $ds = 0$, and this again brings isotropic lines into prominence. It is interesting to consider the integral to be taken round a closed path, and so we shall consider the properties of closed polygons formed of isotropic lines.

THEOREM V. *If a skew quadrilateral in S_4 is formed of isotropic lines, the four lines lie on a sphere.*

Let $PQP'Q'$ be the corners of the quadrilateral, PQ, QP', $P'Q'$, $Q'P$ the isotropic lines. It is clear that P is one of the anti-points of Q, Q' for the plane PQQ'; hence, if C is the middle point of QQ', PC will be perpendicular to QQ' and $CP = iCQ$. Similarly, CP' is perpendicular to QQ' and $CP' = iCQ = CP$. Now since this is the case, it is possible to draw a sphere to touch the planes QPQ', $QP'Q'$ at P and P', and this sphere will contain the isotropic lines PQ, QP', $P'Q'$, $Q'P$, because a sphere which touches a plane at a point P has for generators the two isotropic lines through P which lie in this plane.



* This theorem may be derived immediately from the analysis given in a paper "On the Conformal Transformations of a Space of Four Dimensions," *Proc. London Math. Soc.*, Vol. VII (1909), Ser. 2, p. 70. There is a more general invariant of type $\int V ds \cos \varepsilon$, where ε is the angle between the path and a member of a given family of space-time curves.

Now let R, R' be the two foci of this sphere, then the lines RP, RP', RQ, RQ', $R'P$, $R'P'$, $R'Q$, $R'Q'$ are isotropic lines, and we have a remarkable figure formed of three pairs of points, PP', QQ', RR', in which the twelve lines joining points belonging to different pairs are all isotropic. The twelve lines also lie by fours on three spheres.

The figure which is derived from this by Darboux's transformation is very interesting. The points PP', QQ', RR' correspond to pairs of waves pp', qq', rr', such that the waves of one pair are in contact with the waves of another pair. The point of contact of the waves p, q is the point in which the space $x_4 = 0$ is met by the line PQ, and similarly for the points of contact of the other sphere. Since the lines PQ, QP', $P'Q'$, $Q'P$ lie on a sphere, it follows that the four points of contact of the spheres pp' with the spheres qq' lie on a circle. Similarly, the points of contact of qq' with rr' lie on a circle, and the points of contact of r and r' with p and p' lie on a circle.

If the waves p, p', etc., are representative waves of particles (p), (p'), etc., at their centres, the relations between the particles may be expressed as follows:

(q), (r), (q'), (r')	are	active	relative	to	(p),
(r), (r'), (p')	"	"	"	"	(q),
(q'), (p')	"	"	"	"	(r),
(q'), (p')	"	"	"	"	(r'),
(p')	is	"	"	"	(q').

An interesting extension of this theorem occurs in Steiner's porism relating to a ring of spheres touching two fixed spheres and having their centres in one plane. This theorem is so well-known that it will not be necessary to give an account of it here.*

Another extension of the theorem may be made in connection with spaces of a higher number of dimensions.

If the sides of a skew hexagon in a space of five dimensions S_5 are isotropic lines, the six sides lie on a hyperspherical locus K_4.

To prove this we consider the section of the figure by any linear space S_4. Any corner of the hexagon and its image in S_4 may be regarded as the foci of a hypersphere in S_4. The hyperspheres which correspond to two corners A_1, A_2

* The following references may prove useful: Steiner, *Werke*, Bd. 1, pp. 135, 455; Geiser, *Einleitung in die synthet. Geometrie* (1869); W. Fiedler, *Cyklographie* (1882), p. 233; Holzmüller, *Elem. d. Stereometrie*, Bd. 1, p. 279; A. Schumann, *Berlin, Askanisches Gymn.* (1883); Maennchen, *Grunert's Archiv* (3), 7, p. 232; J. Finsterbusch, *Atti del. IV Congress dei Matematici*, Vol. I, p. 285.

of the hexagon will touch at the point B_{12} in which the line $A_1 A_2$ meets S_4. Now, if six hyperspheres in S_4 touch one another in succession and the contacts are of the proper kind, the six points of contact lie on a hypersphere.* Hence, the six sides of the hexagon are met by any linear space S_4 in six points lying in a hypersphere K_3, and so the six sides must lie on a hypersphere K_4.

To prove the theorem relating to the six hyperspheres in S_4 we invert the space S_4 into a hypersphere Σ_4. The points of contact of the hyperspheres then invert into the points of contact of the sides of a skew hexagon whose sides all touch Σ_4, and the hyperspheres in S_4 invert into the hyperspheres cut out on Σ_4 by the polars of the vertices of the hexagon. Now the points of contact of the sides of the hexagon lie on a linear space L_4, for if $A_1, A_2, A_3, A_4, A_5, A_6$ are the corners of the hexagon, $B_{12}, B_{23}, B_{34}, B_{45}, B_{56}, B_{61}$ the points of contact of the sides, we have relations of the type

$$A_1 B_{12} = A_1 B_{61}$$

which lead to the equation

$$\begin{aligned} \pm A_1 B_{12} \cdot A_2 B_{23} \cdot A_3 B_{34} \cdot A_4 B_{45} \cdot A_5 B_{56} \cdot A_6 B_{61} \\ = A_2 B_{12} \cdot A_3 B_{23} \cdot A_4 B_{34} \cdot A_5 B_{45} \cdot A_6 B_{56} \cdot A_1 B_{61}, \end{aligned} \tag{2}$$

and when the positive sign is taken, this is the condition that the six points of contact should lie in a linear space L_4.

This linear space L_4 will cut Σ_4 in a hypersphere $\overline{K}_3$ containing the six points of contact, and this inverts into a hypersphere K_3 in S_4 containing the six points of contact of the hyperspheres. It should be remarked that the contacts between the hyperspheres which correspond to the corners of a skew hexagon of isotropic lines, are of such a nature that the sign to be taken in equation (2) is always positive; hence, we have the theorem:

THEOREM VI. *If a skew hexagon in S_5 is formed of isotropic lines, its sides lie on a hypersphere K_4.*

There is a corresponding theorem for a polygon of $n+1$ sides in a space of n dimensions ($n > 2$). The case of a skew pentagon in a space of four dimensions may be deduced from Theorem VI by making two of the corners of the hexagon coincident. Since a linear space S_4 can be drawn through five arbitrary points, the sides A_1, A_2, A_3, A_4, A_5 will lie in a liner space S_4, and if A_6 is coincident with A_1, the line $A_1 A_6$ can be taken to be an arbitrary generator of the isotropic

* *Educational Times* (2), 11, Question 16009, pp. 57–61. A solution different from the above was given by D. M. Y. Sommerville.

cone having its vertex at A_1. The theorem then tells us that there are an infinite number of hyperspheres K_4 containing the five lines A_1A_2, A_2A_3, A_3A_4, A_4A_5, for these lines and an *arbitrary* isotropic line through A_1 lie on a hypersphere K_4. Consequently, the five lines A_1A_2, A_2A_3, ... must lie on a hypersphere K_3.

THEOREM VII. *If a skew pentagon in S_4 is formed of isotropic lines, its sides lie on a hypersphere K_3.*

If we apply Darboux's transformation to the corners of this pentagon, we get five spherical waves touching one another in succession, and the theorem tells us that their points of contact lie on a sphere. It should be remarked that, if five spheres touch one another in succession, the five points of contact do not necessarily lie on a sphere; it is only when an *even* number of the contacts are external contacts that the theorem is true.

§ 5.

A very interesting application of Theorem III may be made to a solution of

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},$$

which is a periodic function of t with the period $2T$, and is regular for some domain of values of (x, y, z, t). The integral*

$$U = \int_{t-T}^{t+T} V\,dt$$

is then a solution of $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$, for

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \int_{t-T}^{t+T}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)dt$$

$$= \int_{t-T}^{t+T}\left(\frac{\partial^2 V}{\partial t^2}\right)dt = 0,$$

on account of the periodicity of $\frac{\partial V}{\partial t}$.

Now, Theorem III enables us to express $\int_{t-T}^{t+T} V\,dt$ as an integral taken over the surface of a sphere of radius cT. Hence, we have the theorem that

* A similar theorem is given by Prof. Lamb for the case in which the limits are $-\infty$ and $+\infty$, *Proc. London Math. Soc.*, Vol. VIII, Ser. 2, p. 437.

the mean value of V over a sphere of radius cT is a solution of Laplace's equation, when regarded as a function of the coordinates of the centre of the sphere. It is thought that this theorem may be of fundamental importance in a mathematical theory of gravitation, and that the sphere of radius cT represents the surface of an electron. At present, however, our knowledge is not sufficient to enable us to form any definite theory, and the only use we can make of the theorem is to derive a number of ideas from it and use them as working hypotheses. The ideas which the theorem suggests are:

(1) That the surface of an electron *in motion* can be represented by a sphere in S_4. In the theorem the sphere of integration is situated in a space $t = \text{const.}$, and this sphere may be supposed to represent the surface of an electron at rest. The general form of Theorem III then leads us to consider the mean value of V over an arbitrary sphere in S_4 and a corresponding integral along the line joining the foci.

Let (x_1, y_1, z_1, ict_1), (x_2, y_2, z_2, ict_2), (x, y, z, ict) be the coordinates of the foci and of an arbitrary point on the sphere, and consider the points $F_1(x_1, y_1, z_1)$, $F_2(x_2, y_2, z_2)$, $P(x, y, z)$, which are the centres of the representative waves of these points.

The locus of P is a prolate spheroid having the points F_1 and F_2 as foci. We shall regard P as the position at time t of some point on the surface of an electron. The spheroid then represents a particular *view* of the surface of the electron; it is in fact the view which would be obtained by an observer at one of the foci, *e. g.*, F_2. It should be noticed that different points on the surface of the spheroid have different times associated with them, but this is just what is required in order that disturbances issuing from them should all arrive at F_2 at the same instant. The points of the spheroid will also receive disturbances from F_1 at the same instant.

We shall regard F_1 and F_2 as two positions of the singularity of the electromagnetic field associated with the electron, *i. e.*, of the electromagnetic field considered in § 3. In the case when the spheroid is a sphere, the points F_1 and F_2 coincide and we have supposed that the electron is momentarily at rest. It is possible, however, for the surface to be a sphere when the singularity is describing a periodic orbit with period $2T$. This brings us to the second idea which is suggested by the theorem.

(2) We must look for interpretations of our geometrical theorems in con-

nection with periodic motions. An attempt has already been made to obtain in this way a physical interpretation of Steiner's porisms.*

§ 6.

We shall now prove some fundamental theorems relating to spheres and hyperspheres in S_4 which have a direct bearing upon the kinematics of an electron. According to an idea developed by Minkowski,† Born‡ and Herglotz,§ the motion of a connected system of particles may be represented by equations of the form

$$\left.\begin{aligned} x &= f_1(X, Y, Z, u), \\ y &= f_2(X, Y, Z, u), \\ z &= f_3(X, Y, Z, u), \\ t &= f_4(X, Y, Z, u), \end{aligned}\right\} \quad (1)$$

where (X, Y, Z) are parameters which enable us to identify a particular particle; they vary from particle to particle, but remain constant during the motion. The different values of u determine different *views* of the connected system, a view being obtained when a time t is associated with a point (x, y, z) according to the law

$$u(x, y, z, t) = \text{const.}$$

We shall now investigate the motion which corresponds to the case in which the loci $u = \text{const.}$ are represented in S_4 by hyperspheres and the equations (1) represent the orthogonal trajectories of these hyperspheres.

Using (x_1, x_2, x_3, x_4) in place of the quantities x, y, z, ict, the equation of a singly infinite system of hyperspheres in S_4 may be written in the form

$$f(u)[x_1^2 + x_2^2 + x_3^2 + x_4^2] + 2\alpha_1(u)x_1 + 2\alpha_2(u)x_2 + 2\alpha_3(u)x_3 + 2\alpha_4(u)x_4 + 2\phi(u) = 0,$$

the case of a family of hyperplanes being obtained by putting $f(u) \equiv 0$.

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* *Phys. Zeitschr.* (1910), pp. 318–320. It should be remarked in support of this theory that there are good reasons for supposing that the problem of finding a stable configuration of electrons is poristic, *i. e.*, a configuration will not be stable unless certain conditions (perhaps a single condition) are satisfied, and when these conditions are satisfied an infinite number of configurations will exist. These configurations may be classified into types according to the different ways in which the conditions can be satisfied. Now for each type there will be a gronp of transformations enabling us to pass from one possible configuration to another, and we should expect this transformation to leave the fundamental equations of electromagnetism unaltered in form. This condition is fulfilled in the case of Steiner's porisms, and in fact in the case of any porism relating to spherical waves in contact, for the conditions of the porism are unaltered by any spherical wave transformation which transforms the fixed spherical waves into themselves.

† *Gött. Nachr.* (1908). *Phys. Zeitschr.* (1909).

‡ *Ann. d. Physik*, Vol. XXX (1909). *Gött. Nachr.* (1909).

§ *Ann. d. Physik*, Vol. XXXI (1910).

Differentiating with regard to x_1, x_2, x_3, x_4, we obtain

$$f x_1 + \alpha_1 + \frac{\partial u}{\partial x_1} M = 0, \quad f x_2 + \alpha_2 + \frac{\partial u}{\partial x_2} M = 0, \quad \text{etc.},$$

where

$$M = \tfrac{1}{2} f'(u)\,[x_1^2 + x_2^2 + x_3^2 + x_4^2] + \alpha_1'(u)\,x_1 + \alpha_2'(u)\,x_2 + \alpha_3'(u)\,x_3 + \alpha_4'(u)\,x_4 + \phi'(u).$$

Now, $\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial u}{\partial x_4}\right)$ are proportional to the direction cosines of the normal to the hypersphere $u = \text{const.}$, and so the differential equations of the orthogonal trajectories are

$$\frac{dx_1}{du} = \frac{dx_2}{du} = \frac{dx_3}{du} = \frac{dx_4}{du}.$$
$$f x_1 + \alpha_1 \quad f x_2 + \alpha_2 \quad f x_3 + \alpha_3 \quad f x_4 + \alpha_4$$

Since

$$(f x_1 + \alpha_1)\frac{dx_1}{du} + (f x_2 + \alpha_2)\frac{dx_2}{du} + (f x_3 + \alpha_3)\frac{dx_3}{du} + (f x_4 + \alpha_4)\frac{dx_4}{du} + M = 0,$$

it follows that each of the above ratios is equal to

$$-\frac{M}{(f x_1 + \alpha_1)^2 + (f x_2 + \alpha_2)^2 + (f x_3 + \alpha_3)^2 + (f x_4 + \alpha_4)^2} = -\frac{M}{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 - 2 f \phi} = 2 \varkappa M f,$$

where $\varkappa$ is some function of u.

The infinitesimal transformation by which we may pass from a point (x_1, x_2, x_3, x_4) on the hypersphere u to the point (x_1', x_2', x_3', x_4') in which the orthogonal trajectory through (x_1, x_2, x_3, x_4) meets the consecutive hypersphere, is given by

$$x_1' = x_1 + \frac{dx_1}{du}\delta u, \quad \text{etc.}$$

Substituting the values of $\frac{dx_1}{du}, \frac{dx_2}{du}, \frac{dx_3}{du}, \frac{dx_4}{du}$, we obtain the relations

$$x_1' = x + 2\varkappa\,\delta u\,(f x_1 + \alpha_1)\,[(f\alpha_1' - f'\alpha_1)\,x_1 + (f\alpha_2' - f'\alpha_2)\,x_2 + (f\alpha_3' - f'\alpha_3)\,x_3 + (f\alpha_4' - f'\alpha_4)\,x_4 + f\phi' - f'\phi],$$

where M has been simplified by means of equation (1).

Subtracting the expressions of type

$$\varkappa\,\delta u\,(f\alpha_1' - f'\alpha_1)\,[f(x_1^2 + x_2^2 + x_3^2 + x_4^2) + 2\alpha_1 x_1 + 2\alpha_2 x_2 + 2\alpha_3 x_3 + 2\alpha_4 x_4 + 2\phi] \equiv 0$$

from these equations, we obtain the relations

$$\begin{aligned} x_1' = x + \varkappa\delta u\,[&f(f\alpha_1' - f'\alpha_1)(x_1^2 - x_2^2 - x_3^2 - x_4^2) \\ &+ 2f(f\alpha_2' - f'\alpha_2)x_1x_2 + 2f(f\alpha_3' - f'\alpha_3)x_1x_3 \\ &+ 2f(f\alpha_4' - f'\alpha_4)x_1x_4 + 2f(f\phi' - f'\phi)x_1 \\ &+ 2f(\alpha_1\alpha_2' - \alpha_2\alpha_1')x_2 + 2f(\alpha_1\alpha_3' - \alpha_3\alpha_1')x_3 \\ &+ 2f(\alpha_1\alpha_4' - \alpha_4\alpha_1')x_4 + 2f(\alpha_1\phi' - \phi\alpha_1'), \text{ etc.} \end{aligned}$$

Comparing these with the general equations* of an infinitesimal conformal transformation of S_4, viz.:

$$\begin{aligned} x_1' &= x_1 + \varepsilon\,[p(x_1^2 - x_2^2 - x_3^2 - x_4^2) + 2qx_1x_2 + 2rx_1x_3 + 2sx_1x_4 \\ &\qquad + \mu x_1 + hx_2 + gx_4 + lx_4 + a], \\ x_2' &= x_2 + \varepsilon\,[q(x_2^2 - x_3^2 - x_4^2 - x_1^2) + 2px_1x_2 + 2rx_2x_3 + 2sx_2x_4 \\ &\qquad - hx_1 + \mu x_2 + fx_3 + mx_4 + b], \\ x_3' &= x_3 + \varepsilon\,[r(x_3^2 - x_4^2 - x_1^2 - x_2^2) + 2px_1x_3 + 2qx_2x_3 + 2sx_3x_4 \\ &\qquad - gx_1 - fx_2 + \mu x_3 + nx_4 + c], \\ x_4' &= x_4 + \varepsilon\,[s(x_4^2 - x_1^2 - x_2^2 - x_3^2) + 2px_1x_4 + 2qx_2x_4 + 2rx_3x_4 \\ &\qquad - lx_1 - mx_2 - nx_3 - \mu x_4 + d], \end{aligned}$$

we see that the transformation from one hypersphere to the next can be effected by means of an infinitesimal conformal transformation of S_4.†

This result is important because we know that a conformal transformation of S_4 transforms a sphere into a sphere and a circle into a circle. Hence:

THEOREM VIII. *If a tube of orthogonal trajectories of the hyperspheres $u = \text{const.}$ cuts one hypersphere in a sphere, it will cut every hypersphere in a sphere; if it cuts one hypersphere in a circle, it will cut every hypersphere in a circle; if it cuts one hypersphere in an isotropic line, it will cut every hypersphere in an isotropic line.*‡

It follows from this theorem that, if the points of a material system move in such a way that their paths are represented in S_4 by the orthogonal trajectories of the hyperspheres $u = \text{const.}$, then, if the view determined by one constant value of u is such that the material system is represented by spheres or circles in S_4, then the material system is represented by spheres or circles in S_4 in any other view given by an equation $u = \text{const.}$

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* See p. 243 of my paper on the transformation of the electrodynamical equations, *Proc. London Math. Soc.*, Vol. VIII (1910), Ser. 2, Part 4.

† This theorem is practically proved in Darboux's Leçons sur les systèmes orthogonaux, t. I (1898), Ch. II.

‡ There is a similar theorem for spheres, circles and isotropic lines in a space of three dimensions.

In the particular case when the loci $u = \text{const.}$ are hyperplanes, the figures which are cut out on each by a family of orthogonal trajectories may be derived from one another by displacements in S_4. This is the case considered by Born and Herglotz.

The case when the loci $u = \text{const.}$ are isotropic cones (*i. e.*, hyperspheres of zero radius) is particularly interesting, for then the tangent to an orthogonal trajectory at a point P is also a generator to the isotropic cone through P. For, if the isotropic cones are

$$[x_1 - \xi_1(u)]^2 + [x_2 - \xi_2(u)]^2 + [x_3 - \xi_3(u)]^2 + [x_4 - \xi_4(u)]^2 = 0,$$

the orthogonal trajectories are

$$\frac{dx_1}{x_1 - \xi_1} = \frac{dx_2}{x_2 - \xi_2} = \frac{dx_3}{x_3 - \xi_3} = \frac{dx_4}{x_4 - \xi_4}.$$

Putting $\quad x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict,$

$$\xi_1(u) = \xi(\tau), \quad \xi_2(u) = \eta(\tau), \quad \xi_3(u) = \zeta(\tau), \quad \xi_4(u) = \tau,$$

we obtain the equations

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 = c^2[t - \tau]^2,$$

$$\frac{dx}{x - \xi} = \frac{dy}{y - \eta} = \frac{dz}{z - \zeta} = \frac{dt}{t - \tau}.$$

The first equation shows that a particle at (x, y, z) at time t is active relative to (ξ, η, ζ, τ), or vice versa, and the second equation shows that it is moving with the velocity of light along the line joining the two particle.

Theorem VIII may be generalized by considering the orthogonal trajectories of a singly infinite family of spheres in S_4. We shall show that the orthogonal trajectories map the spheres on one another in such a way that the transition from a figure on one sphere to the corresponding figure on another may be effected by means of a conformal transformation of the space S_4.

Let the axes of coordinates be chosen so that the equations of one of the spheres are $\quad x_1^2 + x_2^2 + x_3^2 = a^2, \quad x_4 = 0.$

Let a consecutive point on the orthogonal trajectory through the point (al_1, al_2, al_3) be given by the equations

$$x_1' = l_1(a + \delta u), \qquad x_3' = l_3(a + \delta u),$$
$$x_2' = l_2(a + \delta u), \qquad x_4' = \theta\, \delta u,$$

and let the equations of the consecutive sphere be

$$(x_1 + \delta\alpha_1)^2 + (x_2 + \delta\alpha_2)^2 + (x_3 + \delta\alpha_3)^2 + (x_4 + \delta\alpha_4)^2 = (a + \delta a)^2,$$
$$x_1\, \delta\xi_1 + x_2\, \delta\xi_2 + x_3\, \delta\xi_3 + x_4(1 + \delta\rho) = \delta p.$$

Since these equations are satisfied by (x_1', x_2', x_3', x_4'), we have

$$l_1(l_1\delta u + \delta\alpha_1) + l_2(l_2\delta u + \delta\alpha_2) + l_3(l_3\delta u + \delta\alpha_3) = \delta a,$$
$$a[l_1\delta\xi_1 + l_2\delta\xi_2 + l_3\delta\xi_3] + \theta\delta u = \delta p,$$

where quantities of the first order only are retained. Since $l_1^2 + l_2^2 + l_3^2 = 1$, the first equation gives

$$\delta u = \delta a - l_1\delta\alpha_1 - l_2\delta\alpha_2 - l_3\delta\alpha_3.$$

The equations of transformation may now be written in the form

$$x_1' = x_1 + l_1(\delta a - l_1\delta\alpha_1 - l_2\delta\alpha_2 - l_3\delta\alpha_3) + \frac{\delta\alpha_1}{2a^2}[x_1^2 + x_2^2 + x_3^2 + x_4^2 - a^2],$$

$$x_2' = x_2 + l_2(\delta a - l_1\delta\alpha_1 - l_2\delta\alpha_2 - l_3\delta\alpha_3) + \frac{\delta\alpha_2}{2a^2}[x_1^2 + x_2^2 + x_3^2 + x_4^2 - a^2],$$

$$x_3' = x_3 + l_3(\delta a - l_1\delta\alpha_1 - l_2\delta\alpha_2 - l_3\delta\alpha_3) + \frac{\delta\alpha_3}{2a^2}[x_1^2 + x_2^2 + x_3^2 + x_4^2 - a^2],$$

$$x_4' = \delta p - a l_1\delta\xi_1 - a l_2\delta\xi_2 - a l_3\delta\xi_3 + \frac{\delta a}{a}x_4 + x_4,$$

where multiples of the zero quantities $x_1^2 + x_2^2 + x_3^2 + x_4^2 - a^2$, x_4 have been added.

Putting $a l_1 = x_1$, $a l_2 = x_2$, $a l_3 = x_3$, we have

$$x_1' = x_1 + \frac{\delta\alpha_1}{2a^2}(x_2^2 + x_3^2 + x_4^2 - x_1^2) - \frac{\delta\alpha_2}{a^2}x_1x_2 - \frac{\delta\alpha_3}{a^2}x_1x_3 + \frac{\delta a}{a}x_1 + x_4\delta\xi_1 - \frac{\delta\alpha_1}{2},$$

$$x_2' = x_2 + \frac{\delta\alpha_2}{2a^2}(x_3^2 + x_4^2 + x_1^2 - x_2^2) - \frac{\delta\alpha_1}{a^2}x_1x_2 - \frac{\delta\alpha_3}{a^2}x_2x_3 + \frac{\delta a}{a}x_2 + x_4\delta\xi_2 - \frac{\delta\alpha_2}{2},$$

$$x_3' = x_3 + \frac{\delta\alpha_3}{2a^2}(x_4^2 + x_1^2 + x_2^2 - x_3^2) - \frac{\delta\alpha_1}{a^2}x_1x_3 - \frac{\delta\alpha_2}{a^2}x_2x_3 + \frac{\delta a}{a}x_3 + x_4\delta\xi_3 - \frac{\delta\alpha_3}{2},$$

$$x_4' = x_4 - x_1\delta\xi_1 - x_2\delta\xi_2 - x_3\delta\xi_3 + x_4\frac{\delta a}{a}.$$

Now, $\delta\alpha_1, \delta\alpha_2, \delta\alpha_3, \delta\xi_1, \delta\xi_2, \delta\xi_3, \delta a$ depend only on the two spheres and not on the coordinates of a particular point (x_1, x_2, x_3, x_4) on the first sphere; hence they may be regarded as constants as far as the transformation is concerned. This being the case, it is easy to see that the above equations represent an infinitesimal conformal transformation of the space S_4, and since such a transformation transforms a circle into a circle, we have the following theorem:

THEOREM IX. *The orthogonal trajectories of a family of ∞^1 spheres in a space S_4 map the spheres conformally on one another in such a way that a circle is always mapped on a circle and an isotropic line on an isotropic line.*

These theorems are of considerable interest in connection with the problem of finding solutions of

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2},$$

having the general form

$$V = W F(X, Y, Z),$$

where F is any solution of a partial differential equation of the second order in the independent variables X, Y, Z.

I have shown elsewhere* that this problem can be reduced to that of determining a transformation

$$X = f_1(x_1, x_2, x_3, x_4), \quad Y = f_2(x_1, x_2, x_3, x_4), \quad Z = f_3(x_1, x_2, x_3, x_4),$$

such that there is an identical relation of the form

$$(p_1^2 + p_2^2 + p_3^2 + p_4^2)(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) - (p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4)^2 \equiv l dX^2 + m dY^2 + n dZ^2, \quad (1)$$

where l, m, n are functions of X, Y, Z only, and x_1, x_2, x_3, x_4 are written instead of x, y, z, ict.

I have also shown that the necessary and sufficient conditions for the existence of a relation of this type are that the following set of equations should be satisfied.

$$\left.\begin{aligned}
&\frac{\partial p_2}{\partial x_3} + \frac{\partial p_3}{\partial x_2} = \xi_3 p_2 + \xi_2 p_3, \quad \frac{\partial p_4}{\partial x_1} + \frac{\partial p_1}{\partial x_4} = \xi_4 p_1 + \xi_1 p_4,\\
&\frac{\partial p_3}{\partial x_1} + \frac{\partial p_1}{\partial x_3} = \xi_1 p_3 + \xi_3 p_1, \quad \frac{\partial p_4}{\partial x_2} + \frac{\partial p_2}{\partial x_4} = \xi_4 p_2 + \xi_2 p_4,\\
&\frac{\partial p_1}{\partial x_2} + \frac{\partial p_2}{\partial x_1} = \xi_2 p_1 + \xi_1 p_2, \quad \frac{\partial p_4}{\partial x_3} + \frac{\partial p_3}{\partial x_4} = \xi_4 p_3 + \xi_3 p_4,\\
&\frac{\partial p_1}{\partial x_1} - \frac{\partial p_2}{\partial x_2} - \frac{\partial p_3}{\partial x_3} - \frac{\partial p_4}{\partial x_4} = 2\xi_1 p_1,\\
&\frac{\partial p_2}{\partial x_2} - \frac{\partial p_3}{\partial x_3} - \frac{\partial p_4}{\partial x_4} - \frac{\partial p_1}{\partial x_1} = 2\xi_2 p_2,\\
&\frac{\partial p_3}{\partial x_3} - \frac{\partial p_4}{\partial x_4} - \frac{\partial p_1}{\partial x_1} - \frac{\partial p_2}{\partial x_2} = 2\xi_3 p_3,\\
&\frac{\partial p_4}{\partial x_4} - \frac{\partial p_1}{\partial x_1} - \frac{\partial p_2}{\partial x_2} - \frac{\partial p_3}{\partial x_3} = 2\xi_4 p_4,
\end{aligned}\right\} \quad (2)$$

where $\xi_1, \xi_2, \xi_3, \xi_4$ are certain functions defined by the last four equations.

* *Cambr. Phil. Trans.*, Vol. XXI (1910), pp. 257–280.

It should be remarked that these equations give the solution of the more general problem of finding solutions of

$$\frac{\partial}{\partial x}\left(K\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial V}{\partial z}\right)=\frac{1}{c^2}\frac{\partial}{\partial t}\left(K\frac{\partial V}{\partial t}\right) \qquad (3)$$

which have the required form. It does not follow that, when the conditions are satisfied, the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \qquad (4)$$

possesses solutions of the required type, but if it does possess solutions of the required type, the conditions must be satisfied, and so in searching for these solutions we must obtain the general solution of the differential equations (2). It should be remarked that the equation (3) possesses the same characteristics as (4), and so also corresponds to a type of wave motion in which disturbances are propagated with the velocity of light.

We shall now obtain the solution of the equations (2) in two cases, first when $p_1\,dx_1+p_2\,dx_2+p_3\,dx_3+p_4\,dx_4$ can be reduced to the form $U\,du$, and secondly, when it can be reduced to the form $U\,du+V\,dv$. These are the only cases which need be considered.

Using suffixes to denote differentiations with regard to the variables x_1, x_2, x_3, x_4, we have in the first case

$$p_1=Uu_1,\quad p_2=Uu_2,\quad p_3=Uu_3,\quad p_4=Uu_4.$$

The equations (2) may be written in the form

$$2\,Uu_{23}=(U\xi_3-U_3)\,u_2+(U\xi_2-U_2)\,u_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$U(u_{11}-u_{22}-u_{33}-u_{44})=2\,(U\xi_1-U_1)\,u_1+(U_1u_1+U_2u_2+U_3u_3+U_4u_4),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

and imply that there is an identical relation of the form

$$\begin{aligned}2\,U[&u_{11}\,dx_1^2+u_{22}\,dx_2^2+u_{33}\,dx_3^2+u_{44}\,dx_4^2+2\,u_{23}\,dx_2\,dx_3+2\,u_{31}\,dx_3\,dx_1\\&+2\,u_{12}\,dx_1\,dx_2+2\,u_{14}\,dx_1\,dx_4+2\,u_{24}\,dx_2\,dx_4+2\,u_{34}\,dx_3\,dx_4]\\\equiv[&dx_1^2+dx_2^2+dx_3^2+dx_4^2]\,[U(u_{11}+u_{21}+u_{33}+u_{44})\\&+(U_1u_1+U_2u_2+U_3u_3+U_4u_4)]+2\,[(U\xi_1-U_1)\,dx_1\\&+(U\xi_2-U_2)\,dx_2+(U\xi_3-U_3)\,dx_3+(U\xi_4-U_4)\,dx_4]\\&[u_1\,dx_1+u_2\,dx_2+u_3\,dx_3+u_4\,dx_4].\end{aligned}$$

Regarding (dx_1, dx_2, dx_3, dx_4) as current coordinates in the neighborhood of a point (x_1, x_2, x_3, x_4) on the manifold $u = \text{const.}$, we see from the above equation that the indicatrix (*i. e.*, the section by a hyperplane parallel to $u_1 dx_1 + u_2 dx_2 + u_3 dx_3 + u_4 dx_4 = 0$) is a sphere at all points of the hypersurface. Now this property is characteristic of a hypersphere and also of a type of isotropic developable. The latter need not be taken into consideration, because it is specified by an equation of the type

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 0,$$

and this would give $p_1^2 + p_2^2 + p_3^2 + p_4^2 = 0$. Hence we may conclude that the hypersurfaces $u = \text{const.}$ are hyperspheres. Conversely, it may be shown that if the manifolds $u = \text{const.}$ are hyperspheres, there is an identical relation of the form

$$(p_1^2 + p_2^2 + p_3^2 + p_4^2)(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) - (p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4)^2 = l dX^2 + m dY^2 + n dZ^2.$$

For, it is known that any family of hyperspheres forms part of a quadruply infinite system of manifolds.* Let these be $u = \text{const.}$, $X = \text{const.}$, $Y = \text{const.}$, $Z = \text{const.}$; then there will be a relation of the form

$$dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = \lambda du^2 + \mu [l dX^2 + m dY^2 + n dZ^2].$$

Since, moreover, the orthogonal trajectories map the manifolds $u = \text{const.}$ conformally on one another, we can choose μ so that l, m, n are independent of u. Dividing out by μ, the equation takes the form

$$\frac{1}{\mu}[dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2] - \frac{\lambda}{\mu}[u_1 dx_1 + u_2 dx_2 + u_3 dx_3 + u_4 dx_4]^2 = l dX^2 + m dY^2 + n dZ^2,$$

and since the Jacobian of the quadratic form on the left is zero, the quantity on the left-hand side must be expressible in the form

$$(p_1^2 + p_2^2 + p_3^2 + p_4^2)(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) - (p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4)^2.$$

This proves the theorem.

In the second case, when

$$p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4 = U du + V dv,$$

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* Cf. Darboux, Leçons sur les systèmes orthogonaux, Ch. II.

the equations (2) become

$$2Uu_{23}+2Vv_{23}=(U\xi_3-U_3)u_2+(U\xi_2-U_2)u_3+(V\xi_3-V_3)v_2+(V\xi_2-V_2)v_3,$$
$$U(u_{11}-u_{22}-u_{33}-u_{44})+V(v_{11}-v_{22}-v_{33}-v_{44})$$
$$=2(U\xi_1-U_1)u_1+2(V\xi_1-V_1)v_1$$
$$+(U_1u_1+U_2u_2+U_3u_3+U_4u_4)+(V_1v_1+V_2v_2+V_3v_3+V_4v_4),$$

and these imply that there is an identical relation of the form

$$2U[u_{11}dx_1^2+\ldots.]+2V[v_{11}dx_1^2+\ldots.]$$
$$=[dx_1^2+dx_2^2+dx_3^2+dx_4^2][U(u_{11}+u_{22}+u_{33}+u_{44})+V(v_{11}+v_{22}+v_{33}+v_{44})$$
$$+(U_1u_1+U_2u_2+U_3u_3+U_4u_4)+(V_1v_1+V_2v_2+V_3v_3+V_4v_4)]$$
$$+2[(U\xi_1-U_1)dx_1+(U\xi_2-U_2)dx_2+(U\xi_3-U_3)dx_3+(U\xi_4-U_4)dx_4]$$
$$[u_1dx_1+u_2dx_2+u_3dx_3+u_4dx_4]$$
$$+2[(V\xi_1-V_1)dx_1+(V\xi_2-V_2)dx_2+(V\xi_3-V_3)dx_3+(V\xi_4-V_4)dx_4]$$
$$[v_1dx_1+v_2dx_2+v_3dx_3+v_4dx_4].$$

Regarding (dx_1, dx_2, dx_3, dx_4) as current coordinates of a point in the neighborhood of the point (x_1, x_2, x_3, x_4) on the surface $u=\text{const.}$, $v=\text{const.}$, we see that the indicatrix of this surface, *i. e.*, the section by a plane parallel to the tangent plane

$$u_1dx_1+u_2dx_2+u_3dx_3+u_4dx_4=0,$$
$$v_1dx_1+v_2dx_2+v_3dx_3+v_4dx_4=0,$$

is a *circle* at every point of the surface. Hence, we may conclude that the surface is a *sphere*. This gives the following theorem:

If there is an identical relation of the form (1), *where*

$$p_1dx_1+p_2dx_2+p_3dx_3+p_4dx_4=Udu+Vdv,$$

the manifolds $u=const.$, $v=const.$ *intersect in a sphere or a number of spheres.*

In the motion represented by the orthogonal trajectories of a family of hyperspheres or spheres in S_4, the component velocities (w_x, w_y, w_z) of the particle which is at the point (x, y, z) at time t and is represented by (x_1, x_2, x_3, x_4) in S_4, are given by the equations

$$\frac{w_x}{p_1}=\frac{w_x}{p_2}=\frac{w_x}{p_3}=\frac{ic}{p_4},$$

and so the relation

$$p_1^2+p_2^2+p_3^2+p_4^2\neq 0$$

implies that $w_x^2 + w_y^2 + w_z^2 \neq c^2$, or that the velocity of the particle is not equal to the velocity of light. It was shown in my former paper that there is an integral invariant of the form

$$(p_1^2+p_2^2+p_3^2+p_4^2)\,[p_1\,dx_2\,dx_3\,dx_4 + p_2\,dx_3\,dx_1\,dx_4 + p_3\,dx_1\,dx_2\,dx_4 - p_4\,dx_1\,dx_2\,dx_3] \equiv \sqrt{lmn}\,dX\,dY\,dZ,$$

or

$$\rho\,[w_x\,dy\,dz\,dt + w_y\,dz\,dx\,dt + w_z\,dx\,dy\,dt - dx\,dy\,dz] = \sqrt{lmn}\,dX\,dY\,dZ.$$

The interpretation of this equation is that the *electric charge** associated with a number of particles remains constant during the motion, *i. e.*, as we pass from one view to another. In particular, if an electron is represented by a sphere in S_4, the electric charge associated with the electron will remain constant.

Again, the equation

$$\frac{\partial}{\partial x_1}[p_1(p_1^2+p_2^2+p_3^2+p_4^2)] + \frac{\partial}{\partial x_2}[p_2(p_1^2+p_2^2+p_3^2+p_4^2)] + \frac{\partial}{\partial x_3}[p_3(p_1^2+p_2^2+p_3^2+p_4^2)] + \frac{\partial}{\partial x_4}[p_4(p_1^2+p_2^2+p_3^2+p_4^2)] = 0,$$

which was obtained on p. 269 of my former paper, takes the form

$$\frac{\partial}{\partial x}(\rho w_x) + \frac{\partial}{\partial y}(\rho w_y) + \frac{\partial}{\partial z}(\rho w_z) + \frac{\partial \rho}{\partial t} = 0,$$

and implies that the equation of continuity is satisfied.

§ 7.

It should be remarked that there is a similar set of theorems connected with Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

and the more general equation

$$\frac{\partial}{\partial x}\left(K\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial V}{\partial z}\right) = 0.$$

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* See p. 229 of my paper on the transformation of the electrodynamical equations, *Proc. London Math. Soc.*, Vol. VIII (1910).

If we seek a set of solutions of the form*

$$V = Wf(X, Y),$$

we find that there must be an identical relation of the form

$$(p^2+q^2+r^2)(dx^2+dy^2+dz^2)-(p\,dx+q\,dy+r\,dz)^2 = a\,dX^2+b\,dY^2+2h\,dX\,dY,$$

where a, b, h are functions of X and Y. The functions p, q, r are determined by differential equations of the form

$$\frac{\partial Q}{\partial z}+\frac{\partial R}{\partial y}=\zeta Q+\eta R, \qquad \frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}-\frac{\partial R}{\partial z}=2\xi P,$$

$$\frac{\partial R}{\partial x}+\frac{\partial P}{\partial z}=\xi R+\zeta P, \qquad \frac{\partial Q}{\partial y}-\frac{\partial R}{\partial z}-\frac{\partial P}{\partial x}=2\eta Q,$$

$$\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}=\eta P+\xi Q, \qquad \frac{\partial R}{\partial z}-\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}=2\zeta R,$$

where

$$P=\frac{p}{\sqrt{p^2+q^2+r^2}}, \quad Q=\frac{q}{\sqrt{p^2+q^2+r^2}}, \quad R=\frac{r}{\sqrt{p^2+q^2+r^2}},$$

and ξ, η, ζ are certain functions which may be defined by means of three of these equations.

Since any quadratic form in two variables, whose discriminant is not zero, may be reduced to the form $\lambda[d\bar{X}^2+d\bar{Y}^2]$, we see that the above equations represent the necessary and sufficient conditions that there should be an identity of the form

$$(p^2+q^2+r^2)(dx^2+dy^2+dz^2)-(p\,dx+q\,dy+r\,dz)^2=\lambda[d\bar{X}^2+dY^2].$$

It can now be shown in the same way as in § 6 that, if $p\,dx+q\,dy+r\,dz$ can be reduced to the form $U\,du$, the surfaces $u=$ const. are spheres, and if it can be reduced to the form $U\,du+V\,dv$, the surfaces $u=$ const., $v=$ const. intersect in circles. Also we have the following theorems, both of which are well-known:

The orthogonal trajectories of a family of ∞^1 spheres map the spheres conformally on one another, a circle being mapped into a circle and an isotropic line into an isotropic line.

* These solutions have been considered from a different point of view by U. Amaldi, *Rend. Palermo*, Vol. XVI (1902), p. 1, and less generally by Levi Civita, *Mem. dell' Acc. di Torino*, s. II, t. XLIX (1899).

The orthogonal trajectories of a family of ∞^1 circles map the circles conformally on one another; i. e., the cross-ratio of four points on one circle is equal to the cross-ratio of four corresponding points on another.

The last theorem occurs in the theory of cyclic systems, and the present theory shows that the theory of cyclic systems should be connected with that of Laplace's equation. We shall not investigate this relation in the present paper, but it may be useful to state the theorem corresponding to the case when $p\,dx + q\,dy + r\,dz = U\,du$, and $u = \text{const.}$ represents a family of spheres.

THEOREM. *If u is defined by the equation*

$$x^2 + y^2 + z^2 + 2\alpha(u)\,x + 2\beta(u)\,y + 2\gamma(u)\,z + 2\phi(u) = 0,$$

then

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right](dx^2 + dy^2 + dz^2) - \left[\frac{\partial u}{\partial x}dx + \frac{du}{\partial y}dy + \frac{\partial u}{\partial z}\partial z\right]^2$$

can be reduced to the form $\lambda(dX^2 + dY^2)$, and any function F which satisfies the equation

$$\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} = 0$$

is a solution of the two equations

$$\frac{\partial}{\partial x}\left(K\frac{\partial F}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial F}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial F}{\partial z}\right) = 0,$$

$$\frac{\partial u}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial F}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial F}{\partial z} = 0,$$

where

$$K = \frac{\chi(u)}{x\alpha'(u) + y\beta'(u) + z\gamma'(u) + \phi'(u)},$$

and $\chi(u)$ is an arbitrary function. If $\chi(u)$ can be chosen so that $\sqrt{K}$ is a solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$, then $V = \sqrt{K}.F(X, Y)$ will also be a solution of this equation.

§ 8.

The group G_{15} of conformal transformations of the space S_4, which is equivalent to the group of spherical wave transformations, is composed of displacements, reflexions and inversions in the space S_4 and of transformations

which may be obtained by compounding these. Thus magnifications and rotations enter as particular transformations of the group. The rotations are of great importance, because, as Minkowski has shown, Lorentz's transformation *

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad z' = z, \quad y' = y$$

corresponds to a rotation

$$x_1' = x_1 \cos\omega + x_4 \sin\omega, \qquad x_2' = x_2, \qquad \tan\omega = \frac{iv}{c}$$
$$x_4' = x_4 \cos\omega - x_1 \sin\omega, \qquad x_3' = x_3,$$

in the four-dimensional space S_4.

When we use Darboux's representation of a point in S_4 by a spherical wave in S_3, the group G_{15} becomes the group of spherical wave transformations which transform a spherical wave into a spherical wave. This group of transformations has been discussed by S. Lie; † it is the group of transformations which transform lines of curvature on a surface enveloped by spherical waves into lines of curvature on the surface enveloped by the corresponding spherical waves.

An inversion of one set of spherical waves into another is a particular instance of a spherical wave transformation, and this corresponds to an inversion in S_4. A dilatation in which the radii of all the spherical waves are increased or diminished by the same amount is another example, and this corresponds to a displacement parallel to the axis of x_4 in the space S_4.

Ribaucour's transformation, which transforms lines of curvature into lines of curvature, is easily seen to correspond to Lorentz's transformation, for, if we take the equations given on p. 254 of Darboux's "Théorie des Surfaces," t. 1, viz.,

$$x' = x, \qquad z' = \frac{1 + k^2}{1 - k^2} z - \frac{2kR}{1 - k^2},$$
$$y' = y, \qquad R' = \frac{2kz}{1 - k^2} - \frac{1 + k^2}{1 - k^2} R,$$

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* *Amsterdam Proceedings*, 1903–04.

† *Gött. Nachr.* (1871). "Transformationsgruppen," Bd. 3, p. 351. "Infinitesimale Berührungstransformationen der Optik." *Leipziger Berichte*, Math.-Phys. Classe, Bd. 48 (1896), pp. 131–133. See also papers by E. O. Lovett, "Contact Transformations and Optics," *Cambr. Phil. Soc.*, Commemoration Volume (1900), p. 256, and P. F. Smith, *Trans. Amer. Math. Soc.*, Vol. 1 (1900), p. 371.

and write ct, ct' in place of R and R', and $\frac{v}{c}$ in place of $\frac{2k}{1-k^2}$, we obtain at once a form of Lorentz's transformation which is appropriate for the case of a uniform motion parallel to the axis of z.

On account of the importance of the transformation theory in electrodynamics, it is desirable to study the group G_{15} of spherical wave transformations in full detail. This may be done very conveniently by means of the following theorem, which is due to Klein:*

The general conformal group in four dimensions is simply isomorphic with the general projective group in three.

This theorem depends on Lie's correspondence between a line and a spherical wave. A projective transformation which transforms a line into a line corresponds to a spherical wave transformation which transforms a spherical wave into a spherical wave.

Spherical wave transformations may thus be studied in connection with projective transformations in S_3. The following examples will illustrate this:

(1) In the general projective transformation in S_3 there are four points and six lines joining them which are transformed into themselves. Now a point in S_3 corresponds to a minimal line. Hence, in the general spherical wave transformation there are four isotropic lines which are transformed into themselves; the six spheres containing these lines in pairs are also transformed into themselves. Each of these spheres touches four of the others.

(2) In a homology in S_3 there is a point (the centre of homology) and a plane (the plane of homology) such that every line through the point and every line in the plane is transformed into itself. In the corresponding spherical wave transformation there are two isotropic lines such that any spherical wave through one of them is transformed into itself.

(3) In a skew reflexion the line joining two corresponding points P, Q meets two fixed lines r, s, and is divided harmonically by them. The line PQ is clearly transformed into itself. In the corresponding spherical wave transformation there are two spherical waves R, S which are transformed into themselves, and every wave touchig R and S is transformed into itself.

(4) If a, b, c, d are four non-intersecting straight lines, a projective transformation which will interchange a and c, b and d may be obtained in the

* *Höhere Geometrie*, Vol. I, p. 487, sqq. See also a paper by J. E. Wright, *Trans. Amer. Math. Soc.* (1906).

following way: Draw lines PAB, PCD through an arbitrary point P to meet a, b; c, d in A, B; C, D respectively. Let AD, BC meet in Q; then Q is the point corresponding to P. It is clear that when P is on a, Q will be on c, and when P is on b, Q will be on d. If the common transversals of the four lines a, b, c, d meet them in $\alpha, \beta, \gamma, \delta$; $\alpha', \beta', \gamma', \delta'$, and points X, Y; Z, W are taken on $(\alpha\,\beta\,\gamma\,\delta)$, $(\alpha'\,\beta'\,\gamma'\,\delta')$ respectively, so that $(\alpha\,\gamma\,X\,Y)$, $(\beta\,\delta\,X\,Y)$; $(\alpha'\,\gamma'\,Z\,W)$, $(\beta'\,\delta'\,Z\,W)$ are harmonic ranges, then X, Y, Z, W are the fixed points of this projective transformation.

The corresponding spherical wave transformation is such that a spherical wave a is transformed into c and another wave b into d. This transformation shows that when we are dealing with the geometrical properties of four spherical waves, the waves may be interchanged in pairs and a corresponding property will still be possessed by the system, provided the property is covariantive for the group of spherical wave transformations.

BRYN MAWR, *December* 24, 1910.

Simple Groups from Order 2001 to Order 3640.

By L. P. Siceloff.

1. The groups of prime order, the alternating groups, and other infinite systems * of finite simple groups include most, but not all, of the known finite simple groups. All of the simple groups having orders less than 2001 have been determined;† but beyond this very limited range a problem of fundamental importance still presents itself: to determine precisely for what orders simple groups exist, and to determine every type of simple group for each such order.

In the field here covered, from order 2001 to order 3640, it turns out that the only possible orders of simple groups are the well-known cases, viz., there is one and only one ‡ simple group of each of the orders 2448, 3420, and there is one of order $2520 = 7! \div 2$. Whether there is more than one of order 2520 has not been determined.

2. In giving a brief account of the methods employed in this paper, it is to be observed at the outset that the procedure is, in general, negative in character: to determine whether there is a simple group of a given order, the initial attack is an attempt to determine whether every group of this order must be composite. This is, of course, due to the fact that the simple group is defined by exclusion: it must not have an invariant subgroup.

In considering whether every group G of given order $g = p^m q^n \ldots$ (p, q, ..., distinct primes) must have an invariant subgroup, we turn our attention to the subgroups that are known to exist for every group of that order. And since the groups of orders p^m, q^n, ..., with their subgroups, are the only subgroups of G whose existence is independent of the type of G,

* For a list, *Bull. Am. Math. Soc.*, 5 (2 ser.), p. 470.

† To order 200 by Hölder, *Math. Annalen*, XL, 55–58; to 660 by Cole, Am. Jour. Math., 14, 378–388, and 15, 303–315; to 1092 by Burnside, *Proc. Lon. Math. Soc.*, 26, 333–338; to 2001, by Ling and Miller, Am. Jour. Math., 22, 13–26.

‡ Frobenius, *Berliner Sitzgsb.*, 1902, 359; Burnside, "Theory of Groups," 317–335.

Sylow's theorems are indispensable for this purpose. In connection with these theorems frequent use is made of the important equation * of Maillet,

$$kp + 1 = 1 + x_1 p + x_2 p^2 + \ldots + x_m p^m,$$

in which $kp + 1$ is the number of subgroups of order p^m in G, and $x_r p^r$ is the number of these subgroups having in commom with a specified one of them precisely p^{m-r} elements. Note, in particular, that if k is not divisible by p, then $x_1 \neq 0$; *i.e.*, some of the subgroups of order p^m have a common subgroup of order p^{m-1}. Of considerable importance also, in this problem, is the well-known isomorphism between a simple group and a group of substitutions on a conjugate set of its subgroups or elements.

Certain other methods and theorems, useful under special conditions, will be explained in connection with their application. In the following section is proven a theorem which considerably simplifies the preliminary reduction of the list of orders in section 4.

3. THEOREM. *A simple group G of order $g = (p+1)p^a n$, where p is prime and is not a factor of n, can not contain a conjugate set of $p+1$ subgroups of order p, unless $a = 1$ and n is a factor of $p - 1$.*

For if there are $(p+1)$ H_{p^a} in G, G is simply isomorphic with a transitive G^{p+1}. Then p^a must divide $(p+1)!$, and $a = 1$. Now if G, of order $(p+1)pn$, has $(p+1)$ H_p, each H_p is invariant in an H_{pn}; and this affects p symbols when G is expressed as a group of substitutions on $p+1$ symbols.† But the symmetric G^p contains precisely $p(p-1)$ substitutions that transform a substitution of order p into its own powers. Then pn must divide $p(p-1)$.

4. We proceed now to examine the list of orders between 2001 and 3640. If the order of a group belongs to any of the following classes of numbers,‡ the group is composite.

I. Odd composite numbers § less than 40,000.

II. Numbers with fewer than six prime factors, ‖ with four exceptions.

III. Even numbers ¶ not divisible by one of the numbers 12, 16, 56.

IV. Numbers ** p^m, $p^m q^n$, $p^4 qr$, $p^2 q^2 r^2$.

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* *Ann. Fac. Sci.*, Toulouse, 9 (1895), D7. See Burnside, "Theory of Groups," 94.

† For several theorems on groups of degree p or $p+1$, consult the memoir referred to in note (‡) p. 361. See also note (§) p. 364.

‡ For a fuller list of such classes, see Easton's "Constructive Group Theory" (Ginn & Co., 1902), 82, 83. See also *Encyclopédie des Sci. Math.*, t. 1, v. 1, fas. 4, p. 590.

§ Burnside, *Proc. Lon. Math. Soc.*, 33, 268. ‖ Burnside, "Theory of Groups", 367. ¶ *Ibid.*, 365.

** On $p^m q^n$, see Burnside, *Proc. Lon. Math. Soc.*, 2 (2 ser.), 432; on $p^4 qr$, Frobenius, *Berl. Sitzgsb.*, 1901, 1329; on $p^2 q^2 r^2$, Maillet, *Quar. Jour. P. and Appl. Math.*, 29, 263.

V. Orders which, for some p and every corresponding k, do not divide $(kp+1)!$, $kp+1$ being the number of subgroups of order p^m. This is an immediate consequence of the isomorphism between the abstract group and a substitution group of degree $kp+1$.

All such numbers being excluded, there remain between 2001 and 3640 the following 45 cases requiring special examination:

$*2016 = 2^5 3^2 7$	$2520 = 2^3 3^2 5 \cdot 7$	$3120 = 2^4 3 \cdot 5 \cdot 13$
$*2040 = 2^3 3 \cdot 5 \cdot 17$	$**2640 = 2^4 3 \cdot 5 \cdot 11$	$3168 = 2^5 3^2 11$
$2080 = 2^5 5 \cdot 13$	$*2688 = 2^7 3 \cdot 7$	$*3240 = 2^3 3^4 5$
$*2100 = 2^2 3 \cdot 5^2 7$	$*2700 = 2^2 3^3 5^2$	$**3276 = 2^2 3^2 7 \cdot 13$
$**2112 = 2^6 3 \cdot 11$	$**2760 = 2^3 3 \cdot 5 \cdot 23$	$*3300 = 2^2 3 \cdot 5^2 11$
$*2160 = 2^4 3^3 5$	$**2772 = 2^2 3^2 7 \cdot 11$	$**3312 = 2^4 3^2 23$
$*2184 = 2^3 3 \cdot 7 \cdot 13$	$*2800 = 2^4 5^2 7$	$3360 = 2^5 3 \cdot 5 \cdot 7$
$**2208 = 2^5 3 \cdot 23$	$*2808 = 2^3 3^3 13$	3420 one simple
$*2240 = 2^6 5 \cdot 7$	$2880 = 2^6 3^2 5$	$*3432 = 2^3 3 \cdot 11 \cdot 13$
$2268 = 2^2 3^4 7$	$*2940 = 2^2 3 \cdot 5 \cdot 7^2$	$*3480 = 2^3 3 \cdot 5 \cdot 29$
$*2280 = 2^3 3 \cdot 5 \cdot 19$	$2976 = 2^5 3 \cdot 31$	$*3520 = 2^6 5 \cdot 11$
$**2376 = 2^3 3^3 11$	$3024 = 2^4 3^3 7$	$*3528 = 2^3 3^2 7^2$
$*2400 = 2^5 3 \cdot 5^2$	$**3040 = 2^5 5 \cdot 19$	$**3564 = 2^2 3^4 11$
2448 one simple	$**3060 = 2^2 3^2 5 \cdot 17$	$*3600 = 2^4 3^2 5^2$
$*2464 = 2^5 7 \cdot 11$	$*3080 = 2^3 5 \cdot 7 \cdot 11$	$*3640 = 2^3 3 \cdot 7 \cdot 13$

A. Orders Easily Rejected.

5. The orders marked with stars are easily shown to belong only to composite groups. In particular, a direct application of the theorem of section 3 to each of the orders marked with a double star removes these 11 numbers from our list of possibilities. Thus, a simple group of order 2112 must contain 12 $H_{11}^{11 \cdot 16}$; but this is impossible, since 16 is not a factor of 10.

We turn then to the examination of the 25 orders marked with a single star.

2016. A simple G_{2016} must have 9, 21 or 63 H_{32}; and if 63, the H_{16} common to $2k+1$ of them is invariant in a subgroup of order $32(2k+1)m$, and has then 21, 9, 7, or fewer conjugates. 7! is not divisible by 32. Thus a simple G_{2016} must be expressible as a transitive group of degree 9 or 21.

There are 8, 36 or 288 H_7. The theorem of section 3 excludes the 8; 36 $H_{7 \cdot 8}$ contain elements of order 14, and these are odd, both in the G^9 and in the G^{21}. Then there must be 288 H_7. These furnish $6 \cdot 288$ elements of order 7, and the

number n of elements whose orders* divide $7 \cdot 32$ is $n = 224x > 6 \cdot 288$; whence x can only be 8, and $n = 1792$. This leaves only $1792 - 6 \cdot 288 = 64$ elements whose orders divide 32. But 3 H_{32} with a common H_{16} furnish 64 different elements, and it is easily seen that the remaining subgroups of order 32 must contain elements other than these 64.

2040. A simple group of this order would contain 120 H_{17} and 51 or more H_5. But these furnish more than 2040 elements.

2100. A simple G_{2100} must have 21 H_{25}. An H_5 is common to 6 or 21 of these; and if to 21, the group is obviously composite. And if 6 H_{25} have a common H_5, this is invariant in a subgroup of order $6 \cdot 25m = 300$ at least ($m = 1$ is excluded because the H_{150} with its invariant H_5 gives a factor group Γ_{30}, which leads to an invariant H_{25} in H_{150}), and has therefore at most 7 conjugates. But 25 does not divide 7!.

2160. There must be 10, 16 or 40 H_{27}; there is, however,† no transitive G^{10}_{2160}. If there are 16 $H^{15}_{27 \cdot 5}$, each has an invariant H_5; but this is impossible, since $5 \cdot 27$ is not a divisor of the order $5^3 4 \cdot 3!$ of the normaliser of H_5 in the symmetric G^{15}.

Thus, there are 40 H_{27}, and the H_9 common to 4 (or more) of them has 20 conjugates (or too few), each invariant in H_{108}. But an H_{108} with 4 H_{27} has‡ an invariant H_4, which, being invariant also in an H_8, is invariant in H_{216m}, and has $10/m$ conjugates. Clearly $m = 1$; and even this fails, since there is no transitive G^{10}_{2160}.

2184. There are 14 $H^{13}_{13 \cdot 12}$; but these are metacyclic,§ and therefore half-odd.

2240. The H_{32} common to $2k + 1$ of the 35 H_{64} has at most 7 conjugates. But $7! \neq 64m$.

2280. There are 20 $H^{19}_{19 \cdot 6}$. But the metacyclic $H^{19}_{19 \cdot 18}$, in which our $H_{19 \cdot 6}$ is contained, is half-odd; and since 6 is not a factor of 9, our $H_{19 \cdot 6}$ is half-odd.

2400. The H_5 common to $5k + 1$ of the 16 or 96 H_{25} has at most (see method for order 2100) 8 conjugates. But $8! \neq 25m$.

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* In a group of order g, the number of elements whose orders divide a factor r of g is a multiple of r. Frobenius, *Berl. Sitzgsb.*, 1895, 984.

† Cole, *Quar. Jour.*, 27, 39. All primitive substitution groups as far as degree 20 have been determined. In most cases where the results of these determinations are applicable to the work of this paper, however, independent proofs are presented, chiefly for the sake of whatever light such independent procedure may throw on the problem here studied.

‡ See p. 63 of M. O. Tripp's dissertation on groups of order p^3q^2 (Columbia University, 1909).

§ The only transitive substitution groups of prime degree p and order less than p^2 are metacyclic. Every transitive group of degree p is either metacyclic or doubly transitive. See Burnside, *Quar. Jour.*, 37, 215.

2464. The greatest subgroup, of order either 16 or 8, common to 7, 11 or 77 of the $(2k+1)$ H_{32} has 22, 14, 11, or too few, conjugates. But there are 56 $H_{11\cdot 4}$, with elements of order 22; and these are either odd or non-constructible when G is expressed as transitive in 22, 14 or 11 symbols.

2688. If some of the 21 H_{128} have a common H_{64}, this has at most 7 conjugates; but $7! \neq 128m$. Then the greatest subgroup common to $2k+1$ of the 21 H_{128} is of order 32, and has 14 conjugates ($k=1$ being the only case that calls for examination). But no simply transitive G^{14} exists, of any order whatsoever.*

2700. The H_5 common to 6 or more of the 36 H_{25} has at most 9 conjugates. But $9! \neq 25m$.

2800. A common H_5 has at most 7 conjugates.

2808. There are 27 $H_{13\cdot 8}$, in each of which any H_8 transforms H_{13} into itself. But since $i^8 \equiv 1 \pmod{13}$ has only the four solutions of $i^4 \equiv 1 \pmod{13}$, viz., $i = \pm 1, \pm 5$, some two elements, s, t, of H_8 transform any element u of H_{13} into the same power, u^i. Then st^{-1} is permutable with u, and $H_{13\cdot 8}$ contains elements of order 26; these will appear as odd substitutions when G is expressed as a transitive G^{27}.

2940. An H_7 is common to all of the 15 H_{49}. Then G is composite.

3080. There are 22 $H_{7\cdot 20}$; each has one H_5. Then G has 11 (or 1) H_5, each invariant in $H^{10}_{5\cdot 56}$; but this can not be, since $5\cdot 56$ does not divide the order $5^2 2\cdot 4$, of the normaliser of H_5 in the symmetric G^{10}.

3240. The number of H_{81} is 10 or 40; and if 40, the H_{27} common to 4 (or 10 or 40) of them has 10 (or 4 or 1) conjugates. Then a simple G_{3240} can be expressed as a transitive G^{10}. But there is no† transitive G^{10}_{3240}.

3300. By the theorem of section 3 there can not be 12 H_{11}. Then there must be 100 $H_{11\cdot 3}$; they furnish 3000 elements of orders 11, 33. But then the number of elements of odd order in G is $3\cdot 5^2 11k > 3000$, whence $k=4$; *i. e.*, all the elements of G are of odd order. This is impossible.

3360. The theorem of section 3 excludes 8 H_7. If there are 15 H_7, there are substitutions of order 14, odd in the 15 symbols; then there are 120 $H_{7\cdot 4}$, and these furnish 720 elements of order 7 and at least 720 of order 14.

There are 16, 21, 56, 96 or 336 H_5. 16 $H_{7\cdot 30}$ yield only 1 or 8 H_7. The case for 21 H_5 fails because the substitutions of 14 would be odd in the G^{21}. The fifth power of a substitution of order 15 in one of the 56 $H_{5\cdot 12}$ is a substi-

* Miller, *Quar. Jour.*, 29, 242. † Cole, *Quar. Jour.*, 27, 39.

tution of order 3, omitting $n = 11$, 26, or 41 of the 56 symbols; and H_3, with 1 or 4 conjugates in the H_{60}, has in G $56m/n$ ($m = 1$ or 4) conjugates, which is impossible. Again, 96 $H_{5 \cdot 7}$ yield only 1 or 8 H_7. Thus a simple G_{3360} must have 336 H_5, giving 1344 elements of order 5. But then the number of elements whose orders divide $5 \cdot 7 \cdot 32$ is $35 \cdot 32x > 1344 + 1440$; whence $x = 3$, and the group has no elements of order 3. This is impossible.

3432. Each of the 78 $H_{11 \cdot 4}$ has an invariant H_2, which therefore omits 34 (s_{22} must be an even substitution, and has then 2 cycles of 22 symbols with 3 cycles of 11) of the 78 symbols, and has $78k/34$ conjugates. But this is impossible, since k, the number of conjugates which H_2 has, under G, in H_{44}, must be either 3 or of the form $11m + 1$ ($m < 3$).

3480. There are 30 $H_{29 \cdot 4}$; and these, being metacyclic, are half-odd, since 4 does not divide 14.

3520. There are 320 H_{11}. Then the number of elements whose orders divide $11 \cdot 64$ is $11 \cdot 64x > 3200$; whence $x = 5$, and there are no elements of order 5. This can not be.*

3528. The H_7 common to 8 or more of the 36 H_{49} has at most† 9 conjugates. But $9! \neq 49m$.

3600. The H_5 common to 6 or more of the 16 or 36 H_{25} is invariant in a subgroup of order 300 or more; and if more than 300, the H_5 has at most 9 conjugates. But $9! \neq 25m$. If the H_5 is invariant in H_{300}, it has 12 conjugates. But there is no transitive G_{3600} on 12 symbols ‡

3640. The theorem of section 3 excludes 14 H_{13}. Then there must be 40 $H_{13 \cdot 7}$, and each has an invariant H_7. The total number of H_7 in G can only be 1 or 8. But $8! \neq 13m$.

B. Orders Requiring more Detailed Examination.

6. Apart from the orders 2448, 2520 and 3420, for each of which there is a simple group, there remain in our list seven orders to be examined. That none of these numbers is the order of a simple group appears as follows.

2080. If there are 16 $H_{5 \cdot 26}$, they contain elements of order 65; but these are not expressible in 15 symbols. 416 H_5 would furnish, with 40 H_{13}, too many

*Attention is called to the fact that this and a few others of our orders could have been rejected by an important theorem of Frobenius: Every transitive G^n of class $n - 1$ has a characteristic H_n. *Berl. Sitzgsb.*, 1901, 1226.

† In Maillet's great memoir referred to in note (*), p. 362, above, is proven the following theorem, which applies to orders 2100, 2940, 3528. If, in Maillet's equation (section 2), for every $i > 1$ $x_i = 0$, all the H_{pm} have a common H_{p^m-1}, and the group is simple. See Vol. 10, A 5, of the *Annales*.

‡ Miller, *Quar. Jour.*, 28, 219.

elements. Then there must be 26 $H_{5\cdot 16}$. Each H_{80} has 5 H_{16}, since only one H_{16} in H_{80}, being invariant also in H_{32}, would be invariant in H_{160m}, and would have 13 conjugates ($m=1$); the 13 H_{160} contain each an H_{80} with 1 H_5; then each H_{160} has an invariant H_5, and G has not its necessary 26 H_5. These 5 H_{16} have in common an H_8 or only an H_4.

If the 5 H_{16} have a common H_8, they furnish 48 different elements of order 2^i. But observe the 40 $H_{13\cdot 4}$: since an s_{26} would be odd in the transitive G^{26}, each H_{52} has 13 cyclical H_4, with only identity in common. Each H_4 effects 38 symbols in G^{40}, since s_4 must be even and its square must not transform s_{13} into itself. Then H_4 has in G $40\cdot 13/2=260$ conjugates. But in the G^{26}, s_4 affects 24 symbols, since its square must transform s_{13} into s_{13}^{-1}. Let k be the number of these 260 H_4 in each H_{80}. $26k/2=260$, $k=20$. These 20 cyclical H_4 furnish 40 s_4 and 10 s_2 in each H_{80}. But this can not be, since the 5 H_{16} with a common H_8 yield only 48 elements of order 2^i.

Then the 5 H_{16} have only a common H_4. Let J denote this H_4, and a, b, c, its elements of order 2 (the H_4 can not be cyclical, since there would then be substitutions of order 20, odd in the G^{26}). a, b, c, being permutable with the elements of H_5, affect 20 symbols in the G^{26}. Then a has $26m/6$ conjugates in G, m being the number of conjugates which a has, under G, in H_{80}. Observe that m must be a multiple of 3. Now a can not be conjugate with an s_2 of the 20 cyclical H_4 in H_{80}, since such an s_2 affects 24 symbols. But these 20 H_4, with J, furnish 53 of the 63 elements of order 2^i in H_{80}. Then we must find the conjugates of a among the remaining 10 elements of the 5 H_{16}. These 10 elements must form, under H_{80}, two conjugate sets of 5 each, or else one set of 10. Thus, a may have in H_{80}, $m=5n+(1, 2, \text{ or } 3)=3$, 6, or 12 conjugates. If $m=3$, G is a doubly transitive* G^{13}. This can not be, for 12 does not divide 2080. If $m=6$ or 12, then either b or c has only one conjugate in H_{80}, and $26/6$ conjugates in G. This is impossible.

2268. In the equation

$$28=1+3x+9y+27z,$$

if $x\neq 0$, an H_{27} common to 4, 7, or 28 H_{81} has at most 7 conjugates; but $7!\neq 81m$. Therefore $x=0$, and the only solutions for the equation are (a) $x=z=0$, $y=3$, and (b) $x=y=0$, $z=1$.

Under the solution (b), each H_{81} has in common with each other H_{81} precisely 3 elements. A common H_3 belongs in 4, 7, or 28 H_{81}. Thus each H_{81} has

* See foot-note (§), p. 364.

at least 62 elements not in any other H_{81}; and the 28 H_{81} furnish more than $28 \cdot 62 = 1736$ elements. But then the number of elements whose orders divide $7 \cdot 81$ is 567 $j > 1736$, whence $j = 4$, and there are no elements of order 2. This is impossible.

Under solution (a), each H_{81} has an H_9 in common with each other H_{81}. Denote by N such an H_9. If N is invariant in an H_{81}, it is invariant in an H_{81k}, with $k \geqq 4$, and has at most 7 conjugates. This can not be. Thus N is invariant in only an H_{27} from each of the 4 or 7 H_{81} in which it lies. If N is common to 7 H_{81}, the 12 $H_{27 \cdot 7}$ yield only 1 H_7. Therefore N is common to precisely 4 H_{81}, is invariant in an $H_{27 \cdot 4g}$, and has $21/g$ conjugates. Obviously, $g = 1$, N has 21 conjugates, each invariant in an H_{108}. If N has, under G, n conjugates in each H_{81}, $28n/4 = 21$, $n = 3$. Moreover, a single H_{27} of an H_{81} contains the 3 conjugates of N that lie in that H_{81}; for an H_{27}, being invariant in an H_{81}, has 28 conjugates, and $28h/4 = 21$ gives $h = 3$. Denote by P, Q the other two conjugates of N in $T \equiv H_{27}$. N is common to the first H_{81} and three other H_{81}; P is common to the first H_{81} and a second set of three others; and similarly for Q. Now if T is abelian, the H_3 common to N and P is invariant in 2 H_{108}, and has 12, 7, or fewer conjugates. These possibilities have already been rejected. Therefore T is not abelian; it has 4 H_9 with a common H_3 which, being the only H_3 invariant in T, is the central H_3 of the first H_{81}.

The arrangement outlined above exhibits the first H_{81} as belonging to three sets of 4 H_{81}, each set having a common H_9. This arrangement involves only 10 H_{81}, and must be accompanied by two other such arrangements, in order that the first H_{81} may appear as having an H_9 in common with *every* H_{81}, as required in the case (a) under discussion. It follows that the first H_{81} has 9 H_9 lying in sets of three in three non-abelian H_{27}. But observe that any two of these 3 H_{27} have a common H_9, which contains the central H_3 of each. Both of these centrals are invariant in H_{81}, and therefore coincide, since the H_9 common to the 2 H_{27} has only 1 H_3 that is invariant in either H_{27}. Then this central H_3 is common to all the 9 H_9, and is thus common to all the 28 H_{81}. Similarly, the central of each H_{81} must be common to all the 28 H_{81}. But since 2 H_{81} must not have more than an H_9 in common, the central of one H_{81} must coincide with the central of another H_{81}; and this central is invariant in an H_{81r}, where $r \geqq 4$. Then it has at most 7 conjugates, which is impossible.

2880. There are 16, 36, 96, or 576 H_5. And there can not be 16 $H_{5 \cdot 36}$, since $5 \cdot 36$ does not divide the order $5^3 4 \cdot 3!$ of the normaliser of H_5 in the symmetric G^{15}.

If there are 96 $H_{6\cdot 5}$, an s_3 permutable with an s_5 omits $k=6, 21, 36, \ldots$, of the 96 symbols, and has $96/k$ conjugates. Then $k=6$; $\{s_2\}$ has 16 conjugates, each invariant in an H_{180}. If the corresponding factor-group Γ_{60} has only 1 H_5, H_{180} has only 1 H_5, and G has fewer than 96 H_5. Thus, Γ_{60} must have 6 H_5, and H_{180} has 6 $H_{15\cdot 2}$. Then each H_5 affects 15 symbols in the transitive G^{16}, and s_{15} is a circular substitution. s_{15} is not permutable with s_2 in H_{30}, since the resulting s_{30} would not be expressible in the 15 symbols. Therefore s_2 consists of seven transpositions and is odd, since it transforms s_{15} into s_{15}^{-1}.

There can not be 576 H_5, since the transitive G^{576} would be of class 575, and would therefore contain * a characteristic H_{576}.

Thus, if there is a simple G_{2880}, it must have 36 H_5, each invariant in an H_{80}. Each H_{80} has 5 H_{16}; for if it had only 1 H_{16}, this would be invariant in an $H_{5\cdot 32m}$ containing our H_{80}, and the $36/m$ conjugates of the H_{160m} would not yield 36 H_5. Now the greatest subgroup S common to the 5 H_{16} of H_{80} is of order 4 or 8. S contains a substitution s_2 permutable with each s_5 of H_5; this s_2 affects 20 of the 36 symbols, and has then $36n/16$ conjugates. To determine n, consider any element t of order 2 in the H_{80} but not in S; t must transform each s_5 into s_5^{-1}. Then t affects 28 symbols at least, and can not be conjugate with our s_2. Thus S contains all the conjugates of our s_2 which are in the H_{80}, when the s_2 is transformed by all the elements of G. Therefore there is in S an s_2 for which $n=1$, 3, 5, or 7, and the number, $36n/16$, of its conjugates in G is not an integer. This is impossible.

2976. The 32 $H_{31\cdot 3}$ obviously can not be cyclic. Then in our transitive G^{32}, each s_3 affects 30 symbols, and G has $32\cdot 31/2 = 496$ H_3.

There are 31 or 93 H_{32}. But a transitive G^{31} must be doubly transitive or metacyclic, both of which are impossible here. Thus there are 93 H_{32}; and if 3 (or 31) of these have a common H_{16}, this has 31 (or 3 or 1) conjugates. This can not be, since there is no transitive G^{31}_{2976}. Then there must be an H_8 common to 3 (or 31) of the 93 H_{32}. This H_8 is invariant in an $H_{3\cdot 16m}$; and if $m>1$, the H_8 has 31, 3, or 1 conjugate. Thus, $m=1$; there are 62 H_{48}, and since G has 496 H_3, each H_{48} has 16 H_3 and therefore only 1 H_{16}, which, being invariant also in an H_{32}, is invariant in an H_{96k} and has 1 or 31 conjugates. Every G_{2976} is therefore composite.

3024. In a simple G_{3024} there must be 36 $H_{7\cdot 12}$. In one of these an s_3 can not be permutable with an s_7, since $\{s_3\}$, affecting 21 symbols in G^{36}, would then have $36k/15$ conjugates, where $k=1$ or 4.

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* See foot-note (*), p. 366.

The solutions of $i^4 \equiv 1 \pmod 7$ are also the solutions of $i^2 \equiv 1 \pmod 7$, viz., ± 1. Therefore any H_4 in an H_{84} has an s_2 permutable with every element of the H_7, and the elements permutable with H_7 form a group S of order 14 or 28 in H_{84}. S is invariant in H_{84}.

If S is of order 28, it has an H_4 which is invariant in H_{84} and also in an H_8. H_4 is then invariant in an H_{168m}. If $m > 2$, H_4 has too few conjugates. And if $m = 1, 2$, then H_{168m} has only 1 H_7, so that G has fewer than its required 36 H_7.

Thus, S must be of order 14. The subgroup T of order 2 in S is invariant in H_{84}. T affects 28 of the 36 symbols. T has, under G, only one conjugate in H_{84}, since any other H_2 in H_{84} transforms an s_7 into its inverse and therefore, affecting at least 30 symbols, can not be conjugate with T. Then S has in G 36/8 conjugates, which is impossible.

3120. There must be 16, 26 or 156 H_5. 16 $H_{5 \cdot 39}$ would yield only 1 H_{13}. If there are 26 H_5, each is invariant in H_{120}, in which an s_3 permutable with an s_5 affects 15 of the 26 symbols; and $\{s_3\}$ therefore has $26k/11$ conjugates, with $k = 1$ or 4 since an H_{120} with an invariant H_5 has 1 or 4 H_3. This can not be. Thus there can only be 156 H_5.

There are 40 $H_{13 \cdot 6}$. An H_{78} contains elements of one or more of the orders 6, 26, 39. An s_{26} would be odd in our G^{40}, and if there is an s_{39} in H_{78}, then H_{78} has an s_2 which consists of 19 transpositions and is odd, since it transforms the circular s_{39} into its inverse. Thus H_{78} has an s_6. This affects 36 or 39 of the 40 symbols, and generates a cyclical H_6 with $40 \cdot 13/(1, 4) = 520$ or 130 conjugates. The case for 520 cyclical H_6 fails, since these, with the 40 H_{13} and the 156 H_5, would furnish too many elements.

Therefore, there can only be 130 cyclical H_6, 130 H_3, 130 H_2, each invariant in an H_{24}. Each H_8 in H_{24} affects an even number of the 130 symbols and therefore occurs in 2 of the 130 H_{24}. If each H_{24} has only 1 H_8, this is invariant in 2 H_{24} and therefore in a group S which contains several H_3, each invariant in an H_{24} in S. Then the order of S is 240 or more; and if more than 240, S has too few conjugates; while if S is of order 240, its 13 conjugates yield fewer than the required 156 H_5 in G. It follows that each H_{24} must have 3 H_8; these have a common H_4 invariant in H_{24}. An H_8 of H_{24}, being in 2 of the 130 H_{24}, contributes an invariant H_4 to each. If these 2 H_4 are the same H_4, this is invariant in a group S as described and rejected above. And again, if the 2 H_4 are different, their common H_2, being invariant under an H_8 and an H_3 in each of the 2 H_{24}, is invariant in each of the 2 H_{24} and therefore in a group S as described and rejected above. Every G_{3120} is composite.

3168. A simple G_{3168} must have 12 or 144 H_{11}. The theorem of section 3 excludes the 12.

There can not be 16 H_9, for 16 $H_{9\cdot 22}$ yield only 1 H_{11} in G. Then there are 22, 88, or 352 H_9; and if 22 or 88, two of these have a common H_3. Moreover, if no two of 352 H_9 had a common H_3, these with 144 H_{11} would furnish too many elements. Hence in all cases a subgroup L of order 3 is common to $3k+1=4$, 16, 22 or more H_9.

If L is common to more than 22 H_9, it has too few conjugates; and if to 22 H_9, the 16 $H_{22\cdot 9k}(k=1)$ yield only 1 H_{11}.

If L is common to 16 H_9, it is invariant in an $H_{16\cdot 9k}$, $k=1$. The factor group Γ_{48} has 16 H_3 and therefore 1 H_{16}, whence H_{144} has an invariant H_{48}. If this H_{48} has an invariant H_{16}, it is invariant in H_{144} and also in an H_{32}, and has too few conjugates. Hence the H_{48} has an invariant H_8, which is invariant in H_{144}. $H_{144}/H_8=\Gamma_{18}$ has 1 H_9; then H_{144} has an H_{72} and only 4 H_9. This can not be.

Thus L is common to only 4 H_9, and is invariant in an $H_{9\cdot 4k}$, $k=1$, 2, or 4.

(a) $k=1$. Each of the 88 H_{36} has an invariant H_4, which is invariant in H_{72m}, $m=1$, 2. If $m=1$, L is invariant in H_{72}; this can not be. If $m=2$, $H_{144}/H_4=\Gamma_{36}$ has an invariant H_9 or H_4; H_{144} has an invariant H_{36} or H_{16}. In the former case, L is invariant in H_{144}; in the latter, H_{16} is invariant in H_{288n}. Both are impossible.

Observe that the case for 352 H_9 is here disposed of, since for them $k=1$.

(b) $k=2$. L has 44 conjugates, each invariant in an H_{72}. If L has m conjugates in each H_9, then for 22 H_9 $22m/4=44$, $m=8$; this is impossible. There must be 88 H_9. $88m/4=44$, $m=2$; each H_9 has 2 conjugates of L. H_{72} has 5 of them, and they affect 39 symbols in G^{44}.

If H_{72} has only 1 H_8, this is invariant in H_{144}, which contains L invariantly. This can not be. Hence H_{72} has 3 or 9 H_8 with* a common quadratic H_4 which is invariant in H_{72}. An H_8 in H_{72} occurs in 2 H_{72}, and contributes to each an invariant H_4. If these 2 H_4 are the same H_4, this is invariant in 2 H_{72}, and therefore in H_{144m}, $m=1$; but then L is invariant in H_{144}, which is impossible. Hence the 2 H_4 are different; they contain 5 H_2, and the H_8 is dihedral or (1, 1, 1). The element s of order 2 common to the 2 H_4 is invariant in an H_{24} from each of the 2 H_{72}, and therefore in a subgroup J which has several H_3, each invariant in an H_{24} of J. J must be of order 96; s has 33 conjugates in G.

Let t denote an s_3 of L. Denote by u an s_3 of L's conjugate in an H_9

* See p. 57 of *loc. cit.* in note (‡), p. 364.

of H_{72}. t is not invariant in H_{72}. For if it were, u would be conjugate (under H_{72}) only with its first power. Then in the H_{18} of H_{72} under which the chosen H_9 is invariant, an H_2 would be invariant under H_9 and an H_4, and therefore under H_{36} or H_{72}. In both cases H_{72} would have only 1 H_9. This is impossible. It follows that t is invariant in H_{36} of H_{72}.

If the H_8 in H_{72} are dihedral, there are nine of them. For an H_2 of the invariant H_4 is invariant in an H_{24} containing L, and L does not transform H_8 into itself; H_{24} has 3 H_8. But a dihedral H_8 does not transform into itself a second H_2 of the invariant H_4. Hence, the 3 H_{24} furnish 9 H_8. These yield $9 \cdot 2 + 3 = 21$ (or more) H_2 conjugate under G. The 44 H_{72} give $44 \cdot 21/m = 33$, $m = 28$; each H_2 of the set omits 28 symbols. But there is in H_{72} an s_3, affecting 39 symbols, which is transformed into its square by such an H_2, and this H_2 can not omit more than 18 symbols.

Hence the H_8 are (1, 1, 1). But there is* no H_{72} with 4 non-cyclic H_9 and 3 abelian H_8.

(c) $k = 4$. L has 22 conjugates, each invariant in H_{144} with 4 H_9. If there are 22 H_9 in G, $22m/4 = 22$, $m = 4$; L has 4 conjugates in each H_9. H_{144} has 13 of these 22 H_3, and L affects 9 symbols. But the 3 H_3 other than L in an $H_9 \equiv N$ of H_{144} affect the other 9 H_3, other than L, in H_{144}. Then N affects only these 9 symbols. But $22 \cdot 4/n = 22$, $n = 4$; each H_9 must affect 18 symbols.

Thus there are 88 H_9 in G. L has only one conjugate in each H_9, and affects 21 symbols. If t (= an s_3 of L) is invariant in H_{144}, this has 3 H_{16}; each of these is invariant in H_{96m}; $m \neq 1$, for L is not invariant in H_{96}; and clearly m is not > 1. Hence, t is invariant in only H_{72} of H_{144}. Every H_2 of this H_{72}, being even and permutable with t, affects just 12 symbols. Hence each H_2 of our H_{72} occurs in 10 H_{144}. Moreover, the H_2 is permutable with the elements of the 10 conjugates of L in the 10 H_{144}, since an s_2 which transforms into its square an s_3 of a conjugate of L affects at least 14 symbols. Thus, the H_2 under observation is invariant in a subgroup which contains precisely 10 conjugates of L. No such subgroup can be constructed.

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*See p. 60 of *loc. cit.* in note (‡), p. 364.

*On Certain Orthogonal Systems of Lines and the Problem of Determining Surfaces Referred to Them.**

By Archer Everett Young.

§ 1. *Introduction.*

Bonnet has shown that the problem of determining surfaces referred to orthogonal lines can be reduced to the solution of the following systems of equations:†

$$\left.\begin{aligned} &\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}-PQ-RS=0,\\ &\frac{\partial P}{\partial v}-\frac{\partial S}{\partial u}+MQ+NR=0,\\ &\frac{\partial R}{\partial v}+\frac{\partial Q}{\partial u}+MS-NP=0, \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} &\frac{R}{\sqrt{E}}+\frac{S}{\sqrt{G}}=0,\\ &M=-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v},\\ &N=\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}, \end{aligned}\right\} \quad (2)$$

where $R^2+P^2=e$, $PS-QR=f$, $S^2+Q^2=g$, and where the first and third quadratic forms for the corresponding surface are respectively:

$$\text{I.}\quad \overline{ds}^2=Edu^2+Gdv^2,$$

$$\text{III.}\quad \overline{d\sigma}^2=edu^2+2fdudv+gdv^2.$$

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*Read before the American Mathematical Society, Chicago Section, April 28, 1911.

†*Journal de l'École Polytechnique*, Vol. XLII, pp. 132–151.

To every set of values of the functions P, Q, R, S, M, N, E, and G satisfying (1) and (2) there corresponds a surface S, uniquely determined as Bonnet has shown,* whose cartesian coordinates can be obtained in the usual way as functions of u and v.

In the application of the general method, Bonnet, in every instance, limited himself to the case where the lines of reference were lines of curvature on the surface, and even then he considered the problem only in very particular cases.*

The general problem of determining surfaces referred to lines of curvature has been treated by us in a previous paper† from the standpoint of the Bonnet equations.

The present paper has to do with the problem of determining surfaces referred to *any set of orthogonal lines.* Assuming the general results obtained by Bonnet, we have sought to determine those systems of orthogonal lines whose choice as lines of reference would either simplify the general problem of determining surfaces, or else enable us to obtain by algebraic operations from every surface found, another, which is, it appears, an associate of the former.

In § 2 the problem of solving the systems of equations (1) and (2) is shown to hinge on the solution of a Laplace equation, if we consider M and N as known, whose invariants involve these latter functions and their derivatives. This result suggests the choice of such lines as lines of reference as will cause an invariant of this Laplace equation, or of one obtained from it by the method of Laplace, to vanish. For then and only then, in general, can the Laplace equation be solved. The lines thus defined we have called *invariant lines.* There are n sets of such lines on every surface where n is arbitrarily large. If, in particular, the invariant lines are also lines of curvature, the surfaces corresponding compose the general class obtained by the solution of the problem‡ of the spherical representation, the problem discussed in our previous paper.§

In § 3 we have determined the equation which defines those orthogonal systems of lines on a surface S which correspond to orthogonal systems on the associates of S. Such lines we have called *principal orthogonal lines,* or more briefly, *principal lines.* Their choice as lines of reference makes possible the determination by algebraic operations of the associates of S, and hence its infinitesimal deformations.

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*See reference above.

†See reference below.

‡See Darboux, *Théorie des Surfaces*, Vol. IV, p. 169.

§See reference below.

In § 4 we have obtained a new method for determining whether a given surface is isothermic, which is applicable whenever the lines of reference are orthogonal. We have also extended the Bour-Darboux theorem for finding a sequence of isothermic surfaces from a given isothermic surface so that, while before it was applicable only when the lines of reference on the known surface were lines of curvature, it is now applicable when the lines of reference are any set of orthogonal lines.

2. *Invariant Lines.*

Eliminating E and G from (2) we have

$$\frac{\partial^2 \log\left(\frac{S}{R}\right)}{\partial u \partial v} + \frac{\partial}{\partial u}\left(M\frac{S}{R}\right) + \frac{\partial}{\partial v}\left(N\frac{R}{S}\right) = 0, \tag{3}$$

which, with (1), gives four equations for the determination of the functions M, N, P, Q, R, and S.

The problem of determining surfaces referred to orthogonal lines reduces, therefore, to the simultaneous solution of (1) and (3), the functions E and G of the first quadratic form being obtained then by quadrature from the following:

$$\left.\begin{aligned} \frac{\partial \log \sqrt{E}}{\partial u} &= -\frac{\partial}{\partial u}\log\left(\frac{S}{R}\right) - N\left(\frac{R}{S}\right), \\ \frac{\partial \log (\sqrt{E})}{\partial v} &= M\left(\frac{S}{R}\right), \\ \sqrt{G} &= -\frac{\frac{\partial \sqrt{E}}{\partial v}}{M}. \end{aligned}\right\} \tag{4}$$

In (1) and (3) we have six functions involved in four equations. If we add to this system a fifth equation involving one or more of the functions and their derivatives, we shall not, in general, limit the result so far as surfaces are concerned,* but shall limit the choice of orthogonal lines as lines of reference on them.

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* An exception to the rule would arise if we should use the equation $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$.

For example, if we associate with (1) and (3) the equation

$$(1°) \quad M = 0,$$

we limit the choice of lines of reference to geodesics and geodesic parallels. We say that this equation *defines* such lines. Again, if we associate with these equations the following,

$$(2°) \quad \frac{\partial M}{\partial u} + \frac{\partial N}{\partial v} = 0,$$

we limit our choice of lines to a class which includes isothermal lines as a particular case.

In the preceding examples we have used an equation involving the functions M and N in defining lines of reference. If in either case we had used also another equation involving the same functions, we might have said that the last equation added, defined a certain class of surfaces referred to the lines defined by the first. For example, if we had used with (1) and (3) the following,

$$\text{(i)} \quad M = 0,$$
$$\text{(ii)} \quad N = f(u),$$

we might have said that (i) defined as lines of reference geodesics and geodesic parallels, and that (ii) defined a certain class of surfaces referred to such lines, namely surfaces of revolution and surfaces applicable to them.

Moreover, it appears from (2) that the determination of the functions M and N by two equations, as in the preceding examples, amounts to the determination of E and G, and hence of the first fundamental form for the surfaces sought.

We proceed to point out certain systems of lines whose choice as lines of reference simplifies the problem of solving (1) and (3), if M and N are assumed to be known, and hence that of determining the corresponding surfaces.

The equation offering the greatest difficulties of solution seemingly is (3). But this can be replaced by

$$\frac{\partial^2 \phi}{\partial u \partial v} - \frac{\partial \log M}{\partial u} \frac{\partial \phi}{\partial v} + MN\phi = 0, \tag{5}$$

making use of the substitution

$$\frac{S}{R} = \frac{1}{\phi}\left(-\int N\phi du + V\right),$$

where V is an arbitrary function of integration.

This is a Laplace equation, and the conditions under which its general solution can be obtained are known.*

Starting with (5) and forming the sequence of Laplace equations in the one way, we have as the invariants of this equation, and the principal invariants of the others written in order, the following:

$$\left.\begin{aligned} k &= -\frac{\partial^2 \log M}{\partial u \partial v} - MN, \quad h = -MN, \\ k_1 &= -\frac{\partial^2 \log (Mk)}{\partial u \partial v} - k, \\ k_2 &= -\frac{\partial^2 \log (Mkk_1)}{\partial u \partial v} - k_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ k_{i+1} &= -\frac{\partial^2 \log (Mk \ldots . k_i)}{\partial u \partial v} - k_i, \quad \text{where } i = 0, 1, 2, \ldots . \end{aligned}\right\}$$

The general solution of (5) can be obtained as a function of M and N, and arbitrary functions of integration when and only when, in general, some one of these invariants vanish.

We take, therefore, as the defining equations† for our lines of reference,

$$\left.\begin{aligned} &h = 0, \text{ or} \\ &k_i = 0, \text{ where } i \text{ has the values } 0, 1, 2, 3, \ldots . \end{aligned}\right\} \quad (6)$$

The problem of solving (1) and (3) is now reduced to the solution of three equations in P, Q, and R, two of which are linear in these, expressing S in terms of known functions and R, the other being an algebraic equation.

The lines defined by (6) we shall call *invariant lines*, as they arise from equating the invariants of Laplace equations to zero.

These lines appear upon all surfaces and are related in this way.‡ The lines defined by $M = 0$ invert into lines included in the class defined by $k = 0$; and, in general, the lines defined by $k_i = 0$ are either lines obtained by inversion from lines defined by $k_{i-1} = 0$, or $k_{i+1} = 0$.

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* Darboux, *Théorie des Surfaces*, Vol. II, pp. 22–30.

† See previous reference, p. 30.

‡ See work of paper referred to just below. The relation between the systems of lines is the same, whether they are lines of curvature or not.

If, in order to particularize the surfaces referred to any set of invariant lines, we use as the added condition, that the lines of reference (invariant lines) shall be lines of curvature on the surface, we obtain some very interesting surfaces. They are in fact the very surfaces which arise from the solution of the problem of the spherical representation proper. The general class, as we have shown,* can be obtained by quadrature and algebraic operations from a particular set having one set of their lines of curvature plane, geodesic lines.

§ 3. *Principal Lines.*

If the functions M, N, P, Q, R, and S satisfy (1) and (3), then likewise the functions M_0, N_0, etc., defined as follows:

$$\left.\begin{aligned} M_0 &= -M, \\ N_0 &= -N, \\ P_0 &= R, \\ Q_0 &= -S, \\ R_0 &= P, \\ S_0 &= -Q, \end{aligned}\right\} \quad (7)$$

will also satisfy (1). For evidently the first equation is satisfied and the second and third interchange when these values are substituted. If, therefore, the ratio of S_0 over R_0 satisfies (3) when M and N are replaced by their negatives, or, in other words, if the ratio of Q over P satisfies (3), M and N retaining their original values, then, to the surface S corresponding to the first solution, there will be another S_0 corresponding to the second, and this last surface will have the same spherical representation of its lines of reference as the first, since they will both have the same third form. Moreover, it is easily shown that the asymptotic lines on one surface correspond to a conjugate system on the other, and vice versa. Hence S and S_0 are *associate surfaces*, and the orthogonal lines of reference on S correspond to orthogonal lines on S_0.

Such lines we have called *principal orthogonal lines*, or, for brevity, *principal lines*.

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*A. E. Young, "On the Problem of the Spherical Representation, etc.," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, pp. 37–51.

In view of the preceding work we may take as the defining equation of *principal* lines

$$\frac{\partial^2 \log\left(\frac{Q}{P}\right)}{\partial u \partial v} + \frac{\partial}{\partial u}\left(M\frac{Q}{P}\right) + \frac{\partial}{\partial v}\left(N\frac{P}{Q}\right) = 0,$$

which, of course, is nothing but (3) with S over R replaced by Q over P.

There are an infinity of principal lines on every surface, corresponding respectively to the infinity of associates which the surface has. For every set of associate surfaces has one (and in general only one) set of orthogonal lines corresponding, and these, as we shall show in the next article, are always defined by an equation similar to (8), when they are chosen for lines of reference, unless the surfaces are isothermic, and related according to the Bour-Darboux theorem.

As soon as a set of *principal* lines are chosen for lines of reference on a given surface, an associate of it can be found by quadrature, and hence:*

The problem of determining the associates of a surface S is the same as that of determining all the systems of principal lines on S.

In closing this article, we may remark that should we use (8) with (1) and (3) in determining surfaces, we would obtain them in pairs, one being the associate of the other, and the lines of reference on both being *principal* lines.

§ 4. *Isothermic Surfaces and Their Determination.*

It remains for us to determine whether there are ways, other than the one noted in the previous article, in which two associate surfaces referred to orthogonal lines can be related, that is in regard to their fundamental functions.

Suppose that S_0 is an associate of S referred to orthogonal lines, and that its fundamental functions are M, N, P, etc.

If it is to have the same third form† as S, then

$$P_0^2 + R_0^2 = P^2 + R^2, \quad P_0 S_0 - Q_0 R_0 = PS - QR, \quad Q_0^2 + S_0^2 = Q^2 + S^2, \quad (9)$$

* Or, since the determination of an associate carries with it the infinitesimal deformation of the original surface, we have the following: The infinitesimal deformation of a surface S is the same problem as the determination of the *principal* systems on S. See the Theorem of Cosserat, Eisenhart, *Differential Geometry*, p. 380.

† See relations given just below system of equations (2).

and if the asymptotic lines on one are to correspond to a conjugate system on the other,* then

$$PRQ_0 S_0 + 2RSR_0 S_0 + QSP_0 R_0 = 0. \tag{10}$$

Considering the first set of functions as given, we have in (9) and (10) four algebraic equations for the determination of four functions, P_0, Q_0, R_0 and S_0. The work of this simple algebraic problem, which is quite long, will be omitted. Eliminating† three of the functions we have a quadratic equation in P_0^2. Solving this and writing down the corresponding possible values for the other functions, we have, if we omit the solution of the preceding article,

$$\left.\begin{aligned} P_0^2 &= \frac{[P(QR+PS)+2SR^2]^2}{(QR+PS)^2+4R^2S^2}, \\ Q_0^2 &= \frac{[Q(QR+PS)+2RS^2]^2}{(QR+PS)^2+4R^2S^2}, \\ R_0^2 &= \frac{R^2(QR-PS)^2}{(QR+PS)^2+4R^2S^2}, \\ S_0^2 &= \frac{S^2(QR-PS)^2}{(QR+PS)^2+4R^2S^2}. \end{aligned}\right\} \tag{11}$$

If we take the ratio of S_0 over R_0 we find that it is the same as S over R, and it follows from the relation of these ratios to those of E_0 over G_0 and E over G, that every set of orthogonal lines on the original surface would correspond to a similar set on any other surface whose functions are related to the former in this way.

It easily follows that the lines of curvature on one correspond to similar lines on the other, and, as one maps conformally on the other, the surfaces are isothermic surfaces related according to the Bour-Darboux theorem.‡

We conclude, therefore, that *if two associate surfaces are referred to orthogonal lines which correspond, their fundamental functions (the parameters being properly chosen) are related as in* (7) *unless they are both isothermic surfaces related according to the Bour-Darboux theorem, in which case their functions are related as in* (11), *provided the lines of reference are not lines of curvature.*

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* This is Bianchi's equation $D''D_0 + DD_0'' + 2D'D_0' = 0$, in our notation. (Bianchi, Lukat's edition, p. 293.)

† We exclude the case where the surfaces are developable, and likewise the case where the lines of reference are lines of curvature.

‡ Darboux, *loc. cit.*, Vol. II, p. 243.

This result gives us a new method of determining whether a surface is isothermic without any consideration of the form which the linear element takes when the lines of reference are lines of curvature. Its advantage lies in the fact that it is applicable whenever the lines of reference are *any set of orthogonal lines*.

Suppose we are given any surface S referred to orthogonal lines and wish to determine whether it is isothermic or not. We have merely to form the functions P_0, Q_0, R_0, and S_0 in terms of the corresponding functions of the given surface according to (11). We can then determine the corresponding values of M_0 and N_0 algebraically from the last two equations of (1), where of course P, etc., have been replaced by P_0, etc. If the new set of values of the six functions, M_0, N_0, etc., satisfy (3) and the first equation of (1), the given surface has a conformal associate and hence is an isothermic surface.*

There are few classes of surfaces which have received more attention from geometers, in respect to their determination, than those which have isothermal lines of curvature (or isothermic surfaces, as they have been named by Darboux.†) The theorem due to Bour and extended in application by Darboux, whereby from a given isothermic surface referred to lines of curvature an infinite sequence of such surfaces can be obtained by algebraic operations and quadrature, has yielded the most valuable results. The limitation of the theorem, that it was applicable only when the lines of reference were lines of curvature, was not in evidence before, as no test‡ existed for determining whether a given surface was isothermic except reference to lines of curvature.

In view of this new test, which is applicable whenever the lines of reference are any orthogonal lines, it would be advantageous to remove the limitation. For although the equation defining the lines of curvature is generally known, in general it is impossible to find the integrating factor which would enable one to change to them as lines of reference. This can be done, and the application of the theorem, extended, may be described as follows:

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*On an isothermic surface any orthogonal system of lines corresponds to orthogonal lines on an associate, but certain systems correspond to orthogonal lines on two associates (one of which is an isothermic surface of course), and these are the *principal orthogonal lines* in this case.

† Darboux, *loc. cit.*, Vol. IV, p. 217.

‡ That is, no test which could be applied to the fundamental functions for a surface in their usual forms.

Being given the cartesian coordinates and fundamental functions M, N, etc., for a surface S referred to any set of orthogonal lines, we may apply our test, and if it is found to be isothermic, write out the corresponding functions for its isothermic associate, which we will refer to as S_0. Obtaining the cartesian coordinates for this last surface in the usual way by quadrature, and applying the principle of inversion to it and the original surface, we have two new ones, S_0' and S' respectively. As before, by algebraic operations and quadrature we can obtain the fundamental functions and cartesian coordinates of the isothermic associates of these last two surfaces. Inversion applied to the surfaces last obtained gives two other isothermic surfaces, etc., etc.

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Wallace's Theorem Concerning Plane Polygons of the Same Area.

By W. H. Jackson.

The object of this paper is to draw attention to Wallace's theorem that *two plane polygons of the same area can in all cases be divided into respectively congruent portions; they are congruent by dissection.* Every student of elementary geometry ought to know this fact, not only on account of its intrinsic interest but also on account of the stimulus it affords towards getting an experimental knowledge of the subject.

This theorem was stated by William Wallace about the year 1807 in the new series of Leybourne's *Mathematical Repository*, Vol. III. Later, in 1831, it appeared in the form of an addition to Playfair's edition of Euclid's Elements.* It is there proved that each construction in Euclid's chain yields figures which are congruent by dissection as well as of the same area. No mention of Theorem I of the present paper is made, though it is necessary for a complete proof.

The present writer was led to the final result by exactly the same steps as those exhibited in Wallace's note. But Wallace adds a corollary that two plane polygons of equal area can in all cases be divided into the *same number of* respectively congruent *triangles.* The rest of this paper remains as it was before a hint given by Dr. F. Morley led to the discovery of Wallace's work. Only the proofs of Theorems IV and VI appear to be novel, but conciseness is gained by seeking as direct a proof of Wallace's theorem as may be possible.

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* "Elements of Geometry"; containing the first six books of Euclid; by John Playfair, with additions by William Wallace: eighth edition, Edinburgh, 1831.

The fundamental theorems concerning areas may be divided into three groups:

(1) The theorems leading to the determination of a square of the same area as any given polygon.

(2) The theorem of Pythagoras.

(3) The theorems corresponding to the distributive law in algebra.

In the usual American and English text-books the proofs of these theorems are probably intended to be based, so far as area is concerned, on the following axioms:

(1) The area of a geometrical figure is a number.

(2) Congruent figures have the same area.

(3) The area of a figure is the sum of the areas of its parts.

(4) If equal areas be added to equal areas, the wholes are of equal area.

(5) If equal areas be taken from equal areas, the remainders are of equal area.

(6) The halves of equal areas are of equal area.

The first three axioms are not always specifically stated. They are of themselves quite sufficient for the basis of the theory of area, for the last three readily follow.

If we turn to Weber and Wellstein's Encyclopædia* of elementary mathematics, we find a quite different treatment. In the first place the algebraic theory of number is dispensed with by means of the purely geometrical method of comparing ratios of straight-line segments. In the second place no use of the axiom of Archimedes is made, so far as area is concerned.

With regard to the first point, if in the propositions briefly indicated above we everywhere replace the phrases "equal in area" or "of the same area" by "congruent by dissection," there is then no occasion for the introduction of a numerical measure of area.

With regard to the second point, Hilbert has shown† that it is impossible to prove that two triangles on the same base and of the same height are necessarily congruent by dissection without assuming the axiom of Archimedes.

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* *Encyklopädie der Elementar-Mathematik*, Vol. II, p. 254, 2nd edition, Leipzig, 1907.

† "Grundlagen der Geometrie."

Any theory of area which omits this axiom is therefore necessarily incomplete. Though Weber and Wellstein prove, assuming the axiom of Archimedes, that triangles on the same base and of the same height are congruent by dissection, they do not state that it follows from this that all polygons of the same area are necessarily congruent by dissection. Incidentally, it may be noted that the rôle played by the axiom of Archimedes in the works last quoted, where it follows the axioms of congruence, is nothing more or less than to exclude null segments.

DEFINITION. *Two figures are congruent by dissection when either can be divided into parts which are respectively congruent with the corresponding parts of the other.*

THEOREM I. *Two figures each congruent by dissection with a third are congruent by dissection with each other.*

Suppose that figures A and B are each congruent by dissection with C. Accordingly let C be composed of portions A'_r $(r = 1, 2, \ldots, m)$ each congruent with the corresponding member of the series of parts A_r $(r = 1, 2, \ldots, m)$ which compose A. Similarly let C be composed of portions B'_r $(r = 1, 2, \ldots, n)$ congruent with the corresponding portions B_r $(r = 1, 2, \ldots, n)$ which compose B.

Let $[A'_r B'_s]$ denote the portion common both to A'_r and B'_s. Since A_r is congruent with A'_r it can be dissected into portions respectively congruent with $\sum_{s=1}^{n} [A'_r B'_s]$, and A is therefore composed of portions respectively congruent with the parts of C denoted by $\sum_{r=1}^{m} \sum_{s=1}^{n} [A'_r B'_s]$. But B can in the same way be divided into portions congruent respectively with these same parts.

That is, A and B are congruent by dissection.

This mode of proof is clearly not confined to polygons. It is not even confined to surfaces, but applies equally to volume spaces.

THEOREM II. *A parallelogram can be found congruent by dissection with any given triangle.*

Choose as base the longest side of the triangle and make one adjacent side of the parallelogram just half an adjacent side of the triangle, so that the triangle and the parallelogram have an *acute* angle in common. [The details are arranged so that by means of Theorems II and III a triangle can in *every* case be divided into three parts congruent with corresponding parts of a rectangle.]

THEOREM III. *A rectangle can be determined congruent by dissection with any given parallelogram.*

Choose as base a side of the parallelogram which is not less than either adjacent side.

THEOREM IV. *A rectangle can be found on any given base congruent by dissection with any given rectangle.* (Fig. 1.)

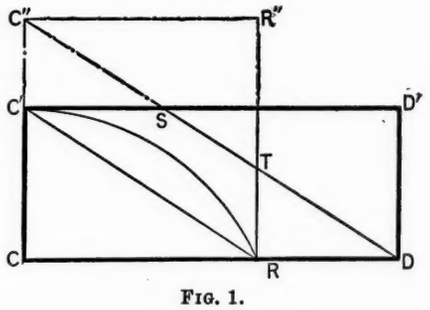

FIG. 1.

Let $ABB'A'$ be the given rectangle and let PQ be the given base. One of three cases must arise:

(i) $AB > 2PQ$,

(ii) $2PQ \lesseqgtr AB \geqq PQ$,

(iii) $PQ > AB$.

In case (i) let $CDD'C'$ be the first rectangle congruent by dissection with $ABB'A'$ and such that $2PQ \geqq CD$, obtained by successively halving the base and consequently doubling the height. By hypothesis $2CD$ does not satisfy the first inequality; that is, $CD > PQ$, and conditions (ii) are satisfied by the new base CD.

Similarly, by successively doubling the base and consequently halving the height we can also reduce the consideration of case (iii) to that of case (ii).

In case (ii) the theorem can be proved as follows: If on CD, $C'D'$ we choose lengths CR, SD' each equal to PQ, the straight line through R parallel to CC' necessarily meets SD in some point T within the rectangle when conditions (ii) are satisfied. Let DS, CC' when produced meet in C''. Then if $CRR''C''$ be a rectangle, it is the rectangle required, congruent by dissection with the given rectangle $ABB'A'$.

First, the rectangle $CRR''C''$ exists in all cases.

Secondly, $CRR''C''$ and $CDD'C'$ are congruent by dissection because the part $CRTSC'$ is common to both, $C'SC''$ and RDT are congruent, $C''TR''$ and SDD' are congruent.

Lastly, by Theorem I $CRR''C''$ is also congruent by dissection with $ABB'A'$.

THEOREM V. WALLACE'S THEOREM. *Two plane polygons of the same area are congruent by dissection.*

First, any polygon can be divided into triangles, and by Theorems I–IV rectangles can be found on a given base respectively congruent by dissection with all these triangles; these rectangles can be so placed as to form a single rectangle on the given base congruent by dissection with the given polygon.

Secondly, if two polygons of the same area are given, there are two rectangles on any given base respectively congruent by dissection with the two polygons, and therefore the two rectangles also have the same area by axiom (3). They must also be congruent; for, if not, one would have an area less than that of the other. It now follows from Theorem I that the two given polygons are congruent by dissection.

COR. I. If two polygons congruent by dissection are taken from polygons congruent by dissection, the remainders are congruent by dissection.

COR. II. The halves of polygons congruent by dissection are congruent by dissection.

THEOREM VI. *A square can be found congruent by dissection with any given rectangle.* (Fig. 2.)

Let $ABB'A'$ be the given rectangle, where $AB > AA'$. Two cases arise according as:

(i) $AB > 4\,AA'$,

(ii) $4\,AA' \geqq AB > AA'$.

As in the last theorem, the consideration of case (i) can easily be led back to case (ii) by halving the base of the rectangle and doubling its height, and repeating the process as often as may be necessary.

Let $CDD'C'$ be a rectangle congruent by dissection with $ABB'A'$ satisfying conditions (ii). From $C'C$ produced suppose CE taken congruent with CD and let O be the center of $C'E$. The circle with center O and radius OC' meets CD in some point R, lying in CD and not in CD produced, because $CR < CO + OR$, and therefore $CR < CD$. With the same construction as in Theorem IV, let SD' in $C'D'$ be congruent with CR and let DS, CC' produced meet in C'' and let $CRR''C''$ be a rectangle.

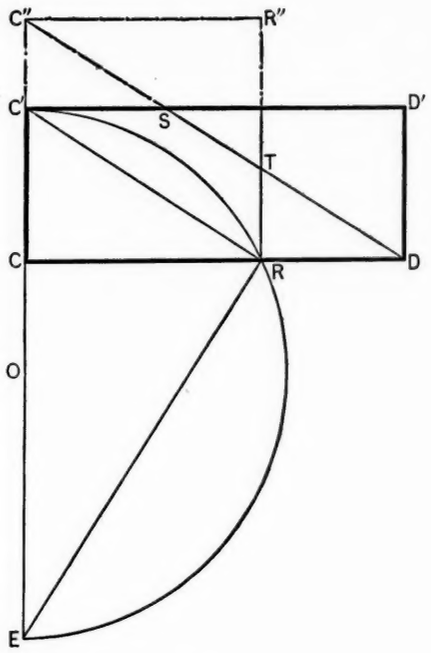

FIG. 2.

It is easily proved (if desired, without depending on the similarity of triangles) that, by virtue of conditions (ii), $2CR \geqq CD$. Therefore RR'' meets SD in some point T, and the rectangle $CRR''C''$ is congruent by dissection with the given rectangle exactly as in the last theorem.

Finally, since the angles CDC'', REC' are congruent, it follows that CC'', CR are congruent. The desired square has therefore been found.

THEOREM VII. *A square can always be found congruent by dissection with any given polygon.*

As in Theorem V a rectangle can be found on any given base congruent by dissection with a given polygon. By Theorem VI a square can be found congruent by dissection with this rectangle, and by Theorem I this square is congruent by dissection with the given polygon.

In conclusion, two questions naturally arise:

(i) *How many subdivisions are in general necessary to exhibit the congruence of two polygons of the same area containing given numbers of sides?*

That it is impossible to fix an upper limit to the number of subdivisions necessary to exhibit the congruence by dissection of two polygons, merely being given the number of sides each contains, will be clear from a simple example.

A rectangle 100 by .01 is congruent by dissection with a unit square. The straightforward way to exhibit this congruence is to divide the longer side of the rectangle into 10 equal parts, placing the 10 rectangles so obtained together so as to form a rectangle 10 by .1, and then to repeat the process in order to obtain the unit square. This method involves 18 cuts dividing the rectangle into 100 parts. If the construction of Theorem VI be followed, the longer side of the rectangle would be halved 6 times and the construction would be completed by 2 more cuts. This method involves 8 cuts dividing the rectangle into 164 parts. In either method the number of parts depends on the shape of the rectangle.

The maximum number of cuts necessary to transform a triangle into a rectangle is 2, dividing it into 3 parts; at least 2 more cuts are necessary to obtain a rectangle on any given base, and the least number of resulting parts is 5.

(ii) *Why should the results obtained with regard to polygons not be capable of extension to curved plane figures or to polyhedra?*

A necessary condition that a polygon should be congruent by dissection with a rectangle is that the sum of its angles must be an integral multiple of a straight angle. This condition is, of course, always satisfied, but the analogous condition is not satisfied for curved plane figures or for polyhedra.

The analogous condition that a curved figure may be congruent by dissection with a rectangle is that the sum of all its angles is an integral multiple of a straight angle and that all its curved arcs should fall into pairs of congruent portions only differing as to the side on which the interior of the figure lies.

The corresponding necessary, but not necessarily sufficient, condition that a polyhedron may be congruent by dissection with a rectangular block is that such of its dihedral angles as bear incommensurable ratios to a straight angle must fall into groups such that their sum, when multiplied by integers respectively inversely proportional to the lengths of their edges, bears a commensurable ratio to a straight angle.

For example, *a tetrahedron containing three intersecting edges mutually perpendicular and of lengths* $1, \sqrt{2}, \sqrt{2}$ *is congruent by dissection with a cube.* Its dihedral angles all bear commensurable ratios to a straight angle. Similarly, *two polyhedra symmetric with respect to a point or to a plane are congruent by dissection.*

MANCHESTER, ENGLAND, *September*, 1911.

*On Harmonic Functions.**

By R. A. Harris.

§ 1.

This paper constitutes an elementary study in harmonic functions, V, and what might be called quasi-harmonic functions, V, where

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \ldots = 0, \text{ or } \nabla^2 V_p = 0, \tag{1}$$

and

$$\frac{\partial^2 \mathrm{V}}{\partial x^2} - \left(\frac{\partial^2 \mathrm{V}}{\partial y^2} + \ldots\right) = 0, \text{ or } \nabla^2 \mathrm{V}_p = 0, \tag{2}$$

p denoting the number of independent variables. While its object is to show that many solutions of (1) and (2) can be found by elementary methods and to point out some of the connections which exist between various solutions, the fact that many, and perhaps all, of the solutions may, in a sense, be regarded as an outgrowth or extension of those harmonic functions, X, Y, with which the theory of functions has to deal, will not be lost sight of.

The partial differential equations (1) and (2) will not be transformed into their equivalents in spherical or cylindrical coördinates even when solutions involving Legendre's coefficients, Bessel's functions, etc., are contemplated. In adapting equation (2) or any of its solutions to physical problems, the quantities x, y, z are generally to be replaced by xt, x, and y, respectively.

§ 2.

The lemmas or propositions given below are, with few exceptions, self-evident or nearly so. But inasmuch as their wording may not always be exact and such as to cause their meaning to be readily apprehended, brief demonstrations and qualifying statements are added.

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* An abstract of the principal parts of this paper was read before Section A of the American Association for the Advancement of Science, 1898. See *Bull. Am. Math. Soc.*, Vol. V (1898), pp. 96–98.

(i) If a solution of any linear partial differential equation having constant coefficients, but not a term of order zero, be differentiated (or integrated) with respect to one or several of its variables, the result is a solution of the given equation.

For, when several operators like $\frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x \partial y}$, ... are applied to any quantity or solution the order of application is immaterial; the result of any number of such operations on the given differential equation is, of course, a new differential equation. So if a partial derivative of the function constituting the given solution be made, by transposing the operators, to occupy the same position in the new differential equation as did the given solution in the original equation, the result must still be a true equation.

Similarly for partial integration; but it is understood that in this case the number of terms in the power-series representing the given solution developed in powers of the variable in question, is generally infinite.

(ii) If in any solution of an equation of the kind just referred to, any two of the variables be replaced by their equivalents in circular coördinates, the result of differentiating (or integrating) the solution with respect to either circular coördinate is a solution of the given equation.

For, suppose $\eta = y + \sqrt{-1}\, z$, $\zeta = y - \sqrt{-1}\, z$, and so

$$y = \tfrac{1}{2}(\eta + \zeta), \quad \sqrt{-1}\, z = \tfrac{1}{2}(\eta - \zeta).$$

$$\therefore \frac{\partial}{\partial \eta} = \frac{\partial}{\partial y}\frac{dy}{d\eta} + \frac{\partial}{\partial z}\frac{dz}{d\eta} = \tfrac{1}{2}\left(\frac{\partial}{\partial y} - \sqrt{-1}\frac{\partial}{\partial z}\right),$$

$$\frac{\partial}{\partial \zeta} = \tfrac{1}{2}\left(\frac{\partial}{\partial y} + \sqrt{-1}\frac{\partial}{\partial z}\right),$$

and so, by lemma (i), these indicated differentiations yield solutions. Similarly for coördinates such that, say, $y + z = \eta_1$, $y - z = \zeta_1$, which may be called hyperbolic coördinates.

The same is evidently true if any other linear substitutions be made for the variables.

(iii) If in any solution of (1) the letters $x, y, z, \ldots$ be interchanged at will, the result is a solution. The same is true for all variables after the first in a solution of (2).

For, the operators in (1) are symmetrical in all variables, and in (2), in all but the first variable.

(iv) If in any solution, V_p, of (1), $y, z, \ldots$ be replaced by $\pm\sqrt{-1}\,y$, $\pm\sqrt{-1}\,z, \ldots$, the result is a solution, V_p, of (2); and conversely.

For, equation (1) must still be a true equation provided the substitution be made in the operators as well as in the original function V_p. But the operators $\frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}, \ldots$ become $-\frac{\partial^2}{\partial y^2}, -\frac{\partial^2}{\partial z^2}, \ldots$, and so the transformed function must satisfy (2).

(v) Given any solution of (1) or (2) in the form

$$f_0(y, z, \ldots)\chi(x) + f_2(y, z, \ldots)\chi''(x) + f_4(y, z, \ldots)\chi^{IV}(x) + \ldots, \quad (3)$$

and such that the entire coefficients of $\chi''(x)$, $\chi^{IV}(x), \ldots$ in (1) or (2) are severally zero, then is

$$f_0(y, z, \ldots)\psi(x) + f_2(y, z, \ldots)\psi''(x) + f_4(y, z, \ldots)\psi^{IV}(x) + \ldots \quad (4)$$

a solution, ψ being an arbitrary function.

This means that the given solution when substituted in (1) or (2) satisfies the same irrespective of the nature of χ, and so χ may be replaced by ψ. To say that the coefficient of $\chi^{(\nu)}$ in (1) or (2) shall be zero, is equivalent to saying that

$$f_{\nu-2}(y, z, \ldots) \pm \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \ldots\right) f_\nu(y, z, \ldots) = 0. \quad (5)$$

Of course this is likewise true for the alternate terms; that is, for the terms having odd subscripts and indices.

(vi) If in any solution of (1) or (2) referred to in lemma (v), $x, y, \ldots$ be replaced by $x \pm x', y \pm y', \ldots$, where $x', y', \ldots$ are constants, the result is also a solution of the given differential equation.

For, let $x_1 = x \pm x'$, $y_1 = y \pm y', \ldots$; the solution will be the same function of $x_1, y_1, \ldots$ as it originally was of $x, y, \ldots$. The operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots$ may be written $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots$. But expression (5) being identically zero, it will remain so when $x_1, y_1, \ldots$ replace $x, y, \ldots$ throughout.

(vii) If for the variables in equation (1) we make an orthogonal substitution, the result is still a true equation.

For, in V, replacing $x, y, \ldots$ by their equivalents in a second set of

variables $x_1, y_1, \ldots$, the form of V will generally be altered, but its value will, of course, remain the same as before. Consequently the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \ldots$$

applied to this new expression will still give a result equal to zero. But the effect of the operator

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \ldots$$

upon either form of V is not zero unless the two operators are equal or proportional. The first operator when expressed in terms of $x_1, y_1, \ldots$ by aid of the matrix of transformation shows that for any orthogonal substitution the two operators are equivalent, ignoring a constant factor, the modulus of the substitution. For suppose

$$\left.\begin{aligned} x &= a_1 x_1 + b_1 y_1 + \ldots, \\ y &= a_2 x_1 + b_2 y_1 + \ldots, \\ &\ldots\ldots\ldots\ldots\ldots, \end{aligned}\right\} \text{ and so } \left\{\begin{aligned} x_1 &= a_1 x + a_2 y + \ldots, \\ y_1 &= b_1 x + b_2 y + \ldots, \\ &\ldots\ldots\ldots\ldots\ldots; \end{aligned}\right.$$

then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1}\frac{dx_1}{dx} + \frac{\partial}{\partial y_1}\frac{dy_1}{dx} + \ldots = a_1\frac{\partial}{\partial x_1} + b_1\frac{\partial}{\partial y_1} + \ldots,$$

and finally,

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \ldots = (a_1^2 + a_2^2 + \ldots)\frac{\partial^2}{\partial x_1^2} + (b_1^2 + b_2^2 + \ldots)\frac{\partial^2}{\partial y_1^2} + \ldots,$$

wherein

$$a_1^2 + a_2^2 + \ldots = b_1^2 + b_2^2 + \ldots = \ldots.$$

Similarly, if only a part or group of the variables of (1) be transformed into a group of corresponding variables by means of a uni-modular orthogonal substitution, the equation will still hold true. This remark is evidently applicable to the second part of equation (2).

(viii) Equation (1) is satisfied by making V equal to

$$\frac{d^{p-2}\log\rho}{d\rho^{p-2}} \text{ or } \frac{1}{\rho^{p-2}}, \tag{6}$$

where p denotes the number of variables in V and where

$$\rho \equiv \sqrt{x^2 + y^2 + \ldots}.$$

For, substituting $1/\rho^{p-2}$ in (1) we have the sum of p quantities like

$$(p-2)(-\rho^{-p}+px^2\rho^{-p-2}),$$

except that x^2 is replaced by $y^2, z^2, \ldots$ in succession; and this sum is zero.

Similarly, or by lemma (iii),

$$\frac{1}{\sigma^{p-2}}, \tag{7}$$

where $$\sigma \equiv \sqrt{x^2-(y^2+\ldots)},$$

is a solution of equation (2).

(ix) If H_n denote an homogeneous solution of (1) of degree n, then is

$$\frac{H_n}{\rho^{2n+p-2}} \tag{8}$$

likewise a solution, n being general.

For, operating upon H_n/ρ^{2n+p-2} with $\frac{\partial^2}{\partial x^2}$, we have

$$\frac{1}{\rho^{2n+p-2}}\frac{\partial^2 H_n}{\partial x^2}-\frac{2x(2n+p-2)}{\rho^{2n+p}}\frac{\partial H_n}{\partial x}-\frac{2n+p+2}{\rho^{2n+p}}H_n + \frac{x^2(2n+p-2)(2n+p)}{\rho^{2n+p+2}}H_n.$$

Replacing x by y, we have the result of the operation $\frac{\partial^2}{\partial y^2}$; and so on. But H_n being a solution, the result of adding together the terms involving $\frac{\partial^2 H_n}{\partial x^2}$, $\frac{\partial^2 H_n}{\partial y^2}, \ldots$ must be zero. Because H_n is homogeneous and of degree n, the sum of the second terms is $-\frac{2n(2n+p-2)}{\rho^{2n+p}}H_n$. The sum of the third terms is $-\frac{p(2n+p+2)}{\rho^{2n+p}}H_n$; the sum of the fourth terms, since $\rho^2=x^2+y^2+\ldots$, is $\frac{(2n+p-2)(2n+p)}{\rho^{2n+p}}H_n$. Hence the total is zero.

Conversely, if (8) be a solution and H_n an homogeneous function, H_n must be a solution.

Similarly, or by lemma (iii), if H_n denote an homogeneous solution of (2),

$$\frac{H_n}{\sigma^{2n+p-2}}, \tag{9}$$

where

$$\sigma=\sqrt{x^2-(y^2+z^2+\ldots)},$$

is a solution.

§ 3. *Application of the Foregoing Lemmas to Spherical Harmonic Functions.*

If the solutions (6), (7) be differentiated n times with respect to x, we obtain solutions of the forms

$$\frac{F_n\left(\frac{x}{\rho}\right)}{\rho^{n+p-2}}, = \frac{\rho^n F_n\left(\frac{x}{\rho}\right)}{\rho^{2n+p-2}}; \quad \frac{F_n\left(\frac{x}{\sigma}\right)}{\sigma^{n+p-2}}, = \frac{\sigma^n F_n\left(\frac{x}{\sigma}\right)}{\sigma^{2n+p-2}}, \tag{10}$$

where $F_n\left(\frac{x}{\rho}\right)$, $F_n\left(\frac{x}{\sigma}\right)$ denote algebraic integral functions of the n-th degree in x/ρ, x/σ.

The truth of this statement becomes evident upon performing a few differentiations and referring to lemma (i). The converse of (ix) shows that the homogeneous functions

$$\rho^n F_n\left(\frac{x}{\rho}\right), \quad \sigma^n F_n\left(\frac{x}{\sigma}\right) \tag{11}$$

are also solutions of equations (1) and (2). These solutions evidently have circular symmetry about the x-axis; they are termed zonal harmonics of the n-th order while (10) are zonal harmonics of order $-n-p+2$.

If the solution $1/\rho^{p-2}$ be differentiated with respect to a circular coördinate, ζ, we have

$$\frac{\partial^n \frac{1}{\rho^{p-2}}}{\partial \zeta^n} = (-1)^n \frac{(p-2)\,p\,(p+2)\,\ldots\,(p+n-1)}{2^n} \frac{\eta^n}{\rho^{p+2(n-1)}}, \tag{12}$$

where η and ζ are conjugate imaginaries.

Similarly, if the solution of (2), $1/\sigma^{p-2}$, be thus differentiated, we have

$$\frac{\partial^n \frac{1}{\sigma^{p-2}}}{\partial \zeta^n} = (+1)^n \frac{(p-2)\,p\,(p+2)\,\ldots\,(p+n-1)}{2^n} \frac{\eta^n}{\sigma^{p+2(n-1)}}. \tag{13}$$

If η_1, ζ_1 denote hyperbolic coördinates such that

$$\eta_1 = x + y \text{ (or } z \text{ or } w, \ldots.),$$
$$\zeta_1 = x - y \text{ (or } z \text{ or } w, \ldots.),$$

the n-th derivative with respect to ζ_1 is

$$\frac{\partial^n \frac{1}{\sigma^{p-2}}}{\partial \zeta_1^n} = (-1)^n \frac{(p-2)\,p\,(p+2)\,\ldots\,(p+n-1)}{2^n} \frac{\eta_1^n}{\sigma^{p+2(n-1)}}. \tag{14}$$

In all of the above cases η and ζ (or η_1 and ζ_1) can be interchanged at will. By (ii) and (vii), the expressions involving ρ are solutions of (1), and those involving σ, solutions of (2).

If the solutions

$$\frac{1}{\rho^{p-2}} \text{ and } \frac{1}{\sigma^{p-2}}, \text{ or } \frac{1}{(x^2 \pm c_1^2)^{\frac{p-2}{2}}},$$

where $c_1^2 = y^2 + z^2 + \ldots$, be integrated one or more times with respect to x, we obtain solutions of (1) and (2).

Making $p = 3$ and performing one integration, we obtain from the solutions $\pm \frac{1}{r}$, $\pm \frac{1}{\sigma}$, the solutions

$$\frac{1}{2} \log \frac{r+x}{r-x}, \quad \frac{1}{2} \log \frac{\sigma+x}{\sigma-x}. \tag{15}$$

The former may be written

$$\frac{1}{2} \log \frac{1+\mu}{1-\mu} = r^0 Q_0(\mu), \tag{16}$$

where $\mu = x/r$. Another integration gives the solutions

$$\frac{1}{2} x \log \frac{r+x}{r-x} - r, \quad \frac{1}{2} x \log \frac{\sigma+x}{\sigma-x} - \sigma. \tag{17}$$

The former may be written

$$r\left(\frac{1}{2}\mu \log \frac{1+\mu}{1-\mu} - 1\right) = r\, Q_1(\mu). \tag{18}$$

Solutions like these are zonal harmonics of the second kind.

Zonal Harmonics, $p = 3$. By lemma (viii) or by direct substitution, we see that

$$V_3 = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{x} - \frac{1}{2}\frac{y^2+z^2}{x^3} + \frac{1.3}{2.4}\frac{(y^2+z^2)^2}{x^5} - \frac{1.3.5}{2.4.6}\frac{(y^2+z^2)^3}{x^7} + \ldots. \tag{19}$$

satisfies equation (1). Moreover it is a solution of the form referred to in (v), $\chi(x)$ being $1/x$. Therefore, by (vi),

$$\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \tag{20}$$

satisfies equation (1), as can be directly verified. The first radical denotes the distance of a variable point from the origin; the second, the distance of this point from a fixed point whose coördinates are x', y', z', both points being

referred to the same coördinate axes. Suppose the fixed point to lie upon the x- (or x'-) axis; *i. e.*, suppose $y' = z' = 0$. The distance of x, y, z from x', 0, 0 is $\sqrt{(x-x')^2 + y^2 + z^2}$ or $\sqrt{r^2 - 2xr' + r'^2}$, where $r^2 = x^2 + y^2 + z^2$, $r'^2 = x'^2 + 0^2 + 0^2 = x'^2$. Expanding by the binomial theorem and arranging the result in powers of r/r' or r'/r, we have

$$\frac{1}{\sqrt{r^2 - 2xr' + r'^2}} = \frac{1}{r'\sqrt{1 - 2\frac{x}{r}\frac{r}{r'} + \frac{r^2}{r'^2}}} = \frac{1}{r\sqrt{1 - 2\frac{x}{r}\frac{r'}{r} + \frac{r'^2}{r^2}}} \tag{21}$$

$$= \frac{1}{r'}\left[P_0 + P_1\frac{r}{r'} + P_2\frac{r^2}{r'^2} + \ldots.\right] \tag{22}$$

$$= \frac{1}{r}\left[P_0 + P_1\frac{r'}{r} + P_2\frac{r'^2}{r^2} + \ldots.\right]. \tag{23}$$

In these developments it is evident that P denotes a rational integral algebraic function of x/r. But (22) or (23) satisfies Laplace's equation (1), regardless of the value assigned to the constant r' or to the ratio r'/r. Consequently since (22) and (23) are arranged in powers of this arbitrary magnitude, each of their terms is likewise a solution. In other words,

$$r^n P_n\left(\frac{x}{r}\right) \text{ or } r^{-n-1} P_n\left(\frac{x}{r}\right), \tag{24}$$

where $n = 0, 1, 2, \ldots.$, is a solution. [Cf. lemma (ix).]

We may write $r^n P_n\left(\frac{x}{r}\right)$ in the form

$$A_n x^n + B_n x^{n-2} r^2 + C_n x^{n-4} r^4 + \ldots., \tag{25}$$

or
$$K_{n,0} x^n + K_{n,2} x^{n-2} r^2 + K_{n,4} x^{n-4} r^4 + \ldots., \tag{26}$$

or
$$\sum_{\nu = 0, 2, 4, \ldots.} K_{n,\nu} x^{n-\nu} r^\nu. \tag{27}$$

Expressions for A_n, B_n, C_n, ... may be obtained from the expansion of (21) in powers of r/r', by picking out the entire coefficient of the n-th power. The sum of the coefficients entering this entire coefficient may be shown to be unity.

Laplace's Coefficients. Returning to solution (20), in which x', y', z' may refer to any fixed point in space, we have, since $r'^2 = x'^2 + y'^2 + z'^2$,

$$\left.\begin{aligned} &\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\ = &\frac{1}{r'\sqrt{1 - 2\frac{xx' + yy' + zz'}{rr'}\frac{r}{r'} + \frac{r^2}{r'^2}}} = \frac{1}{r\sqrt{1 - 2\frac{xx' + yy' + zz'}{rr'}\frac{r'}{r} + \frac{r'^2}{r^2}}}. \end{aligned}\right\} \tag{28}$$

These expanded by the binomial theorem and arranged in powers of r/r' or r'/r may be written

$$\frac{1}{r'}\left[Y_0 + Y_1\frac{r}{r'} + Y_2\frac{r^2}{r'^2} + \cdots\right], \tag{29}$$

$$\frac{1}{r}\left[Y_0 + Y_1\frac{r'}{r} + Y_2\frac{r'^2}{r^2} + \cdots\right]. \tag{30}$$

It is evident that Y denotes the same rational integral function of $\frac{xx'+yy'+zz'}{rr'}$ that P (Legendre's coefficient) denotes of x/r. The Y's are Laplace's coefficients. The expression $\frac{xx'+yy'+zz'}{rr'}$ is the cosine of the angle at the origin between the point x, y, z and the point x', y', z'. Since the value of r'/r is arbitrary, each term of the above expressions is a solution of (1).

Tesseral Harmonics, Associated Forms. Any unimodular orthogonal substitution does not alter the value of r, and the result is by lemma (vii) a solution of equation (1).

In the solutions $r^n P_n$ and $r^{-n-1}P_n$, r and so r^n and r^{-n-1} are unaltered in value if we put $x = l_1 x_/ + m_1 y_/ + n_1 z_/$, where l_1, m_1, n_1 are quantities subject to no other condition than $l_1^2 + m_1^2 + n_1^2 = 1$. Therefore, if $l_1 = 1$, then $m_1^2 + n_1^2 = 0$ or $n_1 = \sqrt{-1}\, m_1$. Let $m_1 = \alpha_1$; then $n_1 = \sqrt{-1}\,\alpha_1$ and $x = x_/ + \alpha_1(y_/ + \sqrt{-1}\, z_/)$. In substituting this value for x in expression (25), we have to consider only the explicit x since r remains unchanged. The result of the substitution arranged in powers of $\alpha_1(y_/ + \sqrt{-1}\, z_/)$ will be a series of expressions of the form

$$\left.\begin{aligned}\alpha_1^s(y+\sqrt{-1}\,z)^s\,[&n(n-1)\ldots(n-s+1)\,A_n x^{n-s}\\ &+(n-2)(n-3)\ldots(n-s-1)\,B_n x^{n-s-2}r^2\\ &+(n-4)(n-5)\ldots(n-s-3)\,C_n x^{n-s-4}r^4+\ldots.],\end{aligned}\right\} \tag{31}$$

wherein all subscripts have been omitted from the variables. Each expression of the above form is a solution of the equation (1); for, the entire expression (25) is still a solution after the above transformation has been made, whatever be the value of α_1. Consequently, α_1 being arbitrary, the entire coefficient of any power of α_1 must be a solution. Further, the real part will be one solution, the imaginary part another.

By writing $x = r\cos\theta$, $= r\mu$, $y = r\sin\theta\cos\phi$, $z = r\sin\theta\sin\phi$, expression (31) when divided by α_1^s becomes

$$r^n \sin^s\theta\,(\cos s\phi + \sqrt{-1}\sin s\phi)\frac{d^s P_n}{d\mu^s}, \tag{32}$$

which may be written

$$r^n(\cos s\phi + \sqrt{-1}\sin s\phi)\,T_n^{(s)}. \tag{33}$$

Expression (27) may have its y^2+z^2 replaced by the equivalent, $\eta\zeta$, in circular coördinates and, by lemma (ii), the result of one or more differentiations with respect to ζ will be a solution.

$$\frac{\partial}{\partial\zeta}\sum_{\nu=0,2,4,\ldots.} K_{n,\nu}\,x^{n-\nu}\,r^{\nu}=\eta\sum_{\nu=2,4,\ldots.}\frac{\nu}{2}K_{n,\nu}\,x^{n-\nu}\,r^{\nu-2}. \tag{34}$$

By making $s=1$ in (31), the latter becomes, to a factor,

$$\eta\sum_{\nu=0,2,4,\ldots.}(n-\nu)\,K_{n,\nu}\,x^{n-1-\nu}\,r^{\nu}. \tag{35}$$

Upon replacing n by $n-1$, ν by $\nu-2$, this becomes

$$\eta\sum_{\nu=2,4,\ldots.}(n-\nu+1)\,K_{n-1,\nu-2}\,x^{n-\nu}\,r^{\nu-2}. \tag{36}$$

Here are two solutions (34), (36) of equation (1), both of which are identical so far as the variables are concerned. Consequently, the coefficients of like powers of the variables must have a constant ratio to each other.

$$\therefore\ \frac{\nu}{2(n-\nu+1)}\,\frac{K_{n,\nu}}{K_{n-1,\nu-2}}=\text{constant},\ =\frac{1}{2}, \tag{37}$$

as any special case will show. This furnishes a ready means of deriving P_n from P_{n-1}; for

$$K_{n,\nu}=-\frac{n-\nu+1}{\nu}\,K_{n-1,\nu-2}, \tag{38}$$

$$=-\frac{(n-1)-(\nu-2)}{\nu}\,K_{n-1,\nu-2}. \tag{39}$$

Example. Given $P_1=\frac{x}{r}$ or μ, to determine P_2, P_3, P_4, ... Here $K_{1,0}=1$;

$$\therefore\ K_{2,2}=-\frac{2-2+1}{2}\,K_{1,0}=-\frac{1}{2}.$$

The sum of all coefficients in a P function is to be taken equal to unity.

$$\therefore\ P_2=\frac{3}{2}\,\frac{x^2}{r^2}-\frac{1}{2}.$$

Similarly,

$$P_3=\frac{5}{2}\,\frac{x^3}{r^3}-\frac{3}{2}\,\frac{x}{r},$$

$$P_4=\frac{35}{8}\,\frac{x^4}{r^4}-\frac{30}{8}\,\frac{x^2}{r^2}+\frac{3}{8}.$$

Other matters concerning spherical harmonics will be found in §§ 4 and 6.

§ 4. *Applications of Lemma* (v).

The solutions $1/\rho^{p-2}$, $1/\sigma^{p-2}$ of equations (1) and (2) become when expanded in powers of x and c_l, where $c_l^2 = y^2 + z^2 + \ldots$,

$$\sum_{\nu=0}^{\infty} (\mp 1)^\nu \frac{(p-2)\,p\,(p+2)\ldots(p+2\nu-4)}{2^\nu} \frac{1}{\nu!} \frac{c_l^{2\nu}}{x^{p+2\nu-2}}. \tag{40}$$

In particular, if $p = 3$,

$$\frac{1}{r} = \frac{1}{x} - \frac{1}{2}\frac{y^2+z^2}{x^3} + \frac{1.3}{2.4}\frac{(y^2+z^2)^2}{x^5} - \frac{1.3.5}{2.4.6}\frac{(y^2+z^2)^3}{x^7} + \ldots\ldots \tag{41}$$

If $\chi(x) \equiv \frac{1}{x}$, then $\chi''(x) = \frac{2!}{x^3}$, $\chi^{\text{IV}}(x) = \frac{4!}{x^5}$, $\ldots$;

$$\therefore \frac{1}{r} = \chi(x) - \frac{1}{2}\frac{1}{2!}(y^2+z^2)\chi''(x) + \frac{1.3}{2.4}\frac{1}{4!}(y^2+z^2)^2\chi^{\text{IV}}(x) - \ldots\ldots \tag{42}$$

Let $\psi(x) = \sinh x$; then $\psi''(x) = \sinh x$; $\psi^{\text{IV}}(x) = \sinh x$, $\ldots$;

$$\therefore \sinh x\left[1 - \frac{1}{2^2}(y^2+z^2) + \frac{1}{2^2 4^2}(y^2+z^2)^2 - \ldots\right], \tag{43}$$

or

$$\sinh x\, J_0(\varpi), \tag{44}$$

where $\varpi^2 = y^2 + z^2$ and J_0 denotes Bessel's function of order zero, is a solution of Laplace's equation (1).

If $\psi(x) = x^n$, then $\psi''(x) = n(n-1)x^{n-2}$,

$$\psi^{\text{IV}}(x) = n(n-1)(n-2)(n-3)x^{n-4}, \ldots;$$

$$\therefore x^n\left[1 - \frac{n(n-1)}{2^2}\frac{y^2+z^2}{x^2} + \frac{n(n-1)(n-2)(n-3)}{2^2 4^2}\frac{(y^2+z^2)^2}{x^4} - \ldots\right] \tag{45}$$

is also a solution. Putting $x = r\mu$, this may be written

$$r^n \mu^n\left[1 - \frac{n(n-1)}{2^2}\frac{1-\mu^2}{\mu^2} + \frac{n(n-1)(n-2)(n-3)}{2^2 4^2}\frac{(1-\mu^2)^2}{\mu^4} - \ldots\right], \tag{46}$$

or

$$r^n P_n(\mu),$$

since, by writing $r^2 - x^2$ for $y^2 + z^2$, (45) is of the form (25) and $r^n P_n(\mu)$ satisfies equation (1).

[It may be well to note that the general expression (45) for a solid zonal harmonic can be obtained by starting with the solution $x^2 - \frac{1}{2}(y^2+z^2)$, where $\chi(x) = x^2$, proceeding to one of the third degree by making $\psi(x) = x^3$, and then to a solution of the fourth degree by making $\psi(x) = x^4$, and so on.

The coefficient of the added term in the solution of the fourth degree, viz., $K_4(y^2+z^2)^2\psi^{IV}(x)$, is found by substituting the solution in equation (1). In general, it will be found, upon equating to zero the entire coefficient of $x^{n-\nu}y^{\nu-2}$ in (1), that $\nu^2 K_\nu = -K_{\nu-2}$. Hence, any solid zonal harmonic may be written

$$x^n - \frac{y^2+z^2}{2^2}\frac{\partial^2(x^n)}{\partial x^2} + \frac{(y^2+z^2)^2}{2^2.4^2}\frac{\partial^4(x^n)}{\partial x^4} - \ldots,$$

which agrees with (45).]

Replacing n by $-n-1$, we obtain the solution

$$\frac{1}{r^{n+1}\mu^{n+1}}\left[1 - \frac{(n+1)(n+2)}{2^2}\frac{1-\mu^2}{\mu^2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2^2\,4^2}\frac{(1-\mu^2)^2}{\mu^4} - \ldots.\right], \quad (47)$$

which, by lemma (ix), must be equivalent to P_n/r^{n+1}.

Since $(y+\sqrt{-1}\,z)^n$ is a solution of equation (1) of the n-th degree and homogeneous, we have by lemma (ix) as another solution this expression divided by r^{2n+1},

$$\frac{(y+\sqrt{-1}\,z)^n}{r^{2n+1}} = \frac{(y+\sqrt{-1}\,z)^n}{x^{2n+1}}\left[1 - \frac{2n+1}{2}\frac{y^2+z^2}{x^2} + \frac{2n+1}{2}\frac{2n+3}{2}\frac{1}{2!}\frac{(y^2+z^2)^2}{x^4} - \frac{2n+1}{2}\frac{2n+3}{2}\frac{2n+5}{2}\frac{1}{3!}\frac{(y^2+z^2)^3}{x^6} + \ldots.\right], \quad (48)$$

which may, by putting $\chi(x) = 1/x^{2n+1}$, be written

$$(y+\sqrt{-1}\,z)^n\left[\chi(x) - \frac{y^2+z^2}{2^2(n+1)}\chi''(x) + \frac{(y^2+z^2)^2}{2^4\,2!\,(n+1)(n+2)}\chi^{IV}(x) - \frac{(y^2+z^2)^3}{2^6\,3!\,(n+1)(n+2)(n+3)}\chi^{VI}(x) + \ldots.\right]. \quad (49)$$

Putting $\psi(x) = \sinh x$, we have as a solution

$$\sinh x\,(y+\sqrt{-1}\,z)^n\left[1 - \frac{y^2+z^2}{2^2(n+1)} + \frac{(y^2+z^2)^2}{2^4\,2!\,(n+1)(n+2)} - \frac{(y^2+z^2)^3}{2^6\,3!\,(n+1)(n+2)(n+3)} + \ldots.\right]. \quad (50)$$

Since $(y+\sqrt{-1}\,z)^n = (\cos n\phi + \sqrt{-1}\sin n\phi)(y^2+z^2)^{n/2}$, this solution may be written

$$c_n \sinh x\,(\cos n\phi + \sqrt{-1}\sin n\phi)\,J_n(\varpi), \quad (51)$$

where c_n denotes a constant dependent upon n.

If $\psi(x) = x^{n-s}$, then (49) becomes

$$(y + \sqrt{-1}\, z)^s x^{n-s}\left[1 - \frac{(n-s)(n-s-1)}{2^2(s+1)}\frac{y^2+z^2}{x^2} + \frac{(n-s)(n-s-1)(n-s-2)(n-s-3)}{2^4\, 2!\,(s+1)(s+2)}\frac{(y^2+z^2)^2}{x^4} - \ldots.\right], \quad (52)$$

$$= (y + \sqrt{-1}\, z)^s (r\mu)^{n-s}\left[1 - \frac{(n-s)(n-s-1)}{2^2(s+1)}\frac{1-\mu^2}{\mu^2} + \frac{(n-s)(n-s-1)(n-s-2)(n-s-3)}{2^4\, 2!\,(s+1)(s+2)}\frac{(1-\mu^2)^2}{\mu^4} - \ldots.\right]. \quad (53)$$

Since $y^2 + z^2 = r^2 - x^2$, (52) can be broken up into terms containing the same powers of the variables as do the terms of (31); therefore their coefficients are proportional and so (53) may be written

$$c_{n,s}\, r^n (\cos s\phi + \sqrt{-1}\sin s\phi)\, T_n^{(s)}, \quad (54)$$

where $c_{n,s}$ denotes a constant dependent upon n and s.

It is obvious that if in (46) divided by r^n, or

$$P_n(\cos\theta) = \cos^n\theta\left[1 - \frac{n(n-1)}{2^2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{2^2\, 4^2}\tan^4\theta - \ldots.\right], \quad (55)$$

we put $\theta = \varpi/n$, and then cause n to become infinite, we shall have

$$\operatorname*{Limit}_{n=\infty} P_n\left(\cos\frac{\varpi}{n}\right) = 1 - \frac{\varpi^2}{2^2} + \frac{\varpi^4}{2^2\, 4^2} - \ldots. = J_0(\varpi). \quad (56)$$

Similarly, if in (53) or

$$r^{n-s}(y + \sqrt{-1}\, z)^s \cos^{n-s}\theta\left[1 - \frac{(n-s)(n-s-1)}{2^2(s+1)}\tan^2\theta + \frac{(n-s)(n-s-1)(n-s-2)(n-s-3)}{2^4\, 2!\,(s+1)(s+2)}\tan^4\theta - \ldots.\right] \quad (57)$$

we put $\theta = \varpi/s$, and then cause s to become infinite in the part which is multiplied by $r^{n-s}(\cos\phi + \sqrt{-1}\sin\phi)^s$, we shall have for this part

$$\varpi^s\left[1 - \frac{\varpi^2}{2^2(s+1)} + \frac{\varpi^4}{2^4\, 2!\,(s+1)(s+2)} - \ldots.\right] = c_s' J_s(\varpi), \quad (58)$$

where c_s' denotes a constant dependent upon s.

According to lemma (ix), x/r^3 is a solution; it is, in fact, the velocity-potential defining the motion of a sphere in an infinite mass of water. x/r^3 is the kind of solution satisfying condition (5). If in

$$\frac{x}{r^3} = \frac{1}{x^2} - \frac{1.3}{2}\frac{y^2+z^2}{x^4} + \frac{1.3.5}{2.2}\frac{1}{2!}\frac{(y^2+z^2)^2}{x^6} - \ldots . \tag{59}$$

we put $\chi(x) = \frac{1}{x^2}$, then $\chi''(x) = \frac{2.3}{x^4}$, $\chi^{IV}(x) = \frac{2.3.4.5}{x^6}$, ..., we have

$$\frac{x}{r^3} = \chi(x) - \frac{1}{2}\frac{1}{2!}(y^2+z^2)\chi''(x) + \frac{1.3}{2.4}\frac{1}{4!}(y^2+z^2)^2\chi^{IV}(x) - \ldots ., \tag{60}$$

which, being of the same form as (42), leads to no solution additional to those which might have been obtained from $1/r$.

In studying Bessel's functions, it is more natural to begin with solutions of (2) than with solutions of (1). We have as an illustration the physical problem of the vibration of a circular drumhead, or of wave-motion in a circular sheet of water, the periodicity of the phenomenon suggesting circular time-functions.

By lemma (viii) we have as a solution of (2)

$$\frac{1}{\sqrt{x^2-(y^2+z^2)}} = \frac{1}{x} + \frac{1}{2}\frac{y^2+z^2}{x^3} + \frac{1.3}{2.4}\frac{(y^2+z^2)^2}{x^5} + \frac{1.3.5}{2.4.6}\frac{(y^2+z^2)^3}{x^7} + \ldots . \tag{61}$$

Putting $\chi(x) = 1/x$, this becomes

$$\chi(x) + \frac{1}{2}\frac{1}{2!}(y^2+z^2)\chi''(x) + \frac{1.3}{2.4}\frac{1}{4!}(y^2+z^2)^2\chi^{IV}(x) + \ldots . \tag{62}$$

Suppose, now, that x, y, z are replaced by t, x, y in this expression and in equation (2). Let $\psi(t) = \cos t$; then $\psi''(t) = -\cos t$, $\psi^{IV}(t) = \cos t$, ..., and we have as a solution of (2) thus altered

$$\cos t\left[1 - \frac{1}{2^2}(x^2+y^2) + \frac{1}{2^2 4^2}(x^2+y^2)^2 - \ldots .\right] = \cos t\, J_0(\varpi), \tag{63}$$

where $\varpi^2 = x^2 + y^2$.

If $\psi(x) = x^n$ we have $\sigma^n P_n\left(\frac{x}{\sigma}\right)$ as a solution of (2), σ^2 denoting $x^2 - (y^2+z^2)$. This solution could have been obtained from $r^n P_n\left(\frac{x}{r}\right)$ by lemma (iv).

By lemma (viii),

$$\frac{1}{t^2-(x^2+y^2+z^2)} \text{ or } \frac{1}{t^2-r^2} \tag{64}$$

is a solution of (2) when x, y, z, w are replaced by t, x, y, z. In

$$\frac{1}{t^2 - r^2} = \frac{1}{t^2} + \frac{r^2}{t^4} + \frac{r^4}{t^6} + \frac{r^6}{t^8} + \ldots . \tag{65}$$

put $\chi(t) = 1/t^2$ and it becomes

$$\frac{1}{t^2 - r^2} = \chi(t) + \frac{r^2}{3!}\chi''(t) + \frac{r^4}{5!}\chi^{\text{IV}}(t) + \ldots . \tag{66}$$

Let $\psi(t) = \sin t$; we then have as a solution of (2)

$$V_4 = \sin t \frac{\sin r}{r}. \tag{67}$$

The solutions obtained in the next section are also of the kind specified in lemma (v), as will be shown presently.

§ 5. *Solutions Obtained by an Application of Taylor's Theorem.*

Suppose $\phi(\alpha x + \beta y + \gamma z + \ldots .)$ to be expanded in powers of $\beta y + \gamma z + \ldots .$ thus,

$$\phi(\alpha x) + \frac{\beta y + \gamma z + \ldots .}{1!\,\alpha}\phi'(\alpha x) + \frac{(\beta y + \gamma z + \ldots .)^2}{2!\,\alpha^2}\phi''(\alpha x) + \frac{(\beta y + \gamma z + \ldots .)^3}{3!\,\alpha^3}\phi'''(\alpha x) + \ldots ., \tag{68}$$

the accent denoting differentiation with respect to x. (If the accent denote differentiation with respect to αx, then, of course, α would not appear in the denominator.)

Equation (1) will be satisfied by this expression if

$$\alpha^2 + \beta^2 + \gamma^2 + \ldots . = 0; \tag{69}$$

and equation (2) if

$$\alpha^2 - (\beta^2 + \gamma^2 + \ldots .) = 0. \tag{70}$$

For, differentiating the coefficient of $\phi^{(\nu)}(\alpha x)$ twice with respect to each of the variables $y, z, \ldots .$ and adding the results together, we have

$$\frac{\beta^2(\beta y + \gamma z + \ldots .)^{\nu-2}}{(\nu-2)!\,\alpha^\nu} + \frac{\gamma^2(\beta y + \gamma z + \ldots .)^{\nu-2}}{(\nu-2)!\,\alpha^\nu} + \ldots .$$

The term

$$\frac{(\beta y + \gamma z + \ldots .)^{\nu-2}}{(\nu-2)!\,\alpha^{\nu-2}}\phi^{(\nu-2)}(\alpha x)$$

twice differentiated with respect to x becomes

$$\frac{\alpha^2(\beta y + \gamma z + \ldots .)^{\nu-2}}{(\nu-2)!\,\alpha^\nu}\phi^{(\nu)}(\alpha x).$$

Thus it is seen that if $\phi(\alpha x + \beta y + \gamma z + \dots)$ be substituted in (1) or (2) the coefficient of the resulting $\phi^{(\nu)}(\alpha x)$ will be zero, and so (1) or (2) will be satisfied when the relation (69) or (70) obtains. Or what amounts to the same thing, the given $(y, z, \dots)$-coefficient of $\phi^{(\nu-2)}(\alpha x)$, added to the $\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \dots\right)$-derivative of $\phi^{(\nu)}(\alpha x)$, is zero, and so this comes under lemma (v).

[It may be noted here that (68) is a solution of the equations

$$\frac{\partial^n V}{\partial x^n} + \frac{\partial^n V}{\partial y^n} + \frac{\partial^n V}{\partial z^n} + \dots = 0, \tag{71}$$

$$\frac{\partial^n V}{\partial x^n} - \left(\frac{\partial^n V}{\partial y^n} + \frac{\partial^n V}{\partial z^n} + \dots\right) = 0, \tag{72}$$

provided that, in the respective equations,

$$\alpha^n + \beta^n + \gamma^n + \dots = 0, \tag{73}$$

$$\alpha^n - (\beta^n + \gamma^n + \dots) = 0. \tag{74}$$

This can be shown by differentiating the coefficient of $\phi^{(\nu)}(\alpha x)$ n times with respect to $y, z, \dots$, and adding the results together, and by differentiating the term

$$\frac{(\beta y + \gamma z + \dots)^{\nu-n}}{(\nu-n)!\,\alpha^{\nu-n}}\,\phi^{(\nu-n)}(\alpha x)$$

n times with respect to x.]

The whole of the expression $\phi(\alpha x + \beta y + \gamma z + \dots)$ satisfies (1) or (2). Consequently, if it can be thrown into parts linearly independent of one another, each such part must be a solution.

For separating the solutions of (1) just referred to, it is convenient to make use of a complex variable $\alpha x + ib y + jcz + kdw + \dots$ of p dimensions, where the original units are

$$1, \quad i = \sqrt{-1}, \quad j = \sqrt{-1}, \quad k = \sqrt{-1}, \quad \dots, \tag{75}$$

and where

$$a^2 = b^2 + c^2 + d^2 + \dots. \tag{76}$$

Let p denote the dimension of the complex or the number of original units; then the total number of units in the $(1, i, j, \dots)$-algebra is

$$p + c_2(p-1) + c_3(p-1) + \dots + c_{p-1}(p-1). \tag{77}$$

Here $c_2(p-1)$ means the number of combinations of $p-1$ things taken two at a time.

Similarly, for separating the solutions of (2) it is convenient to make use of a complex variable $ax + \mathrm{i}by + \mathrm{j}cz + \mathrm{k}dw + \ldots$ of p dimensions, where the original units are

$$1, \quad \mathrm{i} = \sqrt{+1}, \quad \mathrm{j} = \sqrt{+1}, \quad \mathrm{k} = \sqrt{+1}, \quad \ldots, \tag{78}$$

and where

$$a^2 = b^2 + c^2 + d^2 + \ldots.$$

The parts of (68) thus separated may in the first instance be denoted by the Italic symbols

$$R_1, \quad iR_i, \quad jR_j, \quad ijR_{ij}, \quad \ldots, \tag{79}$$

and in the second by the Roman symbols

$$\mathrm{R}_1, \quad \mathrm{i}\mathrm{R}_\mathrm{i}, \quad \mathrm{j}\mathrm{R}_\mathrm{j}, \quad \mathrm{ij}\mathrm{R}_\mathrm{ij}, \quad \ldots. \tag{80}$$

If $p = 2$, the two solutions of (1) are the ordinary X, Y of the function-theory; the two solutions of equation (2) are somewhat analogous functions which may be denoted by Roman X, Y.

In the case $p = 3$, we have as solutions of (1)

$$R_1 + iR_i + jR_j + ijR_{ij} = \phi(ax + iby + jcz) = \phi[ax + i(by - ijcz)]; \tag{81}$$

$$R_1 = \phi(ax) - \frac{b^2y^2 + c^2z^2}{2!\,a^2}\phi''(ax) + \frac{b^4y^4 + 6b^2c^2y^2z^2 + c^4z^4}{4!\,a^4}\phi^{\mathrm{IV}}(ax) - \ldots., \tag{82}$$

$$R_i = \frac{by}{a}\phi'(ax) - \frac{b^3y^3 + 3bc^2yz^2}{3!\,a^3}\phi'''(ax) + \frac{b^5y^5 + 10b^3c^2y^3z^2 + 5bc^4yz^4}{5!\,a^5}\phi^{\mathrm{V}}(ax) - \ldots., \tag{83}$$

$$R_j = \frac{cz}{a}\phi'(ax) - \frac{c^3z^3 + 3cb^2zy^2}{3!\,a^3}\phi'''(ax) + \frac{c^5z^5 + 10c^3b^2z^3y^2 + 5cb^4zy^4}{5!\,a^5}\phi^{\mathrm{V}}(ax) - \ldots., \tag{84}$$

$$R_{ij} = \frac{bcyz}{a^2}\phi''(ax) - \frac{b^3cy^3z + bc^3yz^3}{3!\,a^4}\phi^{\mathrm{IV}}(ax) + \frac{b^5cy^5z + 5b^3c^3y^3z^3 + bc^5yz^5}{5!\,a^6}\phi^{\mathrm{VI}}(ax) - \ldots.. \tag{85}$$

$$\therefore \; R_1 - R_{ij} + \sqrt{-1}\,(R_i + R_j) = \sum_{\nu=0}^{\nu=n}(-1)^{\nu/2}\frac{(by + cz)^\nu}{\nu!\,a^\nu}\phi^{(\nu)}(ax). \tag{86}$$

In all these cases the accent denotes differentiations with respect to x.

For the corresponding solutions of equation (2) we have

$$\mathrm{R}_1 + \mathrm{i}\,\mathrm{R}_i + \mathrm{j}\,\mathrm{R}_j + \mathrm{ij}\,\mathrm{R}_{ij} = \phi(ax + \mathrm{i}by + \mathrm{j}cz) = \phi[(ax + \mathrm{i}by + \mathrm{ij}cz)]. \quad (87)$$

The R's may be obtained from the preceding expressions by changing each minus sign into a plus.

If $\nu = \infty$,

$$R_1 + R_{ij} = \cos\left[\frac{by - cz}{a}\frac{\partial}{\partial x}\right]\phi(ax), \quad (88)$$

$$R_1 - R_{ij} = \cos\left[\frac{by + cz}{a}\frac{\partial}{\partial x}\right]\phi(ax), \quad (89)$$

$$R_i + R_j = \sin\left[\frac{by + cz}{a}\frac{\partial}{\partial x}\right]\phi(ax), \quad (90)$$

$$R_i - R_j = \sin\left[\frac{by - cz}{a}\frac{\partial}{\partial x}\right]\phi(ax). \quad (91)$$

For corresponding combinations of the R's, replace the symbols cos, sin by cosh, sinh.

As already stated, (68) is a solution of (1) or (2) of the form referred to in lemma (v), and so the linearly independent parts R_1, R_i, R_j, R_{ij}, ... are severally solutions of that form.

A lemma similar to (v) is applicable to solutions of (71) and (72); the 2's throughout equation (5) must then be changed to n's.

§ 6. *Applications of Complex Variables of Two or More Dimensions.*

Instances of the use of the ordinary two-dimensional variable $x + iy$, or $x - iy$, and the variable $\log r + i\theta$ $\left(\text{where } r^2 = x^2 + y^2,\ \theta = \tan^{-1}\frac{y}{x}\right)$ for finding or expressing solutions of equation (1) need not here be given, since they frequently occur in cartography, in works on hydrodynamics, electricity and magnetism, and other branches of mathematical physics.

A few examples of such solutions of (1) and (2) will now be given as may be expressed by aid of the other complex variables mentioned in the preceding section. The examples show that solutions of this kind, or aggregates of them, may have physical applications; also, that they are in many ways analogous to solutions made up of the ordinary X- and Y-functions. The conception of these complex quantities may not be of great importance in the obtaining of solutions satisfying prescribed conditions; but when solutions have been once obtained, this idea often facilitates their inspection.

Example (1). Any function ϕ of the variable $x + iy$, where $i = \sqrt{+1}$, gives two solutions of equation (2); that is

$$\phi(x + iy) = R_1 + iR_i \tag{92}$$

gives R_1, R_i as solutions.

If for x we put xt, and for y, x, then R_1, R_i are solutions of

$$\frac{\partial^2 V}{(\partial xt)^2} - \frac{\partial^2 V}{\partial x^2} = 0, \quad \text{or} \quad \frac{\partial^2 V}{dt^2} - x^2 \frac{\partial^2 V}{\partial x^2} = 0. \tag{93}$$

For a vibrating string, the displacement V should be periodic with respect to t. ϕ may be put equal to $A_1 \cos \frac{\pi}{l}$ or $A_1 \sin \frac{\pi}{l}$, where l denotes the length of the cord. Suppose $\phi = A_1 \sin \frac{\pi}{l}$; then

$$R_1 + iR_i = A_1 \sin \frac{\pi}{l}(xt + ix), \tag{94}$$

and so

$$R_1 = A_1 \sin \frac{\pi x t}{l} \cos \frac{\pi x}{l}, \quad R_i = A_1 \cos \frac{\pi x t}{l} \sin \frac{\pi x}{l}. \tag{95}$$

The displacement R_i becomes zero when $x = 0$, $x = l$; and $\frac{\partial R_i}{\partial t} = 0$ when $t = 0$. The problem of having the string initially at rest, and conforming to a prescribed curve whose equation is

$$y \text{ or } R_i = f(x),$$

generally requires for its solution an infinite number of terms like R_i. That is, when $t=0$, y is of the form

$$y = A_1 \sin \frac{\pi x}{l} + A_2 \sin \frac{2\pi x}{l} + A_3 \sin \frac{3\pi x}{l} + \ldots\ldots \tag{96}$$

In this case the function ϕ is really

$$R_1 + iR_i = A_1 \sin \frac{\pi}{l}(xt + ix) + A_2 \sin \frac{2\pi}{l}(xt + ix) + A_3 \sin \frac{3\pi}{l}(xt + ix) + \ldots\ldots \tag{97}$$

Example (2). Using the variable $ax + iby + jcz$, where $i = \sqrt{+1}$, $j = \sqrt{+1}$ and, of course, $a^2 = b^2 + c^2$, any function

$$\phi(ax + iby + jcz) = R_1 + iR_i + jR_j + ijR_{ij} \tag{98}$$

gives four solutions of (2). If for x we put xt, for y, x, and for z, y, the R's are solutions of the equation

$$\frac{\partial^2 V}{\partial t^2} - x^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) = 0. \tag{99}$$

Giving to b, c, the values $\frac{m\pi}{h_1}$, $\frac{n\pi}{h_2}$, and so to a the value $\sqrt{\frac{m^2}{h_1^2}+\frac{n^2}{h_2^2}}\,\pi$, and making $\phi =$ cosine, we have

$$\left.\begin{aligned} R_{ij} &= -\cos \pi x t \sqrt{\frac{m^2}{h_1^2}+\frac{n^2}{h_2^2}} \sin \frac{m\pi x}{h_1} \sin \frac{n\pi y}{h_2}, \\ R_1 &= \cos \pi x t \sqrt{\frac{m^2}{h_1^2}+\frac{n^2}{h_2^2}} \cos \frac{m\pi x}{h_1} \cos \frac{n\pi y}{h_2}. \end{aligned}\right\} \quad (100)$$ *

Functions like these are suitable for representing the free vibrations of a rectangular or square drum-head, or for representing the motion of long stationary waves (*i. e.*, oscillations) in a rectangular or square tank of uniform depth. See example (12) below.

Example (3). Taking as variable $ax + iby + jcz$, where $i = \sqrt{-1}$, $j = \sqrt{-1}$, $a^2 = b^2 + c^2$, and letting

$$\phi(ax + iby + jcz) = e^{-(ax+iby+jcz)}, \quad (101)$$

we have

$$R_{ij} = e^{-ax} \sin by \sin cz. \quad (102)$$

If for b, c we write

$$\frac{(2m+1)}{h_1}\pi, \quad \frac{(2n+1)}{h_2}\pi,$$

and so for a,

$$\sqrt{\frac{(2m+1)^2}{h_1^2}+\frac{(2n+1)^2}{h_2^2}}\,\pi,$$

we have

$$R_{ij} = e^{-\pi x\sqrt{\frac{(2m+1)^2}{h_1^2}+\frac{(2n+1)^2}{h_2^2}}} \sin \frac{2m+1}{h_1}\pi y \sin \frac{2n+1}{h_2}\pi z. \quad (103)$$

This solution of equation (1) is applicable to the potential at points within an infinitely long rectangular prism, where the potential at the sides is zero, and that at the base is a given constant.†

Example (4). Taking $x + iy\cos\beta + jz\sin\beta$ as the variable, and $e^{-i()}$ as the function, we have

$$\phi(ax + iby + jcz) = e^{-i(x+iy\cos\beta+jz\sin\beta)}, \quad (104)$$

which may be written

$$\begin{aligned} & e^{y\cos\beta - ijz\sin\beta}\cos x - i\sin x; \\ \therefore\ & R_1 + ijR_{ij} = e^{y\cos\beta - ijz\sin\beta}\cos x, \\ & iR_i + jR_j = -ie^{y\cos\beta - ijz\sin\beta}\sin x \end{aligned} \quad (105)$$

* See Byerly, "Fourier's Series, etc.," pp. 126–134.

† Byerly, pp. 138, 139.

are solutions of equation (1). If we do not care to distinguish between the 1-terms and the ij-terms, or the i-terms and the j-terms, we may put $ij = \pm 1$ and $j = \pm i$ in the above results. Multiplying by suitable time-factors and combining, we obtain as a solution of (1)

$$e^{y\cos\beta - x\sin\beta}\cos(x - \varkappa t). \tag{106}$$

This denotes the velocity-potential for a train of "edge" waves advancing in the x-direction, the shore sloping at an angle β.

This solution is due to Stokes.*

Example (5). If $a^2 = 1$, $b^2 = \frac{1}{2}$, $c^2 = \frac{1}{2}$, and $\phi =$ cosine, we have

$$\phi(ax + iby + jcz) = \cos\left(x + \frac{i}{\sqrt{2}}y + \frac{j}{\sqrt{2}}z\right); \tag{107}$$

$$\therefore\ R_1 = \cos x \cosh\frac{y}{\sqrt{2}}\cosh\frac{z}{\sqrt{2}}. \tag{108}$$

This expression, when multiplied by the time-factor, denotes the velocity-potential of a standing wave in a V-shaped canal whose sides make angles of 45° with the vertical. This type was discovered by Kelland.†

$$-R_{ij} = \cos x \sinh\frac{y}{\sqrt{2}}\cosh\frac{z}{\sqrt{2}}$$

denotes the velocity-potential of an asymmetrical mode, noticed by Greenhill.‡

Example (6). Considering now two variables $x + ix$, $x + i\frac{\sqrt{3}}{2}y + j\frac{1}{2}z$, whose coefficients evidently satisfy the relation $a^2 = b^2 + c^2$, the R's of

$$A\cos(x + iz) + C\cos\left(x + i\frac{\sqrt{3}}{2}y + j\frac{1}{2}z\right) \tag{109}$$

and of

$$B\sin(x + iz) + D\sin\left(x + i\frac{\sqrt{3}}{2}y + j\frac{1}{2}z\right) \tag{110}$$

are evidently solutions of equation (1); so will be the R_1 of the former added to the R_i of the latter. The result is

$$\cos x\left[A\cosh z + B\sinh z + \cosh\frac{\sqrt{3}}{2}y\left(\cosh\frac{1}{2}z + D\sinh\frac{1}{2}z\right)\right]. \tag{111}$$

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* Lamb, "Hydrodynamics," 2d Ed. (1895), p. 431; 3d Ed. (1905), p. 424.

† *Ibid.*, 2d Ed., p. 432; 3d Ed., p. 424.

‡ *Ibid.*, 3d Ed., p. 425.

This expression, when multiplied by the time-factor, is the velocity-potential for a type of standing waves in a canal whose sides are inclined to the vertical at an angle of 60°. A, B, C, D are determined by the boundary and surface conditions. This comprehensive type was discovered by Macdonald.*

Example (7). If

$$\phi(ax+iby+jcz)=\log(ax+iby+jcz), \tag{112}$$

then

$$R_i-R_j=\tan^{-1}\left(\frac{by+cz}{ax}\right) \tag{113}$$

must be a solution of equation (1), provided, of course, that $a^2=b^2+c^2$.

Example (8). If in

$$\phi(ax+iby+jcz)=(ax+iby+jcz)^2 \tag{114}$$

we make $a=\sqrt{2}$, $b=1$, $c=1$, then

$$\left.\begin{aligned} &R_1=2x^2-y^2-z^2, && R_i=2\sqrt{2}\,xy,\\ &R_j=2\sqrt{2}\,xz, && R_{ij}=2yz. \end{aligned}\right\} \tag{115}$$

If we make $c=0$, and so $a=1$, $b=1$, then

$$R_1=x^2-y^2, \qquad R_i=2xy. \tag{116}$$

These are rational integral functions of degree 2 satisfying equation (1), and so are harmonic functions.

Replacing the exponent 2 by 3, 4, ..., rational functions of those degrees are obtained which satisfy equation (1). This fact has been noted by Laurent.† For the exponent 3 we have

$$\begin{aligned} R_1+iR_i+jR_j+ijR_{ij}&=\sqrt{2}(2x^3-3xy^2-3xz^2)\\ &+i(6x^2y-3yz^2-y^3)+j(6x^2z-3y^2z-z^3)+ij\,6\sqrt{2}xyz. \end{aligned} \tag{117}$$

But

$$R_1=\sqrt{2}\,[2x^3-3x(y^2+z^2)], \tag{118}$$

having circular symmetry about the x-axis, is, like $2x^2-y^2-z^2$, a zonal harmonic.

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* See Lamb, "Hydrodynamics," 2d Ed., p. 434; 3d Ed., p. 426.

† "Il est bon d'observer que, si

$$\alpha^2+\beta^2+1=0, \quad (ax+\beta y+z)^m$$

satisfait à $\Delta_2 u=0$ et $(\alpha\sin\theta\cos\psi+\beta\sin\theta\sin\psi+\cos\theta)^m$ est une fonction sphérique du degré m. Une somme de $2m+1$ semblables fonctions sera la fonction sphérique la plus générale." H. Laurent, *Traité d'Analyse*, Tome VI, pp. 297, 298.

If we raise $\sqrt{2}x + iy + jz$ to the fourth power, the harmonic function R_1 thus obtained no longer possesses circular symmetry about the x-axis, although symmetrical in y^2 and z^2; for then

$$R_1 = 4x^4 - 12x^2y^2 - 12x^2z^2 + 6y^2z^2 + y^4 + z^4. \tag{119}$$

Suppose we raise the variable $y + iz$ to the fourth power; its R_1 or R_i must satisfy (1). [Cf. lemma (iii)].

Adding its R_1, $= -6y^2z^2 + y^4 + z^4$, to twice the preceding R_1, we have a zonal harmonic of the fourth order

$$8x^4 - 24x^2(y^2 + z^2) + 3(y^2 + z^2)^2. \tag{120}$$

Were it desirable to do so, it seems probable that the zonal harmonic of any order could be obtained by combining the R_1's of this power of variables of two and three dimensions.

Example (9). Still considering the case where $p = 3$, it will be noticed that the complex quantity $ax + iby + jcz$ contains one effective variable parameter. Let $a = 1$; then by (76), the quantity may be written $x + iby + j\sqrt{1 - b^2}z$. Hence, whenever in the expansion of $\phi(x + iby + j\sqrt{1 - b^2}z)$ it is possible to pick out the entire coefficient of any power of b, that coefficient is a solution of (1). This can be easily done for

$$(x + iby + j\sqrt{1 - b^2}z)^n, \tag{121}$$

where n is a positive integer. The solutions thus obtained are $2n + 1$ in number, as may be seen upon considering the four kinds (82)–(85), the first containing no odd powers of y or z:

	n odd	n even
From R_1,	$1 + \frac{n-1}{2}$	$1 + \frac{n}{2}$
" R_i,	$\frac{n+1}{2}$	$\frac{n}{2}$
" R_j,	$1 + \frac{n-1}{2}$	$1 + \frac{n-2}{2}$
" R_{ij},	$\frac{n-1}{2}$	$\frac{n}{2}$
Total,	$2n + 1$	$2n + 1$

(122)

The coefficients of the various powers of b (including those of b^0) in the several kinds of terms may therefore be taken as the $2n + 1$ independent solutions of (1) of the n-th degree.

Now since the solution $r^n P_n$ is an homogeneous polynomial of degree n where y and z enter by even powers only, and which satisfies (1), it follows that it can be made up of linear combinations of the solutions of the first kind obtained by the expansion of (121). Hence, the surmise made at the end of example (8) turns out to be correct.

We can assume that if any solution V be a regular function of x, y, z within a finite domain surrounding x', y', z', it can be expanded in powers of X, Y, Z, where $x = x' + X$, etc., and the series will there be absolutely and uniformly convergent. The various powers of the variables x, y, z will give rise to homogeneous polynomials of all degrees (not exceeding n) in X, Y, Z. Each is a solution of (1) because, by hypothesis, the whole series is a solution and because of the homogeneous character of (1). From what has just been shown, it follows that for any such polynomial of degree n there are at most $2n+1$ independent solutions. Hence, if the expansion be carried to the n-th powers of X, Y, Z, we obtain at most $\sum_{\nu=0}^{\nu=n} 2\nu + 1$ or $(n+1)^2$ solutions.*

Example (10). Taken in connection with the foregoing section, it seems highly probable that $\frac{1}{r}$ can be expressed in a similar manner by using the R_1 of an infinite number of terms of the form

$$\frac{1}{ax + iby + jcz}. \tag{123}$$

We have

$$\left.\begin{aligned} &\frac{1}{2} R_1 \left[\frac{1}{2(x+iy)} + \frac{1}{2(x+jz)} + \frac{\sqrt{2}}{\sqrt{2}\,x + iy + jz} \right], \\ &= \frac{1}{x}\left[1 - \frac{1}{2}\frac{y^2+z^2}{x^2} + \frac{1.3}{2.4}\frac{(y^2+z^2)^2}{x^4} + \frac{1.3.5}{2.4.6}\frac{(y^2+z^2)^3}{x^6} + \dots \right], \end{aligned}\right\} \tag{124}$$

which agrees with the value of $1/r$ (41) to the number of terms here given.

It may be noted that a term of the form (123) becomes infinite only when the variable becomes zero or lies upon one of the two lines in the yz-plane whose equations are

$$x = 0, \quad by = \pm cz. \tag{125}$$

* Taken mainly from an important paper (*q. v.*) by Professor E. T. Whittaker, entitled "On the Partial Differential Equations of Mathematical Physics," *Mathematische Annalen*, Vol. LVII. He appears to have been the first to show that (121) gives the general homogeneous polynomial solution of (1).

For, (123) when rationalized in denominator gives

$$\frac{a\,x - i\,b\,y - j\,c\,z}{a^2x^2 + b^2y^2 + c^2z^2 - 2\,ij\,bcyz}$$
$$= \frac{(a\,x - i\,b\,y - j\,c\,z)(a^2x^2 + b^2y^2 + c^2z^2 + 2\,ij\,bcyz)}{(a^2x^2 + b^2y^2 + c^2z^2)^2 - 4\,b^2c^2y^2z^2}. \qquad (126)$$

The denominator of this expression vanishes whenever

$$(a\,x)^2 + (b\,y \mp c\,z)^2 = 0.$$

Example (11). Consider the R_1 terms of (123) or (126). They are

$$\frac{a\,x\,(a^2x^2 + b^2y^2 + c^2z^2)}{(a^2x^2 + b^2y^2 + c^2z^2)^2 - 4\,b^2c^2y^2z^2}. \qquad (127)$$

Putting $a = 1$ and eliminating c by means of the equation $-a^2 + b^2 + c^2 = 0$, the fraction becomes

$$\frac{x\,[x^2 + z^2 + b^2\,(y^2 - z^2)]}{(x^2 + z^2)^2 + 2\,b^2\,(x^2y^2 - x^2z^2 - y^2z^2 - z^4) + b^4\,(y^2 + z^2)^2} \qquad (128)$$

and may, for brevity, be written

$$\frac{x\,[x^2 + z^2 + (y^2 - z^2)\,b^2]}{p + q\,b^2 + r\,b^4}$$
$$= x\,[x^2 + z^2 + (y^2 - z^2)\,b^2]\left[\frac{1}{p} - \frac{q}{p^2}\,b^2 + \frac{1}{p}\left(\frac{-r}{p} + \frac{q^2}{p^2}\right)b^4 - \ldots.\right], \qquad (129)$$

a development in descending powers of $(x^2 + z^2)^2$.

The coefficient of b^0 is

$$\frac{x}{x^2 + z^2}; \qquad (130)$$

the coefficient of b^2 is

$$x\,\frac{z^2 - y^2}{(x^2 + z^2)^2} + 4\,x\,\frac{y^2z^2}{(x^2 + z^2)^3}; \qquad (131)$$

either is a solution of equation (1).

Example (12). For phenomena periodic in time and involving the differential relation

$$\frac{\partial^2}{\partial t^2} = x^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \qquad (132)$$

ϕ should consist of a number of sine or cosine terms or of both. Here suppose ϕ to consist of terms of the form

$$\cos\,(x\,a\,t + \mathrm{i}\,b\,x + \mathrm{j}\,c\,y). \qquad (133)$$

If the phenomena be *simply* harmonic in time, then the square of the a of each term of the function will be the same. The R's become

$$\left.\begin{aligned} R_1 &= \cos \varkappa a t \cos bx \cos cy + \cos \varkappa a t \cos b'x \cos c'y + \ldots, \\ -R_i &= \sin \varkappa a t \sin bx \cos cy + \sin \varkappa a t \sin b'x \cos c'y + \ldots, \\ -R_j &= \sin \varkappa a t \cos bx \sin cy + \sin \varkappa a t \cos b'x \sin c'y + \ldots, \\ -R_{ij} &= \cos \varkappa a t \sin bx \sin cy + \cos \varkappa a t \sin b'x \sin c'y + \ldots, \end{aligned}\right\} \quad (134)$$

where $a^2 = b^2 + c^2 = b'^2 + c'^2 = \ldots$.

As already noted, functions of the form $\cos \varkappa a t \sin bx \sin cy$ or $\cos \varkappa a t \cos bx \cos cy$ are suitable for representing the vibration of a rectangular drum-

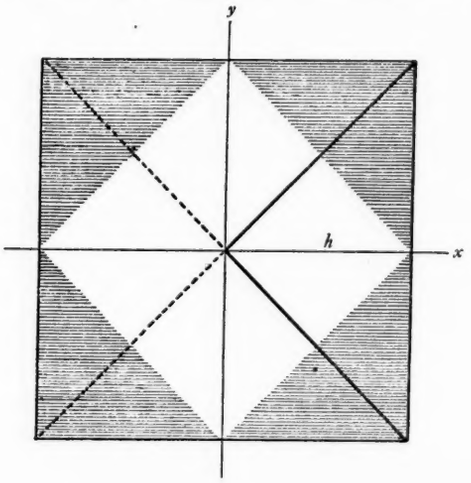

FIG. 1.

head or the oscillation of a shallow rectangular body of water. It remains to suggest here that by taking a sufficient number of similar terms it would probably be possible to thereby represent these phenomena in triangular, circular, or any other shaped areas. For instance, the solution given by Lamé* for the vibration of a membrane whose form is that of an equilateral triangle will be seen to consist of terms of the form (134) as soon as his trilinear coördinates are replaced by Cartesian coördinates conveniently chosen. We shall here confine ourselves to the oscillations of shallow bodies of water.

One mode of oscillation for a 45° triangle is shown in Fig. 1. The vertical displacement, ζ, is proportional to

$$\cos \frac{\pi x t}{h} \left(\cos \frac{\pi x}{h} + \cos \frac{\pi y}{h} \right), \quad (135)$$

* "Leçons sur la théorie mathématique de l'élasticité des corps solides," p. 132.

which is of the form

$$\cos \varkappa a t \, [\cos bx \cos cy + \cos b'x \cos c'y],$$

where $c^2 = b'^2 = 0$, $b^2 = c'^2 = a^2$, and $\varkappa^2 = g \times$ depth of water.

Had the coördinate axes been taken along the legs of the triangle, the above displacement would have been written as one term, viz.,

$$2 \cos \frac{\pi \varkappa t}{h} \cos \frac{\pi x}{\sqrt{2}h} \cos \frac{\pi y}{\sqrt{2}h}, \tag{136}$$

which is of the form

$$2 \cos \varkappa a t \cos bx \cos cy,$$

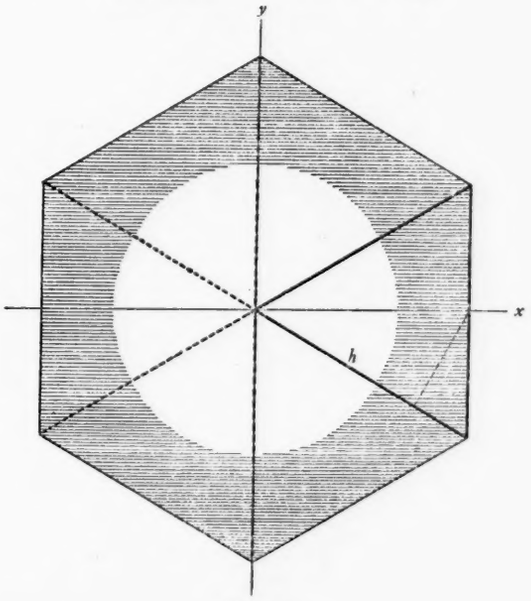

FIG. 2.

where $b^2 = c^2 = \frac{1}{2}a^2$. (Cf. Example 5, above.)

If the above expressions for vertical displacement be expanded in powers of x and y, the results are the same to two terms, viz.,

$$2 \cos \varkappa a t \left(1 - a^2 \frac{x^2 + y^2}{2^2}\right). \tag{137}$$

Considering now a square composed of four triangles, arranged as in the figure, we see that for this mode of oscillation there is a single nodal line consisting of four straight lines in the form of a square. Upon differentiating normally to any boundary of any of these triangles, or to the lines bisecting their right angles, it is easily seen that $\frac{\partial \zeta}{\partial \nu} = 0$.

One mode of oscillation for an equilateral triangle is shown in Fig. 2. Here ζ is proportional to

$$\cos\frac{\pi x t}{h}\left[2\cos\frac{\sqrt{3}\,\pi x}{2h}\cos\frac{\pi y}{2h}+\cos\frac{\pi y}{h}\right],^{*} \tag{138}$$

which is of the form

$$\cos x a t\,[2\cos bx\cos cy+\cos b'x\cos c'y],$$

where $b^2=\frac{3}{4}a^2$, $c^2=\frac{1}{4}a^2$, $b'^2=0$, $c'^2=a^2$. (Cf. Example 6, above.)

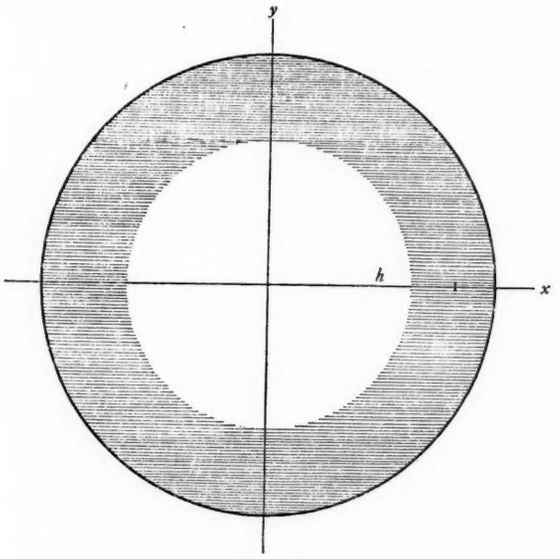

FIG. 3.

This, if expanded, gives for the first three terms

$$3\cos x a t\left[1-a^2\frac{x^2+y^2}{2^2}+a^4\frac{(x^2+y^2)^2}{2^2\,4^2}\right]. \tag{139}$$

Considering now a hexagon composed of equilateral triangles as in the figure, we see that for this mode of oscillation there is a single nodal line resembling the circumference of a circle much more nearly than did the nodal line in the preceding case. Upon differentiating normally to any boundary of any of these triangles or to lines bisecting their angles at the center of the hexagon, it is easily seen that $\frac{\partial\zeta}{\partial\nu}=0$.

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* Coast and Geodetic Survey Report for 1900, p. 589; also Poincaré, "Leçons de Mécanique Céleste," Tome III, p. 371.

The vertical displacement for the slowest symmetrical mode for a circle is proportional to

$$\cos \varkappa a t \, J_0 (a \varpi),$$

or

$$\cos \varkappa a t \left[1 - a^2 \frac{x^2 + y^2}{2^2} + a^4 \frac{(x^2 + y^2)^2}{2^2 \, 4^2} - a^6 \frac{(x^2 + y^2)^3}{2^2 \, 4^2 \, 6^2} + \cdots \right]. \quad (140)$$

That this satisfies the differential relation (132), has been shown under the heading "Application of Lemma (v)".

At the circumference where $\varpi = \varpi_1$, $\partial J(a\varpi)/\partial \varpi$ must be zero. Such a value of $a\varpi$ is $a\varpi_1 = 1.22\,\pi$, or $a = 1.22\,\pi/\varpi_1$; that is, ϖ_1 being given, a becomes fixed, and conversely. If $a\varpi = 0.7655\,\pi$, $J_0(a\varpi)$ becomes zero and so does the displacement ζ.

$$\therefore \; 1.22 \frac{\varpi}{\varpi_1} = 0.7655; \text{ and so } \varpi = 0.628\, \varpi_1$$

is the radius of the nodal circle. Since the motion is symmetrical about the center, the vectorial angle ϕ is not involved; consequently, $\dfrac{\partial \zeta}{\partial \phi} = 0$ everywhere.

Here the "speed" (*i. e.*, $2\pi \div$ period) is $\varkappa a$ or $1.22\,\varkappa\pi/\varpi_1$, while the "speed" for a rectangular trough whose length is ϖ_1 is $\varkappa\pi/\varpi_1$, $\varkappa$ denoting, as usual, the velocity of advance dependent upon the depth. The period for the circle is, therefore, $1/1.22 = 0.820$ times the period for a rectangular trough ϖ_1 in length.

Certain ratios for square, hexagonal and circular areas are given here:

	Height (ζ) at center ÷ Average height along perimeter	Area enclosed by nodal line ÷ Area of polygon	Virtual length (h) ÷ Apothem of polygon
Square (Fig. 1)	− 2.000	0.500	1.000
Hexagon (Fig. 2)	− 2.418	0.399	0.866
Circle (Fig. 3)	− 2.483	0.394	0.820

Comparing expressions (135), (139) and (140), we are led to believe that the R_1 of a sufficient number of terms of the form $\cos(\varkappa a t + \mathrm{i}\, bx + \mathrm{j}\, cy)$ could be made to represent this oscillation of a circular sheet of water, just as in the preceding example we were led to believe that the solution, $1/r$, of Laplace's equation could be regarded as the R_1 of an infinite number of terms of the form

$$\frac{1}{ax + iby + jcz}.$$

Moreover, it has been shown that any zonal harmonic of order n can be regarded as the R_1 of a suitable number of terms of the form

$$(ax + iby + jcz)^n.$$

§ 7. *Differential Relations Connecting the R-Functions.*

If

$$\frac{d(R_1 + iR_i + jR_j + ijR_{ij})}{d(ax + iby + jcz)} \tag{141}$$

has a unique value at a given point, it must be free from the variable quantities dx, dy, dz. Whenever the numerator contains the denominator as a factor, we have the relations

$$\left.\begin{aligned} &\frac{1}{a}\left(\frac{\partial R_1}{\partial x} + i\frac{\partial R_i}{\partial x} + j\frac{\partial R_j}{\partial x} + ij\frac{\partial R_{ij}}{\partial x}\right) \\ &\quad = \frac{1}{ib}\left(\frac{\partial R_1}{\partial y} + i\frac{\partial R_i}{\partial y} \underline{+} j\frac{\partial R_j}{\partial y} + ij\frac{\partial R_{ij}}{\partial y}\right) \\ &\quad = \frac{1}{jc}\left(\frac{\partial R_1}{\partial z} + i\frac{\partial R_i}{\partial z} + j\frac{\partial R_j}{\partial z} + ij\frac{\partial R_{ij}}{\partial z}\right). \end{aligned}\right\} \tag{142}$$

$$\left.\begin{aligned} \therefore\ &\frac{\partial R_1}{a\partial x} = \frac{\partial R_i}{b\partial y} = \frac{\partial R_j}{c\partial z}, \\ &\frac{\partial R_1}{b\partial y} = -\frac{\partial R_i}{a\partial x} = -\frac{\partial R_{ij}}{c\partial z}, \\ &\frac{\partial R_1}{c\partial z} = -\frac{\partial R_j}{a\partial x} = -\frac{\partial R_{ij}}{b\partial y}, \\ &-\frac{\partial R_1}{c\partial z} = -\frac{\partial R_i}{b\partial y} = \frac{\partial R_{ij}}{a\partial x} \end{aligned}\right\} \tag{143}$$

are therefore the nine conditions for a monogenic function of the three-dimensional variable $ax + iby + jcz$.

When the Roman R's and i, j are used, all signs are positive in these relations.

If the variable have more than three dimensions, the additional relations can be readily determined.

For two-demensional variables we have, of course,

$$\left.\begin{aligned} &\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}, \\ &\frac{\partial \mathrm{X}}{\partial x} = \frac{\partial \mathrm{Y}}{\partial y}, \quad \frac{\partial \mathrm{X}}{\partial y} = \frac{\partial \mathrm{Y}}{\partial x}. \end{aligned}\right\} \tag{144}$$

Curves on Quintic Scrolls.

By Frank B. Williams.

Introduction.

The chief aim of this paper is to prove, for scrolls of order 5, Professor Story's formula for the number of intersections of any two curves lying on the same ruled surface, which may be stated as follows: If a curve of order a lies entirely on a ruled surface of order μ, say R_μ, and meets each generator of R_μ in the same number of points, say α, this curve will be designated by the symbol $C_{a,\alpha}$ and α may be called the *rank* of the curve; then the number of intersections* of $C_{a,\alpha}$ with any other curve on R_μ, say $C_{b,\beta}$, of order b and rank β, is given by the formula

$$[C_{a,\alpha}\ C_{b,\beta}] = a\beta + b\alpha - \mu\alpha\beta.$$

In 1883† Professor Story proved this formula for $\mu = 2$ and $\mu = 3$. In 1900 I proved it for $\mu = 4$ and gave, in that paper,‡ Professor Story's three theorems which must be proved before the formula can be established, viz.:

Theorem I. If α be the number of points of C_a on an arbitrary generator, then there are α points of C_a on each generator.

Theorem II. If $C_{a,\alpha}$ is the complete intersection of R_μ and S_ν (S_ν being any surface of order ν) and $C_{b,\beta}$ is any curve on R_μ that has no component in common with $C_{a,\alpha}$, then $[C_{a,\alpha}\ C_{b,\beta}] = a\beta + b\alpha - \mu\alpha\beta$.

Theorem III. If $C_{a,\alpha}$ is irreducible and the partial intersection of R_μ and S_ν, $C_{a',\alpha'}$ being the residual intersection, and if the formula holds for each irreducible component of $C_{a',\alpha'}$ with an arbitrary curve $C_{b,\beta}$ on R_μ, it also holds for $C_{a,\alpha}$ with $C_{b,\beta}$.

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*It is to be understood here that if two curves pass through the same point on a multiple line or multiple curve they are *not* to be regarded as intersecting there unless they lie on the same sheet.

†On the Number of Intersections of Curves Traced on a Scroll of Any Order. *Johns Hopkins University Circulars*, August, 1883.

‡Geometry on Ruled Quartic Surfaces. *Proc. Amer. Acad. of Arts and Sciences*, Vol. XXXVI, No. 3.

The proofs of Theorems II and III are given in the paper cited, and Dr. Story has recently succeeded in proving Theorem I (called the fundamental theorem) for all ruled surfaces, and in also proving his formula for ruled surfaces of any order, μ, that have a multiple linear director of order $\mu - 1$ or $\mu - 2$, and these cases include seven of the fifteen cases of quintic scrolls enumerated by Schwarz.* These cases, however, as well as the proof of Theorem I in many cases, will be given here since these proofs are, in general, somewhat different.

The formula holds for each generator (which is a $C_{1,0}$ since it does not meet an arbitrary generator) with any curve $C_{a,\alpha}$, for $[C_{a,\alpha}\ C_{1,0}] = 0 + \alpha - 0 = \alpha$, and since every plane curve is either the complete intersection of the scroll and the plane or the partial intersection by a plane that cuts out otherwise one or more generators, by Theorems II and III the formula holds for all plane curves on an irreducible scroll of any order. It is not possible for a plane to cut an irreducible scroll in two distinct proper curves and, as Schwarz has pointed out, no scroll of order higher than four can have more than one simple conic, since two conics not in the same plane which correspond, point for point, determine a scroll of the fourth order.

The necessary multiple curve, or aggregate of multiple curves and rectilinear directors, on R_μ (which determine the scroll) will be called the frame and designated by the symbol $F(C^m_{a,\alpha}, C^n_{b,\beta}, \ldots.)$, where m, n, ... denote the multiplicities of the curves or linear directors and are omitted if the curves or linear directors are simple. The order of the curve must be multiplied by the multiplicity in reckoning the number of intersections, *i. e.*,

$$[C^m_{a,\alpha}\ C^n_{b,\beta}] \equiv ma\beta + nb\alpha - \mu a\beta.$$

It will be well to remember that this formula may not give all the actual intersection of two curves in space; if μ be chosen large enough so that the two curves lie on the same scroll, it may happen that both curves pass through the same point on a multiple curve but lie on different sheets there, and this point, which is a point of actual intersection in space, is not one of the points of intersection on the scroll as given by the formula, in accordance with the foot-note on page 421. It may, in like manner, happen that the number of intersections

* Über die geradlinigen Flächen fünften Grades. *J. für Math. von Crelle*, B. 67.

of any two curves considered as lying on one scroll, may be different from the number of their intersections when regarded as lying on another scroll.

If S_ν is a surface of order ν used to cut, from the scroll, any desired curve, the number of points that determine S_ν is $\frac{1}{6}(\nu+1)(\nu+2)(\nu+3)-1$; but if $\nu \geqq 5$ we must be sure that S_ν does not break up in the given scroll, R_5, and a surface $S_{\nu-5}$, and to do this we take $\frac{1}{6}(\nu-4)(\nu-3)(\nu-2)$ points of S_ν off the scroll R_5, *i. e.*, one more than enough to determine the surface $S_{\nu-5}$; as this number vanishes for $\nu = 2$, 3, or 4, we have for $\nu \geqq 2$

$$\left[\tfrac{1}{6}(\nu+1)(\nu+2)(\nu+3)-1\right]-\left[\tfrac{1}{6}(\nu-4)(\nu-3)(\nu-2)\right]=\tfrac{5}{2}\nu(\nu-1)+4$$

arbitrary points available.

For simplicity the scrolls of order 5 will be considered in the order given by Schwarz.

(A) SCROLLS OF DEFICIENCY ZERO. ($\rho = 0$.)

I. *Quintic Scroll with a Quadruple Linear Director.* $F(C^4_{1,1})$.

1. Only one scroll with a quadruple line is given by Schwarz, but there are two distinct species, viz., (I_1) without a simple director and (I_2) with a simple director, the equations in homogeneous coordinates x, y, z and s being of the form

$$U_4 z + V_4 s + W_5 = 0 \qquad \text{for } (I_1)$$

and

$$U_4 z + V_4 s \qquad = 0 \qquad \text{for } (I_2),$$

where U, V, and W are homogeneous polynomials in x and y of degrees indicated by the subscripts, the quadruple line being $x = 0$, $y = 0$ and the simple director in (I_2) being $z = 0$, $s = 0$.

2. *Proof of Theorem* I. In either case, a plane through the quadruple line, $C^4_{1,1}$, cuts out otherwise one generator from the scroll and any curve, C_b, on the scroll is met by this plane in a certain number of points, say β, on the generator and in $b-\beta$ points on $C^4_{1,1}$; as the plane is revolved about $C^4_{1,1}$ it cuts out, in succession, each generator of the scroll, and since there are always $b-\beta$ points of C_b on $C^4_{1,1}$ there are β points of C_b on each generator.

3. The formula holds for the frame, $C^4_{1,1}$, since $[C^4_{1,1}\ C_{b,\beta}] = 4\beta + b - 5\beta = b-\beta$, and we have just seen that $C_{b,\beta}$ has $b-\beta$ points on $C^4_{1,1}$.

Now if a cone of order b, with its vertex at an arbitrary point on $C^4_{1,1}$, be passed through any curve $C_{b,\beta}$, the complete intersection of the scroll and the

cone will be $C_{b,\beta}$, the four generators through the vertex, each of which is a β-tuple edge of the cone, and $C^4_{1,1}$, which is a $(b-\beta)$-tuple edge of the cone and counts for $4(b-\beta)$ lines in the intersection; for this gives $4(b-\beta)+4\beta+b=5b$, the order of complete intersection. Since the formula holds for every generator and for $C^4_{1,1}$, by Theorem III it holds for every curve $C_{b,\beta}$, plane or twisted.

4. *Plane Curves.* On the species (I_2) neither a plane cubic nor a conic can exist, since a plane through two generators must contain also the simple director and consequently also the other two generators through the point where the plane cuts the quadruple line; in other words, a plane through the simple director cuts out, otherwise, the four generators, one on each sheet, that pass through the point where the plane cuts the quadruple line. One and only one conic may exist on species (I_1) if, at some point of the quadruple line, three generators lie in the same plane; the plane of this conic will therefore be a triple tangent plane to the scroll and the conic will pass through the three points of tangency—these being the points where the conic meets the three generators in distinct points, and the other point of common intersection being on the quadruple line at which the conic lies on the sheet not having a generator in this plane. A plane through two generators on (I_1) will cut out a plane cubic $C_{3,1}$ having a double point where the plane cuts the quadruple line, and no other kind of a plane cubic exists on this scroll.

The only kind of a plane quartic that exists on either species is a $C_{4,1}$ having a triple point on $C^4_{1,1}$, whose plane cuts out one generator, and the only kind of a plane quintic that exists is a $C_{5,1}$ having a quadruple point on $C^4_{1,1}$, whose plane does not contain any line on the scroll. The line of intersection of the plane of a quartic and the plane of a quintic pierces the scroll five times; once where the generator in the plane of the quartic meets the quintic, and four times where the curves intersect, $[C_{5,1}\ C_{4,1}] = 5 + 4 - 5 = 4$.

On species (I_1) a plane quartic and plane cubic intersect in $[C_{4,1}\ C_{3,1}] = 4 + 3 - 5 = 2$ points; the other three points where the line common to their planes pierces the scroll being the one where the generator in the plane of the quartic meets the cubic and the two where the two generators in the plane of the cubic meet the quartic. If a conic exists on the scroll (I_1), it meets every plane quintic in two points, every quartic in one point, and does not meet a cubic at all, $[C_{2,1}\ C_{3,1}] = 2 + 3 - 5 = 0$.

5. *Twisted Curves.* Every curve on the scroll must meet the quadruple line; for we have seen that every curve, $C_{b,\beta}$, meets the quadruple line in $b-\beta$

points and $b-\beta \gtreqless 1$ since for species (I_1) two generators lie in a plane and $\beta \gtreqless \frac{b}{2}$, and for species (I_2) four generators lie in a plane and $\beta \gtreqless \frac{b}{4}$. Moreover, in the equation of species (I_1) there are only $5+5+6=16$ terms or 15 arbitrary parameters, and in species (I_2) only nine arbitrary parameters, and in order to make a quintic scroll contain even the cubic we must have $3\times 5+1=16$ arbitrary parameters, unless the quadruple line bears some relation to the curve, *i. e.*, an arbitrary line can not be chosen to be the quadruple line of the scroll and the scroll be made to pass through an arbitrary curve, even of the lowest order.

On species (I_2), no cubic can exist, since there are four generators in a plane and each generator must meet every curve at least once, but there can be a $C_{4,1}$ meeting the quadruple line, $C^4_{1,1}$, in three points, a $C_{5,1}$ meeting $C^4_{1,1}$ four times, etc., up to curves of order 8, of which we may have a $C_{8,1}$ meeting $C^4_{1,1}$ seven times, or a $C_{8,2}$ meeting $C^4_{1,1}$ six times; so that the curves on species (I_2) are arranged in groups of four, according to their ranks.

On species (I_1) every cubic is a $C_{3,1}$, and if there is no conic on the scroll only two generators lie in a plane, so that there may be a $C_{4,2}$ as well as a $C_{4,1}$, whereas if the conic exists there are three generators in one plane and no $C_{4,2}$ can exist. Accordingly, on this species (I_1) $\beta \gtreqless \frac{b}{3}$ if the conic exists and $\beta \gtreqless \frac{b}{2}$ if the conic does not exist (*i. e.*, if the scroll does or does not have a triple tangent plane, respectively).

The number of intersections of any curve $C_{b,\beta}$ with a $C_{5,1}$ is equal to the order of the curve, *i. e.* $[C_{b,\beta}\ C_{5,1}] = b+5\beta-5\beta = b$. The number of intersections of any two curves $C_{a,\alpha}$ and $C_{b,\beta}$, on either species (I_1) or (I_2), is equal to the order of the residual curve of intersection of the two cones, of orders a and b respectively, through these two curves from an arbitrary point on the quadruple line as a vertex: for, the cones intersect in a curve of order ab; each of the four generators through the vertex counts as an α-tuple edge on one cone and a β-tuple edge on the other, while the quadruple line counts for an $(a-\alpha)$-tuple edge on one and a $(b-\beta)$-tuple edge on the other, so that the residual curve is of order

$$ab-[4\alpha\beta+(a-\alpha)(b-\beta)] = a\beta+b\alpha-5\alpha\beta = [C_{a,\alpha}\ C_{b,\beta}].$$

This theorem is true for a scroll of any order, R_μ, having a director line of

multiplicity $(\mu - 1)$, for there are $\mu - 1$ generators through the common vertex of the cones and the residual curve is of order

$$ab - [(\mu - 1)\alpha\beta + (a - \alpha)(b - \beta)] = a\beta + b\alpha - \mu\alpha\beta,$$

which is $[C_{a,\alpha}\ C_{b,\beta}]$.

II. *Quintic Scroll with a Double Sextic Curve having a Triple Point.* $F(C^2_{6,3})$.

Every generator meets the double sextic three times; for, an arbitrary plane through one generator cuts out otherwise a plane quartic which is met four times by the generator, once at the point of tangency of the plane and three times on the double curve. The double sextic is therefore a $C^2_{6,3}$.

1. *Proof of Theorem* I. Pass a cubic surface, S_3, through the double sextic $C^2_{6,3}$. This we can do by taking the triple point and 16 other points of $C^2_{6,3}$ as 17 of the 19 points necessary to determine S_3; $C^2_{6,3}$ will then meet S_3 three times at the triple point and therefore 19 times in all, and consequently will lie entirely on S_3. As we have two more points for the determination of S_3 that can be arbitrarily chosen, a double infinity of cubic surfaces can be passed through $C^2_{6,3}$. Now, every generator meets S_3 three times on $C^2_{6,3}$, so that if we choose our two points, one on each of two generators, these generators must lie entirely on S_3 and the residual intersection of S_3 and the scroll will be $15 - (12 + 2) = 1$, *i. e.*, another generator; moreover, it is easy to see that any point of the intersection of S_3 and the scroll that is not a point of $C^2_{6,3}$, lies on some generator of the scroll, which generator therefore meets S_3 four times, *i. e.*, lies entirely on S_3, so that the residual intersection of S_3 that passes through $C^2_{6,3}$ consists always of three generators, two of which may be arbitrarily chosen. Consider any curve, C_b, on the scroll; it meets S_3 in $3b$ points, of which a fixed number, say k, lie on $C^2_{6,3}$ and the remaining $3b - k$ lie on the three generators; if p is the number of points of C_b on any generator, it is evident that $0 \leqq p \leqq b - 1$ and p can have, at most, b different values. For convenience we shall say that all generators for which p has the same value belong to the same set, a finite set or an infinite set, according as the number of generators in it is finite or infinite. There are, at most, b different sets and, since b is finite, at least one set must be infinite. Since there are only a finite number of generators in all the finite sets taken together, we can always choose S_3 so that no generator of any finite set will form part of the intersection. Suppose there are two infinite sets with values of p equal to β and γ respectively; having two

points at our disposal we can make S_3 cut out two β-generators in such a way as to avoid a finite set, and the third generator cut out will be either a β-generator or a γ-generator; if it is a β-generator then $3\beta = 3b - k$, and if it is a γ-generator $2\beta + \gamma = 3b - k$. In a similar manner we can make an S_3 cut out two γ-generators, and we will have either $3\gamma = 3b - k$ or $2\gamma + \beta = 3b - k$ according as the third generator is a γ- or a β-generator. Either of the first two conditions combined with either of the second two gives $\gamma = \beta$, so that there is only one infinite set. If we assume that there are three infinite sets, β, γ, and δ, and put for convenience $3b - k \equiv h$, the only arrangement differing from the above is:

$$\begin{Bmatrix} 2\beta + \gamma = h \\ 2\gamma + \delta = h \\ 2\delta + \beta = h \end{Bmatrix} \text{ or } \begin{Bmatrix} \gamma = h - 2\beta \\ \delta = h - 2\gamma = -h + 4\beta \\ \beta = h - 2\delta = \ 3h - 8\beta \\ h = 3\beta \end{Bmatrix} \text{ and therefore } \begin{Bmatrix} \gamma = \beta \\ \delta = \beta \\ h = 3\beta = 3b - k \\ k = 3b - 3\beta \end{Bmatrix},$$

which shows that only one infinite set exists. As this method can be extended to any finite number of sets, and b is finite, it is certain that only one infinite set exists. If now one point be chosen on a generator of a finite set, say an α-generator, and the other point so chosen on a β-generator of the infinite set that the third generator is also a β-generator, then

$$2\beta + \alpha = 3b - k = 3\beta, \text{ or } \alpha = \beta.$$

Therefore there is only one set of any kind on the scroll and every generator meets C_b in β points, while $C^2_{6,3}$ meets $C_{b,\beta}$ in $3b - 3\beta$ points. Since

$$[C^2_{6,3}\ C_{b,\beta}] = 12\beta + 3b - 15\beta = 3b - 3\beta,$$

the formula holds for the frame.

2. *Plane Curves.* Every plane quintic, $C_{5,1}$, has six double points, where its plane cuts the double sextic, and intersects the double sextic, therefore, in twelve points, twice at each double point, $[C_{5,1}\ C^2_{6,3}] = 15 + 12 - 15 = 12$. Every plane quartic is a $C_{4,1}$, having three double points, and meets $C^2_{6,3}$ in nine points, twice at each double point and once at each of the three points where the generator in the plane of the quartic meets the $C^2_{6,3}$, and $[C^2_{6,3}\ C_{4,1}] = 12 + 12 - 15 = 9$.

A plane through two generators cuts out a plane cubic, $C_{3,1}$, with a double point, that meets $C^2_{6,3}$ in six points, twice at the double point and once each at two points on each generator, $[C^2_{6,3}\ C_{3,1}] = 12 + 9 - 15 = 6$. The plane of the

cubic is a double tangent plane. If a triple tangent plane exists, it cuts out three generators and a conic, and therefore only one such plane can exist; the conic will meet $C^2_{6,3}$ in three points, one on each generator, and the other point of intersection of the conic with each generator is a point of tangency.

$$[C^2_{6,3}\ C_{2,1}] = 12 + 6 - 15 = 3.$$

3. *Twisted Curves.* Let S_ν be the surface that cuts out any curve $C_{b,\beta}$ from the scroll, and take ν large enough so that S_ν can be made to cut out $C^2_{6,3}$ and at the same time have a residual intersection, outside of generators, of order less than b, *i. e.*, $r \leqq b - 1$. By taking the triple point of $C^2_{6,3}$ as one point of S_ν, it will be necessary to take only $6\nu - 2$ other points of S_ν on $C^2_{6,3}$, or $6\nu - 1$ in all; $b\nu + 1$ points of $C_{b,\beta}$ must lie on S_ν, but as $3(b - \beta)$ points of $C_{b,\beta}$ lie on $C^2_{6,3}$ only $b\nu + 1 - 3(b - \beta)$ additional points of S_ν need be taken on $C_{b,\beta}$. The number of arbitrary points of S_ν remaining is

$$\{\tfrac{5}{2}\nu(\nu - 1) + 4\} - \{(b + 6)\nu - 3(b - \beta)\}.$$

At least two generators lie in one plane, and $\beta \leqq \frac{b}{2}$, so that every cubic is a $C_{3,1}$; if we take $\nu = 4$, we can make S_4 cut out $C_{3,1}$ and $C^2_{6,3}$ and have four arbitrary points left; the residual will consist entirely of generators, since each generator already meets S_4 three times on $C^2_{6,3}$ and once on $C_{3,1}$. Since the formula holds for $C^2_{6,3}$ and the generators it holds for $C_{3,1}$.

In general, if we choose $\nu = \dfrac{2b + 11 + n}{5}$, where $n = 1, 2, 3, 4$ or 5 according as $2b + 11$ is congruent to 4, 3, 2, 1 or 0 (mod. 5), respectively, and make S_ν cut out n generators in each case, we will have for the order of the residual,

$$r = 2b + 11 + n - (b + 12 + n) = b - 1.$$

To do this we must have

$$\tfrac{5}{2}\left(\frac{2b + 11 + n}{5}\right)\left(\frac{2b + 6 + n}{5}\right) + 4 - (b + 6)\left(\frac{2b + 11 + n}{5}\right) + 3b - 3\beta - n\left(\frac{2b + 11 + n}{5} - 2 - \beta\right) \geqq 0,$$

i. e.,

$$2(9 - n)b + (3 - n)n - 10(3 - n)\beta - 26 \geqq 0.$$

Since $\beta \lesseqgtr \frac{b}{2}$, this condition is satisfied, in the five different cases, by the following values of b, viz.:

1st Case: $n = 1$;

$$v = \frac{2b+12}{5};\ b = 4, 9, 14, \ldots \Big\} 16b - 20\beta - 24 \gtreqless 0, \text{ satisfied for } b \gtreqless 4.$$

2nd Case: $n = 2$;

$$v = \frac{2b+13}{5};\ b = 6, 11, 16, \ldots \Big\{ \begin{matrix} 14b - 10\beta - 24 \gtreqless 0 \\ 10\beta \lesseqgtr 5b;\ 9b \gtreqless 24 \end{matrix} \Big\}, \text{ satisfied for } b \gtreqless 3.$$

3rd Case: $n = 3$;

$$v = \frac{2b+14}{5};\ b = 3, 8, 13, \ldots \Big\} 12b - 26 \gtreqless 0, \text{ satisfied for } b \gtreqless 3.$$

4th Case: $n = 4$;

$$v = \frac{2b+15}{5};\ b = 5, 10, 15, \ldots \Big\{ \begin{matrix} 10b + 10\beta - 30 \gtreqless 0 \\ \beta \gtreqless 1;\ 10b \gtreqless 20 \end{matrix} \Big\}, \text{ satisfied for } b \gtreqless 2.$$

5th Case: $n = 5$;

$$v = \frac{2b+16}{5};\ b = 7, 12, 17, \ldots \Big\{ \begin{matrix} 8b + 20\beta - 36 \gtreqless 0 \\ \beta \gtreqless 1;\ 8b \gtreqless 16 \end{matrix} \Big\}, \text{ satisfied for } b \gtreqless 2.$$

Therefore, for all twisted curves on the scroll, we can choose v so that $r \lesseqgtr b-1$, and since we have proved the formula for $b = 3$, it holds for all values of b.

III. *Quintic Scroll with a Triple Linear Director and a Double Twisted Cubic.* $F(C^3_{1,1}\ C^2_{3,1})$.

A plane through the two generators that meet at an arbitrary point on the double cubic, cuts out the triple line, and cuts the cubic in two other points which, being double points of the section, must lie on the triple line; *i. e.*, the triple line meets the double cubic twice, and each generator meets the double cubic once, so that the double cubic is a $C^2_{3,1}$ and the triple line is a $C^3_{1,1}$.

1. *Proof of Theorem* I. Pass a quadric surface, S_2, through the double cubic, $C^2_{3,1}$, by choosing seven points of S_2 on $C^2_{3,1}$. Since two points of the triple line, $C^3_{1,1}$, lie on $C^2_{3,1}$, $C^3_{1,1}$ will lie entirely on S_2, if one more point of S_2 be chosen on $C^3_{1,1}$. The total intersection of S_2 and the scroll will then consist of $C^2_{3,1}$, $C^3_{1,1}$ and one generator $(6 + 3 + 1 = 10)$ and this generator can be selected at will by choosing the ninth point for the determination of S_2 on it. Now any curve, C_b, on the scroll meets $C^2_{3,1}$ and $C^3_{1,1}$ together in a fixed number of points, say k, and therefore must meet each generator in the same number of points, say β, where

$$\beta = 2b - k \text{ or } k = 2b - \beta.$$

2. *Plane Curves.* Every plane quintic is a $C_{5,1}$ having one triple point where its plane cuts the triple line and three double points where its plane cuts the double cubic; if the plane goes through one of the two points common to $C^3_{1,1}$ and $C^2_{3,1}$, the triple point of the quintic is modified by having two branches tangent, making the point equivalent to four double points. Every plane through one and only one generator cuts out a plane quartic, a $C_{4,1}$, having three double points, one on the triple line where the generator in its plane passes through and two on the double cubic where its plane cuts the double cubic not on the generator. Regarded as lying in the plane the generator meets the quartic four times, twice at the double point on the triple line, once where the generator meets the double cubic, and once at the point of tangency of the plane and scroll; but regarded as lying on the scroll, the generator meets the quartic only once, at the point of tangency of the plane and scroll, since this is the only one of the four where the generator and quartic lie on the same sheet, $[C_{4,1}\ C_{1,0}] = 0 + 1 - 0 = 1$. Every plane cubic is a $C_{3,1}$ having a double point where its plane cuts the double cubic, not on either of the two generators in its plane. There is one and only one true conic on this scroll, if there exists a triple tangent plane containing all three generators at some point on the triple line, in which case the conic meets each generator once on the double cubic and once at a point of tangency of plane and scroll.

3. *Twisted Curves.* Pass a cubic surface, S_3, through $C^2_{3,1}$ and $C^3_{1,1}$, by choosing 10 points S_3 on $C^2_{3,1}$ and two additional points of S_3 on $C^3_{1,1}$. Now $\beta \leqq \frac{b}{2}$ so that every cubic curve is a $C_{3,1}$ and has five points on $C^2_{3,1}$ and $C^3_{1,1}$ together; therefore by choosing five additional points of S_3 on $C_{3,1}$, S_3 will cut out $C_{3,1}$, the complete intersection being $C^2_{3,1}$, $C^3_{1,1}$, $C_{3,1}$ and three generators since every generator will already meet S_3 three times on the triple line and two curves together, and no other point of the residual can exist unless the generator through that point lies entirely on S_3. The formula holds for the frame, since the triple line lies on both sheets through the double cubic at each of the two points where it meets the double cubic, and hence these count for four intersections,

$$[C^3_{1,1}\ C^2_{3,1}] = 3 + 6 - 5 = 4;$$

also

$$[C^3_{1,1}\ C_{b,\beta}] = 3\beta + b - 5\beta = b - 2\beta,$$

and

$$[C^2_{3,1}\ C_{b,\beta}] = 6\beta + b - 5\beta = b + \beta,$$

so that any curve $C_{b,\beta}$ has $2b-\beta$ points on $C^2_{3,1}$ and $C^3_{1,1}$ together, as we have already seen. Since the formula holds for the frame, it holds for $C_{3,1}$. In general, since there are two points of $C^2_{3,1}$ on the triple line, and $2b-\beta$ points of $C_{b,\beta}$ on the triple line and $C^2_{3,1}$ together, we can make S_ν contain the triple line, $C^2_{3,1}$ and $C_{b,\beta}$ if we place $\nu+1$ points of S_ν on the triple line, $3\nu-1$ additional points on $C^2_{3,1}$ and $b\nu+1-(2b-\beta)$ additional points on $C_{b,\beta}$. If we choose $\nu=\dfrac{2b+8+n}{5}$, where $n=0$, 1, 2, 3, or 4, according as $2b+8$ is congruent to 0, 4, 3, 2, or 1 (mod. 5) respectively, and make S_ν cut out n generators in each case, we will have for the order of the residual,

$$r=2b+8+n-(b+9+n)=b-1.$$

To do this we must have

$$\frac{5}{2}\left(\frac{2b+8+n}{5}\right)\left(\frac{2b+3+n}{5}\right)+4-(b+4)\left(\frac{2b+8+n}{5}\right)$$
$$+2b-\beta-1-n\left(\frac{2b+8+n}{5}-1-\beta\right)\geqq 0,$$

i. e., $$2(5-n)b-(n+3)n+10(n-1)\beta-10\geqq 0.$$

We have the following five cases:

1st Case: $n=0$;

$$\nu=\frac{2b+8}{5};\ b=6,\ 11,\ 16,\ \ldots.\Big\}\ 10b-10\beta-10\geqq 0,\ \beta\leqq\frac{b}{2},\ b\geqq 2.$$

2nd Case: $n=1$;

$$\nu=\frac{2b+9}{5};\ b=3,\ \ 8,\ 13,\ \ldots.\Big\}\ 8b-14\geqq 0,\qquad b\geqq 2.$$

3rd Case: $n+2$;

$$\nu=\frac{2b+10}{5};\ b=5,\ 10,\ 15,\ \ldots.\Big\}\ 6b+10\beta-20\geqq 0,\ \beta\geqq 1,\ b\geqq 2.$$

4th Case: $n=3$;

$$\nu=\frac{2b+11}{5};\ b=7,\ 12,\ 17,\ \ldots.\Big\}\ 4b+20\beta-28\geqq 0,\ \beta\geqq 1,\ b\geqq 2.$$

5th Case: $n=4$;

$$\nu=\frac{2b+12}{5};\ b=4,\ \ 9,\ 14,\ \ldots.\Big\}\ 2b+30\beta-38\geqq 0,\ \beta\geqq 1,\ b\geqq 4.$$

The condition is therefore satisfied for all values of b, and since the formula holds for $b=3$ it holds for all curves on the scroll.

IV. *Quintic Scroll with a Triple Linear Director, a Double Conic and a Double Generator.* $F(C^3_{1,1}, C^2_{2,1})$.

This scroll is somewhat like the previous one where the double cubic has been replaced by a double conic and a double generator. The double conic has one point on the triple director and one point on the double generator, the double generator meeting both conic and triple director, for a plane through the triple director and double generator cuts the double conic twice, once on the triple director and once on the double generator because the double conic cannot lie in the same plane with either. The triple director is a $C^3_{1,1}$, the double conic a $C^2_{2,1}$ and the double generator a $C^2_{1,0}$, since no generator meets the double generator.

1. *Proof of Theorem* I. This is similar to the proof for Scroll III. If five points of S_2 be chosen on the double conic, $C^2_{2,1}$, two additional points on $C^3_{1,1}$ and one additional point on the double generator, making eight points in all, the S_2 will contain all these and cut out one generator $(10-4-3-2=1)$ which can be chosen at will by placing on it the ninth point for the determination of S_2, since each generator already meets S_2 once on $C^3_{1,1}$ and once on $C^2_{2,1}$. Therefore any curve C_b meets $C^3_{1,1}$, $C^2_{2,1}$ and $C^2_{1,0}$ in a fixed number of points, say $2b-\beta$, and every generator in the same number of points, β. For the number of intersections of $C_{b,\beta}$ with each of these the formula gives:

$$\begin{aligned}
[C^3_{1,1}\ C_{b,\beta}] &= 3\beta + b - 5\beta = b - 2\beta \\
[C^2_{2,1}\ C_{b,\beta}] &= 4\beta + b - 5\beta = b - \beta \\
[C^2_{1,0}\ C_{b,\beta}] &= 2\beta + 0 - 0 = 2\beta \\
\text{Total} \ldots\ldots\ldots\ldots &= 2b - \beta
\end{aligned}$$

Also

$$[C^2_{2,1}\ C^3_{1,1}] = 4 + 3 - 5 = 2, \quad [C^2_{1,0}\ C^2_{2,1}] = 2 \quad \text{and} \quad [C^2_{1,0}\ C^3_{1,1}] = 2,$$

which are correct results, since, at the points where these intersect, each lies on each of the two sheets through the point and there are therefore two intersections at each point.

2. *Plane Curves.* An arbitrary plane will cut out a plane quintic having three double points, one on the double generator and two on the double conic, and one triple point; if the plane of the quintic passes through the intersection of the double conic and triple director, a double point unites with a triple point, forming a tacnode with a branch through it, and if the plane passes through the

point of intersection of the double conic and double generator, two double points unite to form a tacnode. In any case, the quintic intersects the double conic four times, twice at each double point or tacnode, for, in case the tacnode has a branch through it, one of the branches of the quintic lies on a sheet that does not go through the double conic; $[C_{5,1}\ C^2_{2,1}] = 5 + 4 - 5 = 4$. The plane of the double conic cuts out one generator that meets the conic once where the conic meets the triple director and once elsewhere when considered as lying in the plane; but considered as lying on the scroll, the generator, at the point on the triple director, lies on the sheet that does not pass through the conic, and hence there is only one intersection of the generator and the conic which is at the other point in the plane.

In general, a plane through one generator cuts out a plane quartic, having three double points, one on the triple director where the generator meets it, one on the conic where the generator does not meet it and one on the double generator. A plane through two of the three generators that meet in a point on the triple director cuts out a plane cubic with a double point on the double generator. No conic can exist except the double conic and there is therefore no triple tangent plane.

[It seems probable that a conic could exist if there were a cuspidal generator instead of a double generator, which conic would lie in the tangent plane of the cuspidal edge.]

3. *Twisted Curves.* Since two generators lie in a plane, every cubic is a $C_{3,1}$, meeting the triple director once, the double conic twice and the double generator twice. If we pass a quadric surface, S_2, through the $C_{3,1}$, the double generator and the triple director by choosing seven points of S_2 on $C_{3,1}$, one more on the double generator, and one additional point on the triple director, the order of the residual will be $10 - 3 - 2 - 3 = 2$ and the residual must consist of two generators, since every generator already meets S_2 twice. The formula holds therefore for $C_{3,1}$.

We have seen that the triple director and double conic have one point in common and that the double generator meets both, and that altogether they meet $C_{b,\beta}$ in $2b - \beta$ points; we can, therefore, make S_ν contain the triple director, double generator, double conic, and $C_{b,\beta}$ by choosing $\nu + 1$ points of S_ν on the triple director, ν additional points on the double generator, $2\nu - 1$ additional points on the double conic and $b\nu + 1 - (2b - \beta)$ additional points

on $C_{b,\beta}$. The number of arbitrary points remaining for the determination of S_ν will then be

$$\tfrac{5}{2}\nu(\nu-1)+4-[(b+4)\nu-2a+\alpha+1]$$
$$=\tfrac{5}{2}\nu(\nu-1)-(b+4)\nu+2b-\alpha+3,$$

which is the same as for Scroll III, and by taking $\nu=\dfrac{2b+8+n}{5}$ as before, where $n=0$, 1, 2, 3, or 4 according as $2b+8$ is congruent to 0, 4, 3, 2, 1 (mod. 5), respectively, and making S_ν cut out n generators in each case, we will have

$$r=2b+8+n-(b+9+n)=b-1$$

as before, and we can prove, in the same manner as for Scroll III, that the formula holds for all curves on the scroll.

V. *Quintic Scroll with a Triple Director, a Double Director and two Double Generators.* $F(C^3_{1,1}\ C^2_{1,1})$.

The four lines form a *gauch quadrilateral* in which the two directors do not intersect and the two double generators do not intersect;

$$[C^3_{1,1}\ C^2_{1,1}]=3+2-5=0 \quad \text{and} \quad [C^2_{1,0}\ C^2_{1,0}]=0;$$

also $[C^3_{1,1}\ C^2_{1,0}]=0+2-0=2$, which is correct, since the double generator lies on two sheets through the triple director and has two intersections with the triple director at the point where they meet; the same is true for the double director and a double generator, $[C^2_{1,1}\ C^2_{1,0}]=0+2-0=2$.

1. *Proof of Theorem* I. Pass a quadric surface, S_2, through the four vertices and through one additional point in each side of the gauch quadrilateral; it will cut out all four lines, making the order of the residual

$$10-3-2-2-2=1,$$

i. e., the residual is a generator that may be chosen at will by placing on it the ninth point for the determination of S_2. Therefore, as before, every generator meets C_b in the same number of points, say β, and $C_{b,\beta}$ has $2b-\beta$ points on the four fixed lines taken together. The formula gives 2β points of intersection of $C_{b,\beta}$ and each double generator, $b-2\beta$ points for $C_{b,\beta}$ and the triple director, and $b-3\beta$ points for the double director and $C_{b,\beta}$, making in all $2b-\beta$, as before.

2. *Plane Curves.* Every plane quintic has one triple point and three double points unless its plane goes through a vertex of the gauch quadrilateral,

each of the two vertices on the triple director giving rise to a tacnode with a branch through it and each of the two vertices on the double director giving rise to a tacnode. A plane through a single generator cuts out a plane quartic that has three double points, one on the triple director, and one on each double generator, and meets the double director where the generator meets it. Since every plane through two distinct generators that meet in a point on the double director cuts out the triple director, and every plane through two generators that meet in a point on the triple director cuts out the double director and therefore the third generator through the point on the triple director, the plane of every cubic must pass through one of the double generators; conversely, every plane through either double generator that does not contain a director cuts out a plane cubic that has a double point where the plane cuts the other double generator, passes once through the point on the triple line and does not meet the double director at all, and we have

$$[C_{1,1}^3\ C_{3,1}] = 3 + 3 - 5 = 1,\ \ [C_{1,1}^2\ C_{3,1}] = 2 + 3 - 5 = 0.$$

There are then two sets of plane cubics, each set lying in a sheaf of planes whose axis is a double generator. It may be observed that there exists an infinite number of triple tangent planes, since the three generators through the same point on the triple line lie in a plane.

3. *Twisted Curves.* Since three generators lie in a plane, $\beta \leqq \frac{b}{3}$, and every cubic is a $C_{3,1}$ that has one point on the triple director, two points on each double generator and does not meet the double director at all. If we pass a quadric surface, S_2, through $C_{3,1}$ by placing seven points of S_2 on it and choose the remaining two points of S_2 on the triple director, then S_2 will cut out $C_{3,1}$, the triple director and the two double generators $(3 + 3 + 2 + 2 = 10)$ which form the complete intersection. The formula holds therefore for the $C_{3,1}$. By making S_ν pass through the vertices of the gauch quadrilateral and $\nu - 1$ additional points on each of the four sides, and through $b\nu + 1 - (2b - \beta)$ additional points of $C_{b,\beta}$, we will make S_ν cut out $C_{b,\beta}$, the triple and double directors, and the two double generators, and the remaining number of arbitrary points of S_ν will be

$$\tfrac{5}{2}\nu(\nu - 1) + 4 - [(b + 4)\nu - 2b + \beta + 1]$$
$$= \tfrac{5}{2}\nu(\nu - 1) - (b + 4)\nu + 2b - \beta + 3.$$

By choosing $v = \dfrac{2b + 8 + n}{5}$, where $n = 0, 1, 2, 3$ or 4, according as $2a + 8$ is congruent to 0, 4, 3, 2 or 1 (mod. 5), respectively, and making S_v cut out n generators in each case, the residual will be of order

$$r = 2b + 8 + n - (b + 9 + n) = b - 1,$$

and it can be shown in the same manner as for Scroll III, that the formula holds for every curve on the scroll.

VI. *Quintic Scroll with a Double Director and a Double Twisted Quintic with a Triple Point.* $F(C^2_{1,1}, C^2_{5,2})$.

A plane through the double director cuts out three generators that intersect, two and two, in three non-collinear points on the double quintic (the three points forming a triangle, two on each generator). The quintic is therefore a $C^2_{5,2}$ and meets the double director, $C^2_{1,1}$, at two points giving four intersections, $[C^2_{5,2}\ C^2_{1,1}] = 10 + 4 - 10 = 4$, each point counting for two intersections, since both line and curve lie on both sheets.

1. *Proof of Theorem* I. By choosing the triple point and 13 other points of $C^2_{5,2}$ as points of S_3 and two additional points of S_3 on the double director, we can make S_3 cut out $C^2_{5,2}$ and $C^2_{1,1}$. Each generator will then meet S_3 twice on $C^2_{5,2}$ and once on $C^2_{1,1}$, and consequently the residual must consist entirely of three generators which can be chosen at will since there are still three arbitrary points remaining for the determination of S_3. Any two of these generators may be kept fixed and the third one varied so as to include, in succession, all the other generators of the scroll, and since a fixed number of points of a curve, C_b, lie on $C^2_{1,1}$ and $C^2_{5,2}$ together, it is evident that the same number of points, say β, lie on each generator and that $3b - 3\beta$ points of $C_{b,\beta}$ lie on $C^2_{1,1}$ and $C^2_{5,2}$ together;

$$[C^2_{1,1}\ C_{b,\beta}] = 2\beta + b - 5\beta = b - 3\beta$$

and

$$[C^2_{5,2}\ C_{b,\beta}] = 10\beta + 2b - 10\beta = 2b$$

making a total of $3b - 3\beta$.

2. *Plane Curves.* An arbitrary plane cuts out a plane quintic with six double points; if the plane of the proper quintic goes through either of the two points common to the double director and double quintic, two of the double points unite to form a tacnode. A plane through only one generator cuts out a plane quartic having three double points where the plane cuts $C^2_{5,2}$, not on the

generator. A plane through two generators that pass through the same point on the double director cuts out a plane cubic with a double point where the plane cuts $C^2_{5,2}$ not on either generator. No conic exists.

3. *Twisted Curves.* Since three generators lie in a plane, every cubic is a $C_{3,1}$ and does not meet the double director, $[C_{3,1}\ C^2_{1,1}] = 3 + 2 - 5 = 0$; it must therefore meet $C^2_{5,2}$ in $3(b-\beta) = 6$ points, $[C_{3,1}\ C^2_{5,1}] = 6 + 10 - 10 = 6$. If we pass S_4 through 21 points of $C^2_{5,2}$, three additional points of the double director and seven additional points of $C_{3,1}$, S_4 will cut out the two curves and the double director, and the residual will consist entirely of generators, since each generator already meets S_4 twice on $C^2_{5,2}$, once on $C_{3,1}$ and once on the double director, and can not meet it again without lying entirely on it. Therefore the formula holds for every $C_{3,1}$. If, in general, we make S_ν pass through $5\nu - 1$ points of $C^2_{5,2}$ including the triple point, through $\nu - 1$ additional points of the double director and $b\nu + 1 - 3(b-\beta)$ additional points of $C_{b,\beta}$, then S_4 will cut out the frame and $C_{b,\beta}$, and the number of arbitrary points remaining for the determination of S_4 will be

$$\{\tfrac{5}{2}\nu(\nu-1) + 4\} - \{(b+6)\nu - 1 - 3(b-\beta)\}.$$

By choosing $\nu = \dfrac{2b + 11 + n}{5}$, where $n = 1, 2, 3, 4$ or 5, according as $2b + 11$ is congruent to 4, 3, 2, 1, or 0 (mod. 5) respectively, and making S_ν cut out n generators in each case, the residual will be of order $b - 1$ and it can be shown in the same manner as for Scroll II that the formula holds for all curves on the scroll.

VII. *Quintic Scroll with a Double Director, a Double Generator, and a Double Twisted Quartic having a Double Point.* $F(C^2_{1,1}\ C^2_{4,2})$.

A plane through the double director cuts out three generators whose mutual intersections form a triangle with the vertices on the double quartic, two on each generator; as the plane is revolved around the double director it comes to a position in which it passed through the double point of the quartic, and in this position the plane cuts out the double generator and a single generator, two of the generators uniting as two of the vertices of the triangle fall together at the double point of the quartic; the double generator therefore passes through the double point of the quartic and one other point, while the single generator in this plane passes through the double point of the quartic, on a sheet different from those through the double generator; thus the double generator has two intersections with the quartic at the double point and two more at the other point,

$[C^2_{1,0}\ C^2_{4,2}] = 4$, while the single generator has two intersections with the quartic and none with the double generator at the double point of the quartic, $[C_{1,0}\ C^2_{4,2}] = 2$ and $[C_{1,0}\ C^2_{1,0}] = 0$. The double director and double quartic have one point in common.

1. *Proof of Theorem* I. By passing S_3 through the frame and the double generator, *i. e.*, by choosing the double point and 11 other points of $C^2_{4,2}$, three additional points of the double director, and one additional point of the double generator as points of S_3, Theorem I can be proved in exactly the same way as for Scroll VI.

2. *Plane Curves.* An arbitrary plane cuts out a plane quintic with six double points, four on $C^2_{4,2}$, one on the double director and one on the double generator. A plane through one generator cuts out a plane quartic with three double points, one on the double generator and two on $C^2_{4,2}$. A plane through two generators that meet on the double director cuts out a plane cubic with a double point on the double generator, while a plane through the double generator cuts out a plane cubic with a double point on $C^2_{4,2}$, unless the plane also cuts out the double director.

3. *Twisted Curves.* Since the double quartic, $C^2_{4,2}$, and the double generator here, take the place of the double quintic on Scroll VI—counting for 10 in the order of the section by S_ν and requiring the same number of points of S_ν, *i. e.*, 4ν for $C^2_{4,2}$ (including the double point) and $\nu - 1$ additional points for $C^2_{1,0}$, or $5\nu - 1$ in all—the proof of the formula for this scroll is exactly the same as for Scroll VI.

VIII. *Quintic Scroll with a Double Conic and a Double Twisted Quartic having a Double Point.* $F(C^2_{2,1}\ C^2_{4,2})$.

A plane through two generators that meet on the double conic cuts out a cubic; there are six double points in the section, two on the double conic and four on the double quartic; the generators in this plane can not meet the conic again, so that the cubic has a double point on the double conic; the four double points of the section that are on the quartic lie two and two on each generator, through each of which the cubic passes. The double conic is therefore a $C^2_{2,1}$ and the double quartic is a $C^2_{4,2}$.

Schwarz has shown that the double conic goes through the double point of the quartic and meets the quartic in two other points; at each of these three points the conic and quartic intersect twice, since each lies on both sheets, making six intersections and $[C^2_{2,1}\ C^2_{4,2}] = 8 + 8 - 10 = 6$.

1. *Proof of Theorem* I. In the same way as for Scroll II, we can pass S_3 through the frame, obtaining three generators as the residual, and prove Theorem I in exactly the same way as for Scroll II.

2. *Plane Curves.* An arbitrary plane cuts out a plane quintic with six double points, two on $C^2_{2,1}$ and four on $C^2_{4,2}$; if the plane goes through the double point of $C^2_{4,2}$ and does not contain a generator, the section is a plane quintic with a triple point formed by the union of two of the double points on $C^2_{4,2}$ with one double point on $C^2_{2,1}$. The plane of the double conic cuts out one generator, as in the case of Scroll IV, which meets the conic once at the double point of the twisted quartic $C^2_{4,2}$, and once elsewhere when considered as lying in the plane, but has only one intersection with the conic when considered as lying on the scroll, since at the double point of $C^2_{4,2}$ the generator lies on a sheet that does not pass through the conic there. Any plane, except the plane of the double cone that passes through one generator, cuts out a plane quartic having two double points on $C^2_{4,2}$ and one on $C^2_{2,1}$. A plane through two generators that meet on $C^2_{2,1}$ cuts out a plane cubic with a double point on $C^2_{2,1}$, and a plane through two generators that meet on $C^2_{4,2}$ cuts out a cubic with a double point on $C^2_{4,2}$.

3. *Twisted Curves.* Since the double quartic and double conic take the place of the double sextic on Scroll II, counting 12 in the order of the section by S_ν and requiring the same number of points of S_ν—viz., $4\nu+1$ for $C^2_{4,2}$, if the double point is not chosen, and $2\nu-2$ additional points for $C^2_{2,1}$, or $6\nu-1$ in all—the proof of the formula for this scroll is the same as for Scroll II.

IX. *Quintic Scroll with Three Double Conics.* $F(C^2_{2,1}\ C^2_{2,1}\ C^2_{2,1})$.

The three double conics form a degenerate double sextic curve; each conic meets each of the other two in two points, and they have one point in common which takes the place of the triple point of the sextic, or, as Schwarz has pointed out, the planes of the three conics are three sides of a tetrahedron whose vertex is the point common to the three conics and whose base is the plane through the three other points of intersection of the conics, two and two. Since an arbitrary generator can meet the conic once only, each double conic is a $C^2_{2,1}$, and has three intersections with each of the other two, two at a point not common to all, where each conic lies on both sheets, and one at the common point where each conic meets the generator in the plane of the other on one sheet and the conic once on the other sheet; $[C^2_{2,1}\ C^2_{2,1}] = 4+4-5=3$.

Therefore the formula holds for the curves of the frame. The proof of

Theorem I is the same as for Scroll II, since S_3 can be passed through the frame and three arbitrary generators, and the proof of the formula is the same as for Scroll II, since the frame counts for 12 in the section by S_ν and requires only $6\nu - 2$ points of S_ν. $[C^2_{2,1}\ C_{b,\beta}] = 4\beta + b - 5\beta = b - \beta$, so that any curve, $C_{b,\beta}$, meets the three conics together in $3b - 3\beta$ points, which is the same number as that for the proper sextic on Scroll II.

X. *Quintic Scroll with a Double Generator and a Double Twisted Quintic having a Double Point.* $F(C^2_{5,3})$.

The double quintic and double generator together form a degenerate sextic with a triple point formed by the double generator passing through the double point of the quintic, but not in the plane of the tangents to the curve. This scroll differs from Scroll VI by having a double generator instead of a double director, in consequence of which each generator on this scroll meets the double quintic three times and it is therefore a $C^2_{5,3}$. The double generator has six intersections with the double quintic; $[C^2_{1,0}\ C^2_{5,3}] = 6$. The proof of Theorem I is the same as for Scroll II, since S_3 can be made to cut out $C^2_{5,3}$, the double generator and three arbitrary generators, and the proof of the formula is the same as for Scroll II, since the double quintic and double generator count for 12 in the section by S_ν and require $6\nu - 1$ points of S_ν.

(B) SCROLLS OF DEFICIENCY ONE. $(\rho = 1.)$

XI. *Quintic Scroll with Double Twisted Quintic That is Met Three Times by Each Generator.* $F(C^2_{5,3})$.

This scroll differs from Scroll X by having no double point on the quintic and consequently no double generator, which makes every plane curve of *deficiency one.*

1. *Proof of Theorem* I. If 16 points of S_3 be taken on $C^2_{5,3}$, S_3 will contain this curve and will cut out of the scroll, in addition, five generators, since each generator already meets S_3 in three points on $C^2_{5,3}$. There remain three points of S_3 that can be arbitrarily chosen so that three of the five generators can be chosen at will. Therefore by a method employed for Scroll II it can be shown that only one infinite set of generators exists, and each generator meets C_b in the same number of points, β. There must then be $3b - 5\beta$ points of $C_{b,\beta}$ on $C^2_{5,3}$, and the formula gives

$$[C^2_{5,3}\ C_{b,\beta}] = 10\beta + 3b - 15\beta = 3b - 5\beta,$$

so that the formula holds for the frame.

2. *Plane Curves.* Every plane not containing a generator cuts out a plane quintic having five double points, and no other kind of a plane quintic can exist A plane through one generator cuts out a plane quartic having two double points, and a plane through two generators cuts out a plane cubic without a double point.

3. *Twisted Curves.* Since two generators lie in a plane, $\beta \leqq \frac{b}{2}$, and therefore every cubic is a $C_{3,1}$. If 21 points of S_4 be chosen on $C^2_{5,3}$ and nine additional points on $C_{3,1}$, S_4 will cut out these two curves from the scroll and the residual intersection will consist entirely of generators, since each generator already meets S_4 three times on $C^2_{5,3}$ and once on $C_{3,1}$. Therefore the formula holds for $C_{3,1}$. In general S_ν may be passed through $C^2_{5,3}$ and any curve $C_{b,\beta}$ by choosing $5\nu+1$ points of S_ν on $C^2_{5,3}$ and $b\nu+1-(3b-5\beta)$ additional points of S_ν on $C_{b,\beta}$, which will leave

$$\{\tfrac{5}{2}\nu(\nu-1)+4\}-\{(b+5)\nu-3b+5\beta+2\}$$
$$=\tfrac{5}{2}\nu(\nu-1)-(b+5)\nu+3b-5\beta+2$$

arbitrary points for the determination of S_ν. Choosing $\nu=\frac{2b+13+n}{5}$, where $n=0, 1, 2, 3$ or 4 according as $2b+13$ is congruent to 0, 4, 3, 2 or 1 (mod. 5), respectively, and making S_ν cut out $n+4$ generators in each case, we have the condition,

$$\frac{5}{2}\left(\frac{2b+13+n}{5}\right)\left(\frac{2b+8+n}{5}\right)-(b+5)\left(\frac{2b+13+n}{5}\right)$$
$$+3b-5\beta+2-(n+4)\left(\frac{2b+13+n}{5}+1-3-\beta\right)\geqq 0,$$

i. e.,
$$2(5-n)b+(n-1)10\beta-n(n+3)-30\geqq 0,$$

where
$$1\leqq\beta\leqq\frac{b}{2}.$$

If this condition is satisfied, the order of the residual is

$$r=2b+13+n-\{10+b+(n+4)\}=b-1.$$

There are five cases to consider, viz.:

1st Case: $b=6, 11, \ldots;$

$2b+13\equiv 0 \pmod 5$; $n=0$, $\nu=\frac{2b+13}{5}$ — Condition is $10b-10\beta-30\geqq 0$ — $\begin{cases}\beta=\frac{b}{2}, & b\geqq 6\\ \beta=1, & b\geqq 4\end{cases}$.

Satisfied for $b=6, 11$, etc.

2nd Case: $b = 3, 8, 13, \ldots;$

$$2b + 13 \equiv 4 \pmod 5;\ n = 1,\ r = \frac{2b+14}{5}\Big\} \text{Condition is } 8a - 34 \gtreqless 0.$$

Satisfied for $b \gtreqless 5$, *i. e.*, for $b = 8, 13$, etc.

3rd Case: $b = 5, 10, 15, \ldots;$

$$2b + 13 \equiv 3 \pmod 5;\ n = 2,\ r = \frac{2b+15}{5}\Big\} \text{Condition is } 6b + 10\beta - 40 \gtreqless 0. \begin{cases} \text{Satisfied if} \\ \beta = \frac{b}{2},\ b \gtreqless 4 \\ \beta = 1\ ,\ b \gtreqless 5 \end{cases}$$

and $\therefore$ for $b = 5, 10, 15$, etc.

4th Case: $b = 7, 12, 17, \ldots;$

$$2b + 13 \equiv 2 \pmod 5;\ n = 3,\ r = \frac{2b+16}{5}\Big\} \text{Condition is } 4b + 20\beta - 48 \gtreqless 0. \begin{cases} \text{Satisfied if} \\ \beta = \frac{b}{2},\ b \gtreqless 4 \\ \beta = 1\ ,\ b \gtreqless 7 \end{cases}$$

and $\therefore$ for $b = 7, 12, 17$, etc.

5th Case: $b = 4, 9, 14, 19, \ldots.$

$$2b + 13 \equiv 1 \pmod 5;\ n = 4,\ r = \frac{2b+17}{5}\Big\} \text{Condition is } 2b + 30\beta - 58 \gtreqless 0. \begin{cases} \text{Satisfied if} \\ \beta = \frac{b}{2}, b \gtreqless 4 \\ \beta = 1\ , b \gtreqless 14 \\ \beta \gtreqless 2\ , b \gtreqless 0 \end{cases}$$

and $\therefore$ for $b = 14, 19, 24$, etc.

The only curves not included in this list which satisfy the condition are $C_{3,1}$ under case 2, for which the proof has already been given, and $C_{4,1}$, $C_{9,1}$, under case 5. Now $C_{4,1}$ can be cut out by an S_4 that passes through $C^2_{5,3}$ (and the residual will consist entirely of generators), for it only requires 21 points of S_4 on $C^2_{5,3}$ and 10 additional points on $C_{4,1}$ and we have 34 points available. $C_{9,1}$ can be cut out by an S_5 that passes through $C^2_{5,3}$, for it only requires 26 points of S_5 on $C^2_{5,3}$ and 24 additional points on $C_{9,1}$ and we have 54 points available; the residual is of order 6, for which the formula holds. Therefore the formula holds for all curves on the scroll.

XII. *Quintic Scroll with a Double Director and a Double Conic.* $F(C^3_{1,1}\ C^2_{2,1})$.

A plane through the triple director cuts out two generators that meet on the double conic, and since the generators can not meet the double conic again, the double conic meets the triple director in one point, which counts for two

intersections since the two sheets through the double conic also pass through the triple director; $[C^3_{1,1}, C^2_{2,1}] = 3 + 4 - 5 = 2$, and therefore the formula holds for the frame.

1. *Proof of Theorem* I. By choosing five points of S_2 on $C^2_{2,1}$ and two additional points of S_2 on $C^3_{1,1}$, S_2 will cut out the frame and three generators, two of which may be chosen at will as there are still two arbitrary points for the determination of S_2, and each generator already meets S_2 twice on the frame. Therefore, by the method employed for Scroll II, Theorem I is proven for this scroll. Every curve, $C_{b,\beta}$, on the scroll meets S_2 in $2b$ points, of which 3β lie on the three generators cut out by S_2, and consequently $2b - 3\beta$ points lie on the frame;

$$[C_{b,\beta}, C^3_{1,1}] = b + 3\beta - 5\beta = b - 2\beta,$$

and

$$[C_{b,\beta}, C^2_{2,1}] = b + 4\beta - 5\beta = b - \beta,$$

making in all $2b - 3\beta$.

2. *Plane Curves.* An arbitrary plane cuts out a plane quintic with a triple point and two double points; a plane through one generator (except the plane of the double conic) cuts out a plane quartic with two double points, one on $C^3_{1,1}$ and one on $C^2_{2,1}$; a plane through two of the three generators that meet in a point on the triple director cuts out a plane cubic without a double point, the three points where a generator in its plane meets it being the point of tangency, one point on $C^3_{1,1}$ and one point on $C^2_{2,1}$. No conic except the double conic can exist, and the plane of the double conic cuts out one generator in the same manner as the plane of the double conic on Scroll IV.

3. *Twisted Curves.* Since the triple director and double conic have one point in common, and any curve, $C_{b,\beta}$, has $2b - 3\beta$ points on the frame, S_ν can be made to cut out the frame and $C_{b,\beta}$ if

$$\{\tfrac{5}{2}\nu(\nu - 1) + 4\} - \{2\nu + 1 + \nu + b\nu + 1 - (2b - 3\beta)\} \overline{\geq} 0.$$

If we choose $\nu = \dfrac{2b + 6 + n}{5}$, where $n = 0$ or 5, 1, 2, 3, 4, according as $2b + 6$ is congruent to 0, 4, 3, 2, 1 (mod. 5) respectively, and make S_ν cut out n generators in each case, we have the following condition:

$$\frac{5}{2}\left(\frac{2b + 6 + n}{5}\right)\left(\frac{2b + 1 + n}{5}\right) - (b + 3)\left(\frac{2b + 6 + n}{5}\right) + 2b - 3\beta + 2 - n\left(\frac{2b + 6 + n}{5} - \beta - 1\right) \overline{\geq} 0,$$

i. e.,

$$2(5 - n)b + 10(n - 3)\beta - n(n + 1) - 10 \overline{\geq} 0,$$

and

$$r = 2b + 6 + n - 6 - 7 - n = b - 1.$$

This condition is satisfied in the five cases as follows:

1st Case: $b = 7, 12, 17, \ldots;\ 2b + 6 \equiv 0 \pmod 5$.

If $n = 0$, $\nu = \dfrac{2b+6}{5}$ and $10b - 30\beta - 10 \gtreqless 0$, which is satisfied for $b \gtreqless 4$ if $\beta = 1$.

If $n = 5$, $\nu = \dfrac{2b+11}{5}$ and $20\beta - 40 \gtreqless 0$, which is satisfied for $b \gtreqless 0$ if $\beta \gtreqless 2$.

In this case, therefore, for $\beta = 1$, $\nu = \dfrac{2b+6}{5}$, and for $\beta > 1$, $\nu = \dfrac{2b+11}{5}$.

2nd Case: $b = 4, 9, 14, \ldots;\ 2b + 6 \equiv 4 \pmod 5$.

$n = 1$, $\nu = \dfrac{2b+7}{5}$, $8b - 20\beta - 12 \gtreqless 0$, which is satisfied if $\beta \lesseqgtr \dfrac{2b-3}{5}$.

When $\beta \gtreqless \dfrac{2b-3}{5}$, take $\nu = \beta + 2$, in which case the residual will consist entirely of generators (since each generator meets the frame twice), and the condition becomes

$$\tfrac{5}{2}(\beta+2)(\beta+1) - (b+3)(\beta+2) + 2b - 3\beta + 2 \gtreqless 0.$$

i. e.,
$$5\beta^2 + 3\beta - 2b\beta + 2 \gtreqless 0, \qquad \text{(Eq. 1.)}$$

which for $\beta \gtreqless \dfrac{2b-3}{5}$ gives $10 \gtreqless 0$, and hence the condition is satisfied.

3rd Case: $b = 6, 11, 16, \ldots;\ 2b + 6 \equiv 3 \pmod 5$.

$n = 2$, $\nu = \dfrac{2b+8}{5}$, $6b - 10\beta - 16 \gtreqless 0$, which is satisfied for $\beta \lesseqgtr \dfrac{3b-8}{5}$.

When $\beta \gtreqless \dfrac{2b-8}{5}$, take $\nu = \beta + 2$, and the condition above (Eq. 1) becomes $3b^2 - 23b + 50 \gtreqless 0$, which is satisfied for $b \gtreqless 0$.

4th Case: $b = 3, 8, 13, \ldots;\ 2b + 6 \equiv 2 \pmod 5$.

$n = 3$, $\nu = \dfrac{2b+9}{5}$, $4b - 22 \gtreqless 0$; satisfied for $b \gtreqless 6$.

For $b = 3$, take $\nu = \beta + 2 = 3$, since $\beta \lesseqgtr \dfrac{b}{2}$ must be 1. Eq. 1, above, becomes $4 \gtreqless 0$, which is satisfied.

5th Case: $b = 5, 10, 15, \ldots;\ 2b + 6 \equiv 1 \pmod 5$.

$n = 4$, $\nu = \dfrac{2b+10}{5}$, $2b + 10\beta - 30 \gtreqless 0$, satisfied if $\begin{Bmatrix} \beta = 1, & b \gtreqless 10 \\ \beta \gtreqless 2, & b \gtreqless 5 \end{Bmatrix}$.

For $b=5$, $\beta=1$, take $\nu=\beta+2=3$, and Eq. 1 becomes

$$5+3-10+2\gtrless 0,$$

which is satisfied.

Therefore, for all curves on the scroll, ν may be so chosen that the residual will consist of a curve of order less than b, or else entirely of generators (*e. g.* when $\nu=\beta+2$, which applies to the case $b=3$). The formula holds therefore for all curves on this scroll.

XIII. *Quintic Scroll with a Triple Director, a Double Director and a Double Generator.* $F(C^3_{1,1}\ C^2_{1,1})$.

This scroll differs from Scroll V by having only one double generator instead of two; the number of intersections of this double generator with each director is the same as for Scroll V, $[C^3_{1,1}\ C^2_{1,0}]=2$ and $[C^2_{1,1}\ C^2_{1,0}]=2$.

1. *Proof of Theorem* I. If a quadric surface be passed through $C^3_{1,1}$ and $C^2_{1,1}$ and the double generator by placing three points of the quadric on each director and one on the double generator, the residual intersection will consist of three generators, two of which may be chosen at will by placing on them (one point on each) the two remaining points for the determination of the quadric. Therefore, by the reasoning employed for Scroll II the theorem is proved, and there will be $2b-3\beta$ points of $C_{b,\beta}$ on the frame and the double generator together.

2. *Plane Curves.* Every plane quintic will have one triple and two double points unless its plane goes through the point of intersection of the double generator and a director, in which case the plane quintic has a tacnode if on the double director, and a tacnode with a branch through it if the point is the intersection of the triple director and the double generator. The plane quartics have two double points, and the plane cubics have none and exist only in the planes through the double generator.

3. *Twisted Curves.* By passing S_ν through $\nu+1$ points on the triple director, $\nu+1$ points on the double director and $\nu-1$ additional points on the double generator, and $b\nu-(2b-3\beta)+1$ additional points on $C_{b,\beta}$, we make S_ν cut out both directors, the double generator, and $C_{b,\beta}$. This can be done if

$$\{\tfrac{5}{2}\nu(\nu-1)+4\}-\{(b+3)\nu-(2b-3\beta)+2\}\gtrless 0.$$

This condition is exactly the same as that for Scroll XII, and the proof of the formula is therefore the same as for that scroll.

XIV. *Quintic Scroll with a Double Director and a Double Twisted Quartic.* $F(C^2_{1,1}\ C^2_{4,2})$.

This scroll is similar to Scroll VII, but here no double point exists on the twisted quartic, and no double generator exists on this scroll. A plane through the double director cuts out three generators that form a triangle by intersecting two and two on three points of the twisted quartic, the other point of the twisted quartic in the plane lying on the director, as in Scroll VII.

1. *Proof of Theorem* I. Pass S_3 through the double director and twisted quartic by choosing four points of S_3 on $C^2_{1,1}$ and twelve additional points of S_3 on $C^2_{4,1}$. Since each generator meets the quartic twice and the director once, the residual intersection of S_3 and the scroll will consist of five generators, three of which may be chosen at will since three points of S_3 are still to be determined. Therefore, by a method analogous to the one employed for Scroll II, Theorem I may be proved, and therefore there are $3b - 5\beta$ points of $C_{b,\beta}$ on the double director and the double twisted quartic together.

2. *Plane Curves.* The plane quintic has five double points, the plane quartic has two double points and the plane cubic has no double point.

3. *Twisted Curves.* The double director and double twisted quartic form a degenerate twisted quintic, and the proof of the formula for this scroll is the same as that for Scroll XI, since S_ν can be made to pass through the frame and $C_{b,\beta}$ by placing $\nu + 1$ points of S_ν on $C^2_{1,1}$, 4ν additional points of S_ν on $C^2_{4,1}$ and $b\nu + 1 - (3b - 5\beta)$ additional points of S_ν on $C_{b,\beta}$, leaving

$$\{\tfrac{5}{2}\nu(\nu-1)+4\} - \{(b+5)\nu - 3b + 5\beta + 2\}$$
$$= \tfrac{5}{2}\nu(\nu-1) - (b+5)\nu + 3b - 5\beta + 2$$

arbitrary points for the determination of S_ν as before.

(C) SCROLL OF DEFICIENCY TWO. ($\rho = 2$.)

XV. *Quintic Scroll with a Triple Director and a Double Director.* $F(C^3_{1,1}\ C^2_{1,1})$.

This is the only quintic scroll of deficiency two. A plane through the triple director cuts out two generators that pass through the same point on the double director, and a plane through the double director cuts out three generators that pass through the same point on the triple director, and this scroll differs from Scroll XIII by not having a double generator.

1. *Proof of Theorem* I. By passing a quadric surface, S_2, through both directors, placing three points of S_2 on each, the residual intersection will be five

generators, three of which may be chosen at will by placing on each one point of the remaining three points for the determination of S_2. Therefore, by a method similar to the one employed for Scroll II, Theorem I can be proved and there are therefore $2b - 5\beta$ points of $C_{b,\beta}$ on the two directors taken together. By the formula $[C_{b,\beta}\, C^3_{1,1}] = b - 2\beta$ and $[C_{b,\beta}\, C^2_{1,1}] = b - 3\beta$.

2. *Plane Curves.* Every plane quintic has one triple and one double point, and every quartic has one double point, and no other plane curves exist.

3. *Twisted Curves.* Let S_ν, which cuts out the curve $C_{b,\beta}$, have the triple director as a line of multiplicity $\nu - \beta$; the residual in this case must consist entirely of generators, since each generator will meet S_ν β times on $C_{b,\beta}$ and $\nu - \beta$ times on the director, *i. e.*, ν times in all, and can not meet S_ν again without lying on it and forming a part of the residual. It is only necessary then to show that a value for ν can be found such that S_ν will cut out the curve $C_{b,\beta}$ and will have the triple director as a line of multiplicity $\nu - \beta$ and shall not break up into the scroll itself, *i. e.*, a surface of order 5 with a triple line, and another surface of order $\nu - 5$ with this same line as a line of multiplicity $\nu - \beta - 3$.

If we let $x = 0$, $y = 0$, in homogeneous coordinates, be the line of multiplicity k on a surface of order m, the number of terms in the equation of the surface will be the sum of all the terms of degree m in x and y together, plus all the terms down to those of degree k in x and y, *i. e.*,

$$\begin{aligned}&(m+1) + 2m + 3(m-1) + 4(m-2) + \ldots + (m-k+1)(k+1)\\ &= (m+1)[1 + 2 + 3 + \ldots + m - k + 1]\\ &\qquad - 2\left[1 + 3 + 6 + \ldots + \frac{(m-k+1)(m-k)}{2}\right]\\ &= (m+1)\binom{m-k+2}{2} = 2\binom{m-k+2}{3}.\end{aligned}$$

$$\left[\text{Using here the notation } \binom{p+q}{r} \equiv \frac{(p+q)(p+q-1)\ldots(p+q-r+1)}{1\,.\,2\,.\,3\ldots r}.\right]$$

Since a plane through the triple director cuts out two generators, there are $b - 2\beta$ points of $C_{b,\beta}$ on the triple director, which count for $(b - 2\beta)k$ points of intersection with a surface having this line of multiplicity k. In the present case $m = \nu$ and $k = \nu - \beta$ and

$$(\nu+1)\binom{\beta+2}{2} - 2\binom{\beta+2}{3} - 1 - b\beta - 2\beta\nu + 2\beta^2 - 1 \gtreqless 0.$$

Now when $\beta = 1$, it is impossible for S_ν to have a component of order $\nu - 5$ with a line of multiplicity $\nu - \beta - 3 = \nu - 4$, so that for $\beta = 1$,

$$(\nu + 1)\binom{3}{2} - 2\binom{3}{3} - 1 - b - 2\nu + 2 - 1 \geqq 0,$$

or

$$\nu \leqq b - 1.$$

For $\beta = 2$, the only way in which a surface of order $\nu - 5$ could have a line of multiplicity $\nu - \beta - 3 = \nu - 5$ is for the surface to break up into $\nu - 5$ planes through the line. If then we impose the condition that S_ν shall contain a line that is not on the scroll, S_ν can not have the scroll as a component, so that for $\beta = 2$

$$(\nu + 1)\binom{4}{2} - 2\binom{4}{3} - 2b - 4\nu + 6 - (\nu + 1) \geqq 0,$$

$$\nu \geqq 2b - 3.$$

For $\beta \geqq 3$, S_ν may break up into the scroll and $S_{\nu-5}$ with the triple director as a line of multiplicity $\nu - \beta - 3 \leqq \nu - 6$ on $S_{\nu-5}$; to prevent this we must make S_ν pass through a number of arbitrary points not on the scroll equal to one more than the number of arbitrary coefficients in the equation of an $S_{\nu-5}$ having a line of multiplicity $\nu - \beta - 3$, which is (since $m = \nu - 5$, $k = \nu - \beta - 3$, and $m - k + 2 = \beta$)

$$(\nu + 4)\binom{\beta}{2} - 2\binom{\beta}{3};$$

we have therefore

$$(\nu + 1)\binom{\beta + 2}{2} - 2\binom{\beta + 2}{3} - 1 - (\nu - 4)\binom{\beta}{2} + 2\binom{\beta}{3}$$
$$- b\beta - 2\beta\nu + 2\beta^2 - 1 \geqq 0,$$

from which

$$\nu \geqq b\beta - \frac{5\beta^2 - \beta - 2}{2}; \quad 3 \leqq \beta \leqq \frac{b}{3}.$$

Therefore a value for ν can always be found such that the residual will consist entirely of generators, and therefore the formula holds for all curves on the scroll.

On the Structure of Forms, and the Algebraical Theory of n-Lines.

By O. E. Glenn.

Factorable ternary quantics, representing n-lines, have an invariant theory which is in some respects analogous to binary invariant theory. The theory of degenerate forms lends considerable aid also to the study of non-factorable forms. In this paper we propose to develop the theory of factorable forms considerably farther than was done in a paper on the subject published by the present writer in Vol. XXXII, No. 1, of this Journal,* and to make a series of applications of this theory. Sections 1, 2, 3, 4 are devoted to ternary factor theorems. In § 5 an introduction to an invariant theory of n-lines from the standpoint of matrices is given. Resultant and discriminant matrices of forms representing n-lines are constructed. Section 6 contains a theory of rational partial fractions from the point of view of the Aronhold operator.

§ 1. *Multiple Linear Factors.*

The number of terms in an elementary symmetric function of m groups of any p homogeneous variables is equal to the number of distinct permutations of the variables occurring in any one term, when the subscripts are removed. Thus if the groups are

$$P_1(q_1, r_1, s_1),$$
$$P_2(q_2, r_2, s_2),$$
$$\cdots\cdots\cdots\cdots,$$
$$P_m(q_m, r_m, s_m),$$

we are led by a simple proof to the relation

$$\Sigma \frac{\partial}{\partial r_1}(\Sigma q_1 q_2 \cdots q_r r_{r+1} r_{r+2} \cdots r_{m-k} s_{m-k+1} \cdots s_m)\, s_1$$
$$= (k+1)\Sigma q_1 q_2 \cdots q_r r_{r+1} r_{r+2} \cdots r_{m-k-1} s_{m-k} \cdots s_m. \quad (1)$$

* "The Theory of Degenerate Algebraical Curves and Surfaces." The paper contains a short bibliography of the subject, to which may be added, Brioschi: "Sulla Condizioni, etc.," *Annali di Mathematica*, Ser. 12, Vol. VII; Thaer: "Ueber die Zerlegbarkeit einer ebenen Linie, u. s. w.," *Math. Annalen*, Vol. XIV; Bes: "Décomposition de la forme ternaire du troisième degré," *Math. Annalen*, Vol. LIX (1904).

Hence, when the ternary form

$$f_{3m} = a_{0m}\, y^m + \sum_{r=0}^{m-1} (a_{m-rr}\, x^{m-r} + a_{m-r-1r}\, x^{m-r-1} + \ldots + a_{0r})\, y^r,$$

which we write under the non-homogeneous notation

$$f_{3m} = x^m \phi_{0\, y/x} + x^{m-1} \phi_{1\, y/x} + \ldots + x\, \phi_{m-1\, y/x} + \phi_m,$$

is decomposable into m linear factors, its coefficients are connected with the coefficients of these linear forms, as

$$\rho_i = q_i\, y + r_i\, x + s_i \qquad (i = 1, 2, \ldots, m),$$

by the following relations, among others:

$$\Sigma \frac{\partial a_{m-r-kr}}{\partial r_1} s_1 = (k+1)\, a_{m-r-k-1r} \begin{pmatrix} r = 0, 1, \ldots, m-1 \\ k = 0, 1, \ldots, m-1 \\ \pi q_i = a_{0m} = 1 \end{pmatrix}. \tag{2}$$

Assume that r_1 is a root of multiplicity α_1 of

$$\phi_{0-r} \equiv r^m - a_{1\,m-1} r^{m-1} + a_{2\,m-2} r^{m-2} - \ldots + (-1)^m a_{m0} = 0.$$

Then evidently r_1 is a root of multiplicity $\alpha_1 - \varkappa$ of

$$\phi_{\varkappa-r} = 0 \qquad (\varkappa = 0, 1, 2, \ldots, \overline{\alpha_1 - 1}).$$

Now, when the coefficients a of $\phi_{\varkappa-r}$ are expressed in terms of the variables r_i, s_i, we write

$$\phi_\varkappa = \phi_{\varkappa-r}(a) = I_\varkappa(r_i, s_i; r) = I_\varkappa,$$

and then $I_\varkappa(r_i, s_i; r_1) \equiv 0$. By replacing r_i, r in $I_\varkappa$ by $r_i + \lambda s_i$, $r + \lambda s$ respectively, expanding by Taylor's theorem and using (2), we get

$$\begin{aligned}
&\sum_{h=0}^{\infty} \frac{\lambda^h}{\underline{|h}} \left(\Sigma \frac{\partial}{\partial r_1} s_1 + \frac{\partial}{\partial r} s\right)^h I_\varkappa \\
&\quad = \sum_{h=0}^{\infty} \frac{\lambda^h}{\underline{|h}} \left(\Sigma \frac{\partial}{\partial a_{m-r-kr}} (k+1)\, a_{m-r-k-1r} + \frac{\partial}{\partial r} s\right)^h \phi_\varkappa \\
&\quad = \sum_{h=0}^{\infty} \frac{\lambda^h}{\underline{|h}} \sum_{i=0}^{h} \binom{h}{i} s^{h-i} \frac{\partial^{h-i} \Delta_k^i\, \phi_\varkappa}{\partial r^{h-i}} \\
&\quad = \sum_{h=0}^{\infty} \frac{\lambda^h}{\underline{|h}} \sum_{i=0}^{h} (-1)^i \binom{h}{i} (\varkappa+1)(\varkappa+2)\ldots(\varkappa+i)\, s^{h-i} \frac{\partial^{h-i} \phi_{\varkappa+i}}{\partial r^{h-i}},
\end{aligned} \tag{3}$$

where

$$\Delta_k = \sum_k \left(a_{m-k-10} \frac{\partial}{\partial a_{m-k0}} + a_{m-k-21} \frac{\partial}{\partial a_{m-k-1_1}} + \ldots + a_{0\,m-k-1} \frac{\partial}{\partial a_{1\,m-k-1}}\right)(k+1).$$

The first non-vanishing term of this series, when $r = r_1$, is the term where $h = \alpha_1 - \varkappa$. Placing $\varkappa = 0$, we have from this term the result that the roots s_i corresponding to the α_1 equal r_i are the α_1 roots of the equation

$$\frac{\partial^{\alpha_1}\phi_{0-r_1}}{\partial r_1^{\alpha_1}} s^{\alpha_1} - \alpha_1 \frac{\partial^{\alpha_1-1}\phi_{1-r_1}}{\partial r_1^{\alpha_1-1}} s^{\alpha_1-1} + \alpha_1(\alpha_1-1)\frac{\partial^{\alpha_1-2}\phi_{2-r_1}}{\partial r_1^{\alpha_1-2}} s^{\alpha_1-2} - \ldots + (-1)^{\alpha_1}\underline{|\alpha_1}\,\phi_{\alpha_1-r_1} = 0. \tag{4}$$

Similar results hold for a root r_i of ϕ_{0-r} of multiplicity α_i $(i = 1, 2, \ldots, t)$. That is, we have for the homogeneous

$$f_{3m} = x_2^m \phi_{0\,x_1/x_2} + x_2^{m-1} x_3 \phi_{1\,x_1/x_2} + x_2^{m-2} x_3^2 \phi_{2\,x_1/x_2} + \ldots + \phi_{m\,x_1/x_2} x_3^m :$$

THEOREM 1. *The linearly factorable homogeneous ternary form f_{3m}, whose leading binary ϕ_{0-r} has r_i as a root of multiplicity α_i $(i = 1, 2, \ldots, t;\ \alpha_1 + \alpha_2 + \ldots + \alpha_t = m)$, can be factored into factors of the respective orders $\alpha_1, \alpha_2, \ldots, \alpha_t$, which are rational and integral in the coefficients of the form f_{3m} itself on the one hand, and in the quantities r_i on the other, linear in the coefficients; according to the formula*

$$f_{3m} = \prod_{i=1}^{t}\left[\frac{\partial^{\alpha}\phi_{0-r_i}}{\partial r_i^{\alpha_i}} x^{\alpha_i} - \alpha_i \frac{\partial^{\alpha_i-1}\phi_{1-r_i}}{\partial r_i^{\alpha_i-1}} x^{\alpha_i-1} y_i + \alpha_i(\alpha_i-1)\frac{\partial^{\alpha_i-2}\phi_{2-r_i}}{\partial r_i^{\alpha_i-2}} x^{\alpha_i-2} y_i^2 + \ldots + (-1)^{\alpha_i}\underline{|\alpha_i}\,\phi_{\alpha-r_i} y_i^{\alpha_i}\right] \quad (x_1 + r_i x_2/x_3 = x/y_i). \tag{5}$$

As an application of this theorem consider the problem of resolving the factorable ternary quartic f_{34}, where

$$x_2^4 \phi_{0\,x_1/x_2} = a_{400} x_1^4 + a_{310} x_1^3 x_2 + a_{220} x_1^2 x_2^2 + a_{130} x_1 x_2^3 + a_{040} x_2^4, \text{ etc.,}$$

and

$a_{400} = 1$, $a_{301} = a + 3$, $a_{202} = 3a - 4$, $a_{103} = -4a - 12$, $a_{004} = -12a$,
$a_{310} = 9$, $a_{211} = 7a + 19$, $a_{112} = 13a - 22$, $a_{013} = -14a - 24$,
$a_{220} = 30$, $a_{121} = 16a + 40$, $a_{022} = 14a - 28$,
$a_{130} = 44$, $a_{031} = 12a + 28$,
$a_{040} = 24$.

We find the roots of $\phi_{0\,x_1/x_2} = 0$ to be $+r_1 = +2$, $+r_2 = 3$; $\alpha_1 = 3$, $\alpha_2 = 1$. Equation (4) and the corresponding one for s_2, α_2 are then computed. They are

$$s^3 - (a+1)s^2 + (a-6)s + 6a = 0, \qquad s - 2 = 0.$$

Their roots are $s_1 = -2$, $s_2 = 3$, $s_3 = a$; $s_4 = 2$. Hence,

$$f_{34} = (x_1 + 2x_2 - 2x_3)(x_1 + 2x_2 + 3x_3)(x_1 + 2x_2 + a x_3)(x_1 + 3x_2 + 2x_3).$$

§ 2. *Extension of Method for Quaternary Forms.*

For elementary symmetric functions of m groups of four variables we have

$$\Sigma \frac{\partial}{\partial r_1}(\Sigma q_1 q_2 \cdots q_{m-j-k-l}\, r_{m-j-k-l+1}\, r_{j-k-l+2} \cdots r_{m-j-k}\, s_{m-j-k+1} \cdots s_{m-j}\, t_{m-j+1} \cdots t_m)\, t_1$$
$$= (j+1)\,\Sigma q_1 q_2 \cdots q_{m-j-k-l}\, r_{m-j-k-l+1}\, r_{m-j-k-l+2} \cdots r_{m-j-k-1}\, s_{m-j-k} \cdots s_{m-j-1}\, t_{m-j} \cdots t_m,$$

$$\Sigma \frac{\partial}{\partial r_1}(\Sigma q_1 q_2 \cdots q_{m-j-k-l}\, r_{m-j-k-l+1}\, r_{m-j-k-l+2} \cdots r_{m-j-k}\, s_{m-j-k+1} \cdots s_{m-j}\, t_{m-j+1} \cdots t_m)\, s_1$$
$$= (k+1)\,\Sigma q_1 q_2 \cdots q_{m-j-k-l}\, r_{m-j-k-l+1}\, r_{m-j-k-l+2} \cdots r_{m-j-k+1}\, s_{m-j-k} \cdots s_{m-j}\, t_{m-j+1} \cdots t_m,$$

and with

$$f_{4m} = \sum_{l=0}^{m} \Big[\sum_{k=0}^{m-l} (a_{l\,m-k-l\,k}\, x^l + a_{l-1\,m-k-l\,k}\, x^{l-1} + \cdots + a_{0\,m-k-l\,k})\, z^k\Big]\, y^{m-k-l}$$
$$= \prod_{i=1}^{m} (q_i y + r_i x + s_i z + t_i),$$

there results

$$\left.\begin{aligned} \Sigma \frac{\partial a_{l\,m-j-k-l\,k}}{\partial r_1}\, t_1 &= (j+1)\, a_{l-1\,m-j-k-l\,k}, \\ \Sigma \frac{\partial a_{l\,m-j-k-l\,k}}{\partial r_1}\, s_1 &= (k+1)\, a_{l-1\,m-j-k-l\,k+1} \end{aligned}\right\} \quad (6)$$

$$(j = 0, 1, \ldots, m-1;\ k = 0, 1, \ldots, m-1;\ l = 1, 2, \ldots, m).$$

The terms of a quaternary form may be arranged, under a homogeneous notation, as follows:

$$f_{4m} = x_2^m \Phi_{0\,x_1/x_2} + x_2^{m-1} x_3 \Phi^{(1)}_{1\,x_1/x_2} + x_2^{m-2} x_3^2 \Phi^{(1)}_{2\,x_1/x_2} + \cdots + x_3^m \Phi^{(1)}_{m\,x_1/x_2}$$
$$+ x_2^{m-1} x_4 \Phi^{(2)}_{1\,x_1/x_2} + x_2^{m-2} x_4^2 \Phi^{(2)}_{2\,x_1/x_2} + \cdots + x_4^m \Phi^{(2)}_{m\,x_1/x_2}$$
$$+ \Psi_{4m},$$

where

$$x_2^{m-k} \Phi^{(j)}_{k\,x_1/x_2} = a_{m-k\,0\,\underbrace{0\ldots0}_{j-1}\,k\,\underbrace{0\ldots0}_{2-j}}\, x_1^{m-k} + a_{m-k-1\,\underbrace{0\ldots0}_{j-1}\,k\,\underbrace{0\ldots0}_{2-j}}\, x_1^{m-k-1} x_2 + \cdots$$
$$+ a_{m-k-h\,\underbrace{0\ldots0}_{j-1}\,k\,\underbrace{0\ldots0}_{2-j}}\, x_1^{m-k-h} x_2^h + \cdots + a_{0\,m-k\,\underbrace{0\ldots0}_{j-1}\,k\,\underbrace{0\ldots0}_{2-j}}\, x_2^{m-k}.$$

Hence, by means of (6) and the methods of the previous section, we have, after writing the results in homogeneous form, with

$$f_{4m} = \prod_{i=1}^{m} (x_1 + r_{i}\, x_2 + r_{3i}\, x_3 + r_{4i}\, x_4) \qquad (a_{m000} = 1),$$

THEOREM 2: *If r_i is a root of $\phi_{0-r}=0$ of multiplicity α_i $(i=1, 2, \ldots, t;$ $\alpha_1+\alpha_2+\ldots+\alpha_t=m)$ of a quaternary form f_{4m}, then the coefficients r_{j+2} of its linear factors are the roots of the equation*

$$\prod_{i=1}^{t}\left[\frac{\partial^{\alpha_i}\phi^{(j)}_{0-r_i}}{\partial r_i^{\alpha_i}}r_{j+2}^{\alpha_i}-\alpha_i\frac{\partial^{\alpha_i-1}\phi^{(j)}_{1-r_i}}{\partial r_i^{\alpha_i-1}}r_{j+2}^{\alpha_i-1}+\alpha_i(\alpha_i-1)\frac{\partial^{\alpha_i-2}\phi^{(j)}_{2-r_i}}{\partial r_i^{\alpha-2}}r_{j+2}^{\alpha_i-2}\right.$$
$$\left.+\ldots+(-1)^i\lfloor\alpha_i\,\phi^{(j)}_{\alpha_i-r_i}\right]=0 \quad (j=1, 2;\ \phi^{(1)}_{0-r}=\phi^{(2)}_{0-r}=\phi_{0-r}). \qquad (7)$$

§ 3. *Multiple Roots of Functions $\Phi^{(m)}_{\xi}$, $\Psi^{(m)}_{\eta}$.*

Let the m roots of the equation

$$f(x)=a_0x^m+a_1x^{m-1}+\ldots+a_m=0$$

be $x_1, x_2, \ldots, x_m$. Represent the totality of the sums of these roots taken two at a time by $\xi_i\,(i=1, 2, \ldots, (m)=\frac{1}{2}m(m-1))$, and the totality of the products of the roots taken in pairs by $-\eta_j\,(j=1, 2, \ldots, (m))$. Let the equations having the sets of quantities ξ_i, η_j for roots be respectively

$$\Phi^{(m)}_{\xi}=\phi_0\xi^{(m)}+\phi_1\xi^{(m)-1}+\ldots+\phi_{(m)}=0,$$
$$\Psi^{(m)}_{\eta}=\psi_0\eta^{(m)}+\psi_1\eta^{(m)-1}+\ldots+\psi_{(m)}=0.$$

Then, for convenience of statement in this paper, let the following terminology be adopted: Let $x^2-\xi_1x-\eta_1$ be a quadratic factor of $f(x)$. This factor will be said to be of sum-multiplicity α if $f(x)$ has exactly α quadratic factors in which the coefficient of x is the same number, $-\xi_1$. The factor will be said to be of product-multiplicity β if just β of the factors of $f(x)$ have the same absolute term $-\eta_1$. Again, the factor will be called of multiplicity γ if it is repeated as a whole just γ times in $f(x)$. Evidently if the factor is of multiplicity γ, its sum-multiplicity or its product-multiplicity, or both, may be equal to γ or greater than γ.

Let $x^2-\xi_1x-\eta_1$ be a factor of $f(x)$ of multiplicity α. Suppose its sum-multiplicity and product-multiplicity are both α also. Then $x_1, x_2\,(x_1+x_2=\xi_1)$ are roots of multiplicity α of $f(x)$.

Let the roots of $\Phi^{(m)}_{\xi}=0$ be arranged in a triangular array as follows:

$$\begin{array}{llll} x_1+x_2, & x_1+x_3, & x_1+x_4, & x_1+x_5, \quad \ldots, \\ & x_2+x_3, & x_2+x_4, & x_2+x_5, \quad \ldots, \\ & & x_3+x_4, & x_3+x_5, \quad \ldots, \\ & & & x_4+x_5, \quad \ldots \end{array}$$

Then it is evident that ξ_1 is, *formally*, a root of $\Phi^{(m)}_{\xi}=0$ of multiplicity

$$1+3+5+\ldots+(2\alpha-1)=\alpha^2.$$

Similarly, η_1 is a root of $\Psi_\eta^{(m)} = 0$ of multiplicity α^2. More generally, if we assume

$$\left.\begin{aligned} &\xi_1 = \xi_2 = \cdots = \xi_{\alpha_1+1} = \cdots = \xi_{\alpha_1+\alpha_2} = \xi_{\alpha_1+\alpha_2+1} = \cdots = \xi_{\alpha_1+\alpha_2+\alpha_3} = \cdots = \xi_{\sum_{i=1}^{k}\alpha_i}, \\ &\eta_1 = \eta_2 = \cdots = \eta_{\alpha_1} \neq \eta_{\alpha_1+1} = \cdots = \eta_{\alpha_1+\alpha_2} \neq \eta_{\alpha_1+\alpha_2+1} = \cdots = \eta_{\alpha_1+\alpha_2+\alpha_3} \neq \cdots = \eta_{\sum_{i=1}^{k}\alpha_i}, \end{aligned}\right\} \quad (8)$$

where $\sum_{i=1}^{k} \alpha_i = \alpha$, it results that ξ_1 is a root of $\Phi_\xi^{(m)} = 0$ of multiplicity

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_k^2,$$

whereas $\eta_{\alpha_1+\alpha_2+\cdots+\alpha_h}$ is, *formally*, a root of $\Psi_\eta^{(m)}$ of multiplicity $\alpha_h^2 (h = 1, 2, \ldots, k)$.

A dual result of the same nature is obtained by interchanging the rôles of ξ_i, η_i in this paragraph.

§ 4. *Multiple Quadratic Factors.*

Assume that the general ternary form f_{3m} is decomposable into $\frac{1}{2}m = g$ quadratic factors (m even):

$$\tau_i = n_i x^2 + o_i xy + p_i y^2 + q_i x + r_i y + s_i \quad (i = 1, 2, \ldots, g).$$

Then

$$\prod_{i=1}^{g} (p_i y^2 + o_i yx + n_i x^2) = x^m \phi_{0\, y/x}.$$

The coefficients of the quadratic forms τ_i and those of f_{3m} itself are connected by the relations

$$\left(\Sigma \frac{\partial}{\partial n_1} q_1 + \Sigma \frac{\partial}{\partial o_1} r_1 + 2\Sigma \frac{\partial}{\partial q_1} s_1\right) a_{m-k-r\,r} = (k+1)\, a_{m-k-r-1\,r} \quad (9)$$
$$(k = 0, 1, 2, \ldots, m-1).$$

Consider now the case where $r^2 + o_1 r + n_1$ is a factor of multiplicity 1, and of product-multiplicity 1 and sum-multiplicity α of ϕ_{0r}. Then o_1 is a root of multiplicity α of $\Phi_{-o}^{(m)}$, whereas n_1 is a simple root of $\Psi_{-n}^{(m)}$. (We have taken $p_1 = p_2 = \cdots = a_{0m} = 1$.)

Let
$$\Phi = \Phi_{-o}^{(m)}(a) \equiv I(n_i, o_i, q_i, \ldots; o) = I,$$
$$\Psi = \Psi_{-n}^{(m)}(a) \equiv J(n_i, o_i, q_i, \ldots; n) = J.$$

Then
$$I(n_i, o_i, q_i, \ldots; o_1) \equiv J(n_i, o_i, q_i, \ldots; n_1) \equiv 0.$$

In $\Phi = I$ let us replace n_i, o_i, q_i by $n_i + \lambda q_i$, $o_i + \lambda r_i$, $q_i + 2\lambda s_i$ respectively, and o by $o + \lambda r$. Then Taylor's expansion becomes, by virtue of (9),

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\underline{j}} \left(\Sigma \frac{\partial}{\partial n_1} q_1 + \Sigma \frac{\partial}{\partial o_1} r_1 + 2\Sigma \frac{\partial}{\partial q_1} s_1 + \frac{\partial}{\partial o} r\right)^j I$$
$$= \sum_{j=0}^{\infty} \frac{\lambda^j}{\underline{j}} \left(\Sigma \frac{\partial}{\partial a_{m-k-l\,l}} (k+1)\, a_{m-k-l-1\,l} + \frac{\partial}{\partial o} r\right)^j \Phi. \quad (10)$$

When $o = o_1$, $r = r_1$, the first non-vanishing term of the right-hand series is the term where $j = \alpha$. Hence

THEOREM 3: *The α values of r constituting the set of r-coefficients of the α forms τ_i whose binary quadratic parts are factors of sum-multiplicity α of $x^m \phi_{0\,y/x} = 0$, are the roots of the following equation of order α:*

$$r^\alpha \frac{\partial^\alpha \Phi^{(m)}_{-o_1}}{\partial o_1^{\,\alpha}} + \binom{\alpha}{1} r^{\alpha-1} \frac{\partial^{\alpha-1} \Delta_k \Phi^{(m)}_{-o_1}}{\partial o_1^{\,\alpha-1}} + \binom{\alpha}{2} r^{\alpha-2} \frac{\partial^{\alpha-2} \Delta_k^2 \Phi^{(m)}_{-o_1}}{\partial o_1^{\,\alpha-2}} + \ldots + \binom{\alpha}{\alpha} \Delta_k^\alpha \Phi^{(m)}_{-o_1} = 0, \quad (11)$$

where

$$\Delta_k = \sum_k \left(a_{0\,m-k-1} \frac{\partial}{\partial a_{1\,m-k-1}} + a_{1\,m-k-2} \frac{\partial}{\partial a_{2\,m-k-2}} + \ldots + a_{m-k-1\,0} \frac{\partial}{\partial a_{m-k\,0}} \right) (k+1).$$

The corresponding values of q are the roots of the α linear equations

$$q_i \frac{\partial \Psi^{(m)}_{-n_i}}{\partial n_i} + \Delta_0 \Psi^{(m)}_{-n_i} = 0 \quad (i = 1, 2, \ldots, \alpha). \quad (12)$$

In the functions Φ, Ψ used here the coefficients $a_0, a_1, a_2, \ldots, a_m$ of § 3 are replaced by $a_{0m}, a_{1\,m-1}, \ldots, a_{m0}$, respectively.

Next let $r^2 + o_1 r + n_{\alpha_1+\alpha_2+\ldots+\alpha_h}$ be a factor of ϕ_{0r} of multiplicity α_h, and of product-multiplicity α_h $(h = 1, 2, \ldots, k)$, and let these factors be all of sum-multiplicity $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k$. Then equations (8), § 3, hold $(\xi_1 = -o_1)$, so that $n_{\alpha_1+\alpha_2+\ldots+\alpha_h}$ is a root of $\Psi^{(m)}_{-n}$ of multiplicity α_h^2 $(h = 1, 2, \ldots, k)$. Also $r^2 + o_1 r + n_{\alpha_1+\alpha_2+\ldots+\alpha_h}$, being of multiplicity α_h in ϕ_{0r}, will be of multiplicity $\alpha_h - \varkappa$ in $\phi_{\varkappa r}$ $(\varkappa = 0, 1, \ldots, \alpha_h - 1)$. Its product-multiplicity will be the same and its sum-multiplicity will be

$$\sum_{h=1}^{k} (\alpha_h - \varkappa) = \alpha - \varkappa k.$$

Then o_1 is a root of $\Phi^{(m-\varkappa)}_{-o}$ of multiplicity

$$\mu_\varkappa = (\alpha_1 - \varkappa)^2 + (\alpha_2 - \varkappa)^2 + \ldots + (\alpha_k - \varkappa)^2,$$

whereas $n_{\alpha_1+\alpha_2+\ldots+\alpha_h}$ is a root of $\Psi^{(m-\varkappa)}_{-n}$ of multiplicity $(\alpha_h - \varkappa)^2$.

Now we have

$$\Psi_{-n} = \Psi^{(m)}_{-n}(a) = J(n_i, o_i, \ldots; n) = J,$$

and $J(n_i, o_i, \ldots; n_j) \equiv 0$ $(j = \alpha_1 + \alpha_2 + \ldots + \alpha_h)$. In J let n_i, o_i, q_i, n be replaced by $n_i + \lambda q_i$, $o_i + \lambda r_i$, $q_i + 2\lambda s_i$, $n + \lambda q$ respectively, and we get

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\underline{j}} \left(\Sigma \frac{\partial}{\partial n_1} q_1 + \Sigma \frac{\partial}{\partial o_1} r_1 + 2\Sigma \frac{\partial}{\partial q_1} s_1 + \frac{\partial}{\partial n} q \right)^j J$$
$$= \sum_{j=0}^{\infty} \frac{\lambda^j}{\underline{j}} \left(\Sigma \frac{\partial}{\partial a_{m-k-l-1\,l}} (k+1)\, a_{m-k-l-1\,l} + \frac{\partial}{\partial n} q \right)^j \Psi^{(m)}_{-n}. \quad (13)$$

The first term of this series which does not vanish identically when $n = n_j$ is the one where $j = \alpha_h^2$. We get a similar result for r by replacing Ψ_{-n} by Φ_{-o}, $\frac{\partial}{\partial n} q$ by $\frac{\partial}{\partial o} r$, and α_h^2 by $\mu_0 = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_k^2$. Thus we have

THEOREM 4: *The complete determination of the quadratic factors τ_i of a form f_{3m}, of general multiplicity, involves the solution for the q- and r-coefficients, of the equations*

$$r^{\mu_0} \frac{\partial^{\mu_0} \Phi_{-o_1}^{(m)}}{\partial o_1^{\mu_0}} + \binom{\mu_0}{1} r^{\mu_0 - 1} \frac{\partial^{\mu_0 - 1} \Delta_k \Phi_{-o_1}^{(m)}}{\partial o_1^{\mu_0 - 1}} + \binom{\mu_0}{2} r^{\mu_0 - 2} \frac{\partial^{\mu_0 - 2} \Delta_k^2 \Phi_{-o_1}^{(m)}}{\partial o_1^{\mu_0 - 2}} + \ldots + \binom{\mu_0}{\mu_0} \Delta_k^{\mu_0} \Phi_{-o_1}^{(m)} = 0, \quad (14)$$

$$q^{\alpha_h^2} \frac{\partial^{\alpha_h^2} \Psi_{-n_j}^{(m)}}{\partial n_j^{\alpha_h^2}} + \binom{\alpha_h^2}{1} q^{\alpha_h^2 - 1} \frac{\partial^{\alpha_h^2 - 1} \Delta_k \Psi_{-n_j}^{(m)}}{\partial n^{\alpha_h^2 - 1}} + \ldots + \binom{\alpha_h^2}{\alpha_h^2} \Delta_k^{\alpha_h^2} \Psi_{-n_j}^{(m)} = 0 \quad (15)$$

$$(h = 1, 2, \ldots, k;\ \mu_0 = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_k^2).$$

§ 5. *Semi-Resultant and Discriminant Matrices of Forms Representing n-Lines.*

By a semi-resultant we shall mean a necessary and sufficient matrix condition that an m-line and an n-line, each represented by a ternary form, should have a 1-line in common. The name will probably justify itself inasmuch as a semi-resultant of two ternary forms will be seen to be the natural analogue of a binary resultant. It is, however, seminvariantive instead of invariantive.

The author has proved in another paper* that the general m-line is represented by the ternary form

$$f_{3m} = x_2^m \phi_{0\, x_1/x_2} + x_2^{m-1} x_3 \phi_{1\, x_1/x_2} + D^{-1} (-1)^{\frac{1}{2}m(m-1)} \sum_{j=2}^{m} x_3^j \sum_{i=0}^{m-j} \frac{\Delta_1^{m-j-i} \Delta_2^i R_m}{\underline{|m-j-i|}\ \underline{|i|}} x_1^{m-j-i} x_2^i = 0, \quad (16)$$

where

$$x_2^{m-k} \phi_{k\, x_1/x_2} = a_{k0} x_1^{m-k} + a_{k1} x_1^{m-k-1} x_2 + \ldots + a_{k\, m-k} x_2^{m-k} \quad (k = 0, 1),$$

R_m is the resultant of $x_2^m \phi_{0\, x_1/x_2}$ and $x_2^{m-1} \phi_{1\, x_1/x_2}$, D is the discriminant of $x_2^m \phi_{0\, x_1/x_2}$, and

$$\Delta_1 = m a_{00} \frac{\partial}{\partial a_{10}} + (m-1) a_{01} \frac{\partial}{\partial a_{11}} + \ldots + a_{0\, m-1} \frac{\partial}{\partial a_{1\, m-1}},$$

$$\Delta_2 = m a_{0\, m} \frac{\partial}{\partial a_{1\, m-1}} + (m-1) a_{0\, m-1} \frac{\partial}{\partial a_{1\, m-2}} + \ldots + a_{01} \frac{\partial}{\partial a_{10}}.$$

This result is a consequence of Theorem 1, § 1, when $\alpha_i = 1$.

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*(Note added June 10, 1912.) This paper has been published in *Transactions Amer. Mathematical Society*, Vol. XII, No. 3 (1911), p. 373.

It is therefore natural to expect that *any joint invariant of an m-line and an n-line will be expressible entirely in terms of the two binary forms* $x_2^{m-k}\phi_{k\,x_1/x_2}$ *and the analogous forms from the equation of the n-line.* That this is really the case also follows from the results in § 1. That is, we may write the m-line and n-line equations as follows:

$$f_{3m} = x_2^m \phi_{0\,x_1/x_2} + x_2^{m-1} x_3 \phi_{1\,x_1/x_2} + \cdots,$$
$$g_{3n} = x_2^n \psi_{0\,x_1/x_2} + x_2^{n-1} x_3 \psi_{1\,x_1/x_2} + \cdots \quad (n \leqq m).$$

Then we have (Theorem 1)

$$f_{3m} = \prod_{i=1}^{m} \left(x_1 + r_i x_2 - \phi_{1-r_i} \Big/ \frac{\partial \phi_{0-r_i}}{\partial r_i} \right),$$
$$g_{3m} = \prod_{j=1}^{n} \left(x_1 + r_j x_2 - \psi_{1-r_j} \Big/ \frac{\partial \psi_{0-r_j}}{\partial r_j} \right).$$

It now follows without difficulty that the condition for a common 1-line is equivalent to the condition that the following three binary equations have a common root:

$$\left.\begin{aligned} &\phi_{0x} = 0, \qquad \psi_{0x} = 0, \\ &X_x^{m+n-2} = \phi_{1x} \frac{\partial \psi_{0x}}{\partial x} - \psi_{1x} \frac{\partial \phi_{0x}}{\partial x} = 0. \end{aligned}\right\} \quad (17)$$

This condition can be expressed by means of a matrix, by using Stuyvaert's generalization* of the dialytic eliminant. Stuyvaert has shown that a necessary and sufficient condition that three polynomials in x of respective orders λ, μ, ν, $\nu \leqq \lambda + \mu - 1$, may have a common root, is that a certain matrix of $\lambda + \mu$ rows and $\lambda + \mu + 1$ columns should be of rank† $\lambda + \mu - 1$. We may express the condition here desired in the form of such a matrix with

$$\lambda = m - 1, \quad \mu = n, \quad \nu = m + n - 2.$$

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* Stuyvaert: *Cinq études de géométrie analytique* (1908), p. 60.

† Conditions more desirable than $J_2 = J_3 = 0$ in § 1, part II, of my former paper can be obtained by constructing from the elements of J_2, J_3, a Stuyvaert matrix, $\lambda = 2$, $\mu = 2$, $\nu = 3$, *e. g.*,

$$M = \begin{Vmatrix} a & 0 & a' & 0 & a_0 \\ b & a & b' & a' & a_1 \\ c & b & c' & b' & a_2 \\ 0 & c & 0 & c' & a_3 \end{Vmatrix},$$

where the sets a, b, c, and a', b', c' are respectively the non-zero elements of the third rows of J_2 and J_3. The necessary conditions $J_2 = J_3 = 0$ are also sufficient if $a^3{}_x = 0$ has but one real root. But the rank of M being 3 is both a necessary and sufficient condition, free from any assumption.

Thus, suppose $m = 3$, $n = 2$. Then

$$\phi_{0x} = a_{00}x^3 + a_{01}x^2 + a_{02}x + a_{03}, \quad \psi_{0x} = b_{00}x^2 + b_{01}x + b_{02},$$
$$X_x^3 = X_0x^3 + X_1x^2 + X_2x + X_3,$$

where

$$X_0 = 2a_{10}b_{00} - 3a_{00}b_{10},$$
$$X_1 = a_{10}b_{01} - 3a_{00}b_{11} + 2a_{11}b_{00} - 2a_{01}b_{10},$$
$$X_2 = a_{11}b_{01} - 2a_{01}b_{11} + 2a_{12}b_{00} - a_{02}b_{10},$$
$$X_3 = a_{12}b_{01} - a_{02}b_{11}.$$

Then the semi-resultant of f_{33} and g_{32} is*

$$\rho_{32} \equiv \begin{Vmatrix} a_{01}b_{00} - a_{00}b_{01} & a_{02}b_{00} - a_{00}b_{02} & a_{03}b_{00} & 0 \\ 0 & a_{01}b_{00} - a_{00}b_{01} & a_{02}b_{00} - a_{00}b_{02} & a_{03}b_{00} \\ b_{00} & b_{01} & b_{02} & 0 \\ 0 & b_{00} & b_{01} & b_{02} \\ X_0 & X_1 & X_2 & X_3 \end{Vmatrix}.$$

THEOREM 5: *A necessary and sufficient condition† that f_{3m}, g_{3n} may have a common linear factor is that ρ_{mn} should be of rank $m + n - 2$.*

It is worthy of notice that the coefficients of ϕ_{1x}, ψ_{1x} occur only in the X_i, so that two fourth-order determinants of ρ_{32} are linear in these coefficients. Therefore two of the latter are expressible rationally in terms of the remaining coefficients in the matrix.

THEOREM 6: *In general, if an m-line and an n-line have a 1-line in common, there will be $2m - 1$ independent coefficients in f_{3m} and $2n - 1$ in g_{3n}. Moreover, the remaining $\frac{1}{2}(m^2 - m + 2)$ coefficients of the first form and $\frac{1}{2}(n^2 - n + 2)$ of the second are all rationally expressible in terms of the $2(m + n - 1)$ independent ones.*

The condition that a given m-line $f_{3m} = 0$ may have a double 1-line can be expressed as a Stuyvaert matrix condition that the three binary equations

$$\phi_{0x} = 0, \quad \phi'_{0x} = 0, \quad \nabla_x^{2m-4} = \phi_{1x}'^2 - 2\phi''_{0x}\phi_{2x} = 0 \ddagger \tag{18}$$

---

* The negative condition

$$\begin{Vmatrix} a_{01}b_{00} - a_{00}b_{01} & a_{02}b_{00} - a_{00}b_{02} & a_{03}b_{00} \\ b_{00} & b_{01} & b_{02} \end{Vmatrix} \neq 0$$

is to be understood.

† As Stuyvaert points out, a certain negative condition concerning non-vanishing first minors of a definite determinant of order $\mu + \nu$ of the matrix is to be understood (see ρ_{32} above).

‡ Theorem 1 with $a_1 = 2$, $a_2 = 1$, $m = 3$.

may have a common root. When $m = 3$, this matrix is

$$\delta_3 \equiv \begin{Vmatrix} 3a_{00} & 2a_{01} & a_{02} \\ 6a_{00}a_{02} - 2a_{01}^2 & 9a_{00}a_{03} - a_{02}a_{01} & 0 \\ 0 & 6a_{00}a_{02} - 2a_{01}^2 & 9a_{00}a_{03} - a_{02}a_{01} \\ 4a_{10}^2 - 12a_{00}a_{20} & 4a_{10}a_{11} - 12a_{00}a_{21} - 4a_{01}a_{20} & a_{11}^2 - 4a_{01}a_{21} \end{Vmatrix}.$$

THEOREM 7: *A necessary and sufficient condition that the m-line $f_{3m} = 0$ may have a double 1-line is that the matrix δ_m should be of rank $2m - 4$. The form f_{3m} then has but $2m - 2$ independent coefficients and the remainder are all rationally expressible in terms of those of ϕ_{0x}, ϕ_{1x}.*

Inasmuch as δ_m gives the condition for a repeated linear factor of f_{3m}, it might be called the *ultra-discriminant.* Generalizations giving conditions for a common 1-line for more than two m-lines, and corresponding results for p-ary forms $p > 3$, can be made readily.

Moreover, the method given may be extended in the direction of finding a necessary and sufficient condition for a 1-line of multiplicity greater than 2 of an m-line. Thus from Theorem 1 with $\alpha_1 = 3$ the condition for a triple 1-line is expressible as a Stuyvaert matrix (generalized) condition that the following five binary equations should have a common root:

$$\left.\begin{aligned} &\phi_{0x} = 0, \quad \phi'_{0x} = 0, \quad \phi''_{0x} = 0, \\ &\rho_x^{2m-6} = \phi''^2_{1x} - 2\phi'''_{0x}\phi'_{2x} = 0, \\ &\sigma_x^{2m-6} = 2\phi'^2_{2x} - 3\phi''_{1x}\phi_{3x} = 0. \end{aligned}\right\} \quad (19)$$

These are derived from the Hessian of (4) with $\alpha_1 = 3$.

§ 6. *Resolution into Rational Partial Fractions.*

The results developed in sections 1, 3, 4 give us general formulas for the resolution of an ordinary rational proper algebraical fraction into the various standard types of partial fractions. For evidently the problem of resolving $x_2^{m-1}\phi_{1\,x_1/x_2}/x_2^m\phi_{0\,x_1/x_2}$ into partial fractions with real (linear and quadratic) denominators is identical, in part, with that of factoring a factorable form

$$x_2^m\phi_{0\,x_1/x_2} + x_2^{m-1}\phi_{1\,x_1/x_2} + \cdots$$

into its linear and quadratic real ternary factors (see § 3). We can consider

the general partial fraction theorem, therefore, as one interpretation of theorems 3, 4. A sufficiently general statement of it is embodied in the following:

THEOREM 8: *Let* $x_2^m \phi_{0\,x_1/x_2} = \prod_{i=1}^{h} (x_1 + r_i x_2) \prod_{j=1}^{l} (x_1^2 + \xi_j x_1 x_2 + \eta_j x_2^2)$, *where* $x_1 + r_i x_2$ *is a single (not multiple) factor and* $x_1^2 + \xi_j x_1 x_2 + \eta_j x_2^2$ *is of multiplicity* 1, *sum-multiplicity* α_j *and product-multiplicity* 1. *Then we have the resolution*

$$\frac{x_2^{m-1} \phi_{1\,x_1/x_2}}{x_2^m \phi_{0\,x_1/x_2}} = \sum_{i=1}^{h} \frac{\nu_i}{x_1 + r_i x_2} + \sum_{j=1}^{l} \frac{\lambda_j x_1 + \mu_j x_2}{x_1^2 + \xi_j x_1 x_2 + \eta_j x_2^2},$$

where

$$\nu_i = \frac{\Delta_k \phi_{0-r_i}}{\Delta \phi_{0-r_i}}, \quad \mu_j = -\Delta_k \Psi_{-\eta_j}^{(m)} / \Delta \Psi_{-\eta_j}^{(m)},$$

and the numbers λ_j *corresponding to the* α_j *factors of sum-multiplicity* α_j *are the* α_j *roots of*

$$\lambda^{\alpha_j} \frac{\partial^{\alpha_j} \Phi_{-\xi_j}^{(m)}}{\partial \xi_j^{\alpha_j}} + \binom{\alpha_j}{1} \lambda^{\alpha_j - 1} \frac{\partial^{\alpha_j - 1} \Delta_k \Phi_{-\xi_j}^{(m)}}{\partial \xi_j^{\alpha_j - 1}} + \binom{\alpha_j}{2} \lambda^{\alpha_j - 2} \frac{\partial^{\alpha_j - 2} \Delta_k^2 \Phi_{-\xi_j}^{(m)}}{\partial \xi_j^{\alpha_j - 2}} + \cdots + \binom{\alpha_j}{\alpha_j} \Delta_k^{\alpha_j} \Phi_{-\xi_j}^{(m)} = 0,$$

where

$$\Delta = \frac{\partial}{\partial r}, \quad \Delta_k = \sum_k \left(a_{0\,m-k-1} \frac{\partial}{\partial a_{1\,m-k-1}} + a_{1\,m-k-2} \frac{\partial}{\partial a_{2\,m-k-2}} + \cdots + a_{m-k-1\,0} \frac{\partial}{\partial a_{m-k\,0}} \right) (k+1).$$

These methods may be extended to the case of rational p-ary fractions $p > 2$.

PHILADELPHIA, PA., *July*, 1910.

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FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS, CAMBRIDGE, 1912.

The fifth International Congress of Mathematicians will be held in Cambridge, England, from the 22nd to the 28th of August, 1912.

The preceding Congresses were held at Zürich (1897), Paris (1900), Heidelberg (1904), Rome (1908); and it was at Rome that Cambridge was chosen for the next Congress.

A series of lectures is being planned, which shall give some idea of the present state and progress of the principal branches of Mathematical Science, including its applications to Physics and Engineering. Invitations to deliver lectures have already been accepted by E. Borel, E. W. Brown, A. Kneser, E. G. H. Landau, Sir J. Larmor, Sir W. White.

The definite Program of the Congress, including a statement of the receptions which it is proposed to offer to the members who attend, will be issued subsequently.

Special arrangements will be made for the consideration of the Reports of the International Committee appointed at the Congress in Rome to enquire into the teaching of Mathematics.

The Congress will be divided into four sections:

I. Arithmetic, Algebra, Analysis.
II. Geometry.
III. Mechanics, Physical Mathematics, Various applications of Mathematics.
IV. Philosophical, historical and didactical questions.

The fee for admission to membership of the Congress is £ 1. All members will be entitled to receive the volume of the Proceedings.

The families of members will be entitled to participate in the Congress on payment of a fee of 12 s. for each person, but this fee will not entitle them to the volume of the Proceedings.

For all information in regard to the Congress, apply to the General Secretary of the Organizing Committee, Professor E. W. Hobson, Christ's College, Cambridge, England.